Mathematical and Computational Applications, Vol. 15, No. 3, pp. 394-403, 2010.

METHOD FOR MULTIPLE ATTRIBUTE DECISION-MAKING WITH CONTINUOUS RANDOM VARIABLE UNDER RISK BASED ON PROJECTION MODEL

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Abstract- A rank approach based on projection model is proposed to deal with multiple attribute decision-making[MADM] problems under risk and with attribute value as continuous random variable on bounded intervals. Firstly, risk decision matrix is normalized by density function, and weights of attributes are calculated based on exception value of random variable by using projection pursuit model and genetic algorithm. Next, through calculating weighted correlation coefficients between alternatives and ideal solutions, weighted grey correlation projection models on ideal solutions are developed by grey correlation projection method for every alternative. Furthermore, alternatives are ranked by grey correlation projection value. Finally, an MADM example with interval numbers is provided to demonstrate the steps and effectiveness of the proposed approach

Key Words- Projection pursuit, Density function, Grey correlation, Multiple attribute decision-making

1.INTRODUCTION

Multiple attribute decision making has found wide application in society, economic, management, military and hard technology, and specifically plays an important role in fields of investment decision, project evaluation, economic benefit evaluation, Staff appraisal and so on. So far many mature methods have been proposed to solve multiple attribute decision making problem[1], but most of them need to know the exact attribute values of alternatives from the first . Conversely, the decisions in real life often need to be made in uncertain environment, and its attribute values are random variables varying with the natural state. In this case, decision-makers cannot know the actual state in future accurately, so they give every possible natural state, and quantify the randomicity by setting probability distribution. Thus wise the decisions satisfying the above conditions would be called multiple attribute decision making under risk problem[2]. From this, it can be seen that the research about multiple attribute decision making under risk problem has great theoretical and realistic significance. Now, the research aiming at multiple attribute decision making with continuous random variable under risk are relatively less. Literature [3] studies the risky multiple attribute decision making with continuous random variable on finite interval problems, and uses closeness degree with ideal solution to propose a random multiple attribute decision making method with incomplete weight information. Literature [4] presents TOPSIS rank method to solve risky multiple attribute decision making with weight information known and attribute values in the form of continuous random variables. Literature [5] discusses multiple Criteria decision making problem with weight information incomplete and Criteria values in the form of normally distributed random variables, and then develops a multiple criteria decision making method based on WC-OWA operator. Literature [6] places emphasis on risky multiple attribute decision making with attribute values in the form of continuous random variable on finite interval, and develops a new approach to handle this kind of problem. In the approach, maximizing deviations method is used to get the objective weights of attributes, meanwhile subjective and objective comprehensive weight model is constructed, and then alternatives are ranked based on the thought of weighted method. About risky multiple attribute decision making problem, so far there is no such research that uses projection pursuit model to determine objective weights of evaluation criteria and adopts grey correlation projection to rank alternatives. So this paper, to solve risky multiple attribute decision making problem with attribute values in the form of continuous random variable on finite interval, uses projection pursuit model to determine objective weights and then adopts grey correlation projection to rank alternatives.

2. DECISION PROBLEM DESCRIPTION

Suppose a risky multiple attribute decision making problem has m alternatives $A=(a_1,a_2,\cdots,a_m)$ with n attributes $C=(c_1,c_2,\cdots,c_n)$. w_j is the weight of attribute c_j, and satisfies the condition: $0\le w_j\le 1$, $\sum_{j=1}^{n} w_j = 1$. Weight vector $W=(w_1,w_2,\ldots,w_n)$ is constituted by subjective weight λ_j and objective weight ω_j. X_{ij} $(1\le i\le m;1\le j\le n)$ represents the evaluation value of alternative a_i with respect to attribute c_j. And $f_{ij}(x_{ij})$ is the probability density function of X_{ij}, which is known and takes values on closed interval $[x_{ij}^L,x_{ij}^U]$. x_{ij}^L,x_{ij}^U respectively represents the maximum and minimum values of random variable X_{ij}. In terms of above hypotheses, alternatives of risky multiple attribute decision will be comprehensively evaluated.

3. EVALUATION METHOD

3.1 Decision data normalization

There is need to normalize the decision data to eliminate the influence of different dimensions of physical quantity. The conventional attribute types are benefit attribute(I_1) and cost attribute(I_2). Suppose R_{ij} is the normalized value of X_{ij} and takes value on closed interval $[r_{ij}^L,r_{ij}^U]$. $g_{ij}(R_{ij})$ is the probability density function of R_{ij}. So the normalized method are as follows[5]:

For attribute c_j $(1\le j\le n)$, let

$M_j=\max_i(x_{ij}^U)$, $S_j=\min_i(x_{ij}^L)$.

If c_j is benefit attribute(I_1), let

$$R_{ij} = \frac{X_{ij} - S_j}{M_j - S_j}\ ,\ \ (1 \le i \le m; 1 \le j \le n)$$

(1)

For probability density function of $X_{ij} : f_{ij}(x_{ij})$ is known, it is easy to get probability density function of R_{ij} ($R_{ij} = [r_{ij}^L, r_{ij}^U]$) as follows:

$$g_{ij}(R_{ij}) = (M_j - S_j) f_{ij}(S_j + (M_j - S_j) \times R_{ij}) \quad (1 \le i \le m; 1 \le j \le n)$$

(2)

$$r_{ij}^L = \frac{x_{ij}^L - S_j}{M_j - S_j}, r_{ij}^U = \frac{x_{ij}^U - S_j}{M_j - S_j}$$

(3)

If c_j is cost attribute(I_2), let

$$R_{ij} = \frac{M_j - X_{ij}}{M_j - S_j}\ ,\ \ (1 \le i \le m; 1 \le j \le n) \quad .$$

(4)

And in the same way, the probability density function of R_{ij} ($R_{ij} = [r_{ij}^L, r_{ij}^U]$) is as follows:

$$g_{ij}(R_{ij}) = (M_j - S_j) f_{ij}(M_j - (M_j - S_j) \times R_{ij}) \quad (1 \le i \le m; 1 \le j \le n)$$

(5)

$$r_{ij}^L = \frac{M_j - x_{ij}^U}{M_j - S_j}, r_{ij}^U = \frac{M_j - x_{ij}^L}{M_j - S_j}$$

(6)

Obviously, there is :

$$\int_0^1 g_{ij}(R_{ij}) dR_{ij} = \int_{r_{ij}^L}^{r_{ij}^U} g_{ij}(R_{ij}) dR_{ij} = 1 \quad (1 \le i \le m; 1 \le j \le n) \tag{7}$$

3.2 Calculate objective weight of evaluation index with projection pursuit model

(1) Basic thought of projection pursuit

Projection pursuit model("PP" in abbreviation) is a kind of dimension reduction analysis method for high dimensional data proposed by Friedman and Tukey[9]. Its basic thought is as follows: high dimensional data is projected onto low dimensional subspace(from one dimension to three dimensional) by some combination, and then data structure on low dimensional is analyzed to achieve the purpose that study and analysis of the high dimensional data. In decision field, projection pursuit model is mainly applied in certain multiple attribute decision making[10][11]. This paper applies projection pursuit model in risky multiple attribute decision making with expected value of random variable.

(2) Construct projection index function $Q(w)$

According to normalized method, optimum solution corresponding to different attributes is: $V^{+}=\underbrace{(1,1,\cdots,1)}_{n}$. And weighted optimum solution is: $V^{*}=(w_1,w_2,\cdots,w_n)$.

For R_{ij} is random variable, let the expected value of R_{ij} : $E(R_{ij})$ represent attribute value,

$$E(R_{ij})=\int_{-\infty}^{+\infty}R_{ij}g_{ij}(R_{ij})dR_{ij}=\int_{r_{ij}^{L}}^{r_{ij}^{U}}R_{ij}g_{ij}(R_{ij})dR_{ij} \tag{8}$$

Projection pursuit method is integrating n dimensional data $E(R_{ij})(i=1,2,\cdots,m;j=1,2,\cdots,n)$ into one dimension $z_i(w)$ which is in the projection direction of weighted optimum solution $V^{*}=(w_1,w_2,\cdots,w_n)$, namely the direction of $W=(w_1,w_2,\cdots,w_n)$, so

$$z_i(w)=\sum_{j=1}^{n}w_jE(R_{ij}) \tag{9}$$

Then according to one dimension scatter diagram of $z_i(w)$ $(i=1,2,\cdots,m)$, the analysis and evaluation are made. About comprehensive projection index values, scatter characteristics of projection values $z_i(w)$ are as follows: local projection points are as dense as possible, and had better agglomerate into some points or groups; but for the whole values, projection points and groups are as dispersed as possible, so projection index function can be represented as follows:

$$Q(w)=S_z(w)\times D_z(w) \tag{10}$$

In the above equation, $S_z(w)$ is the standard deviation of $z_i(w)$; $D_z(w)$ is the local density of projection value $z_i(w)$. And the computing formulas of $S_z(w)$ and $D_z(w)$ are as follows:

$$S_z(w)=\sqrt{\frac{\sum_{i=1}^{m}(z_i(w)-\overline{z}(w))^2}{m-1}} \tag{11}$$

$$D_z(w)=\sum_{i=1}^{m}\sum_{j=1}^{m}(T-t(i,j,w))\times u(T-t(i,j,w)) \tag{12}$$

In the above two equations, $\overline{z}(w)$ is the mean value of $z_i(w)$ $(i=1,2,\cdots,m)$. T is the window radius of local density. The value of T should make sure that mean number of projection points in window can not be too less to avoid that deviation of moving average is too big and ensure that it will not increase too much with m. T can be determined by experiment, generally takes value as 0.1 S_z, $t(i,j,w)$ represents the distance between samples, $t(i,j,w)=\left|z_i(w)-z_j(w)\right|$, $u(x)$ is the unit step function, and its definition is as follows:

$$u(x)=\begin{cases}1 & x\geq 0\\ 0 & x<0\end{cases} \tag{13}$$

(3) Optimized projection index function

When the sample set of every index is determined, projection index function $Q(w)$ only changes with the direction of projection W. Different projection directions reflect different data structure characteristics. optimal projection direction is the projection direction which most probably reveals some kind of structure characteristic of high dimensional data. So, by solving maximization problem of projection index function, the optimal projection direction can be determined, namely:

$$\begin{aligned}&\max\ Q(w)=S_z(w)\times D_z(w)\\ &s.t.\\ &\begin{cases}\sum_{j=1}^{n}w_j=1\\ 0\leq w_j\leq 1\end{cases}\end{aligned}$$

(14)

This is a nonlinear optimization problem with $w_j(j=1,2,\cdots,n)$ as optimization variable. So genetic algorithm can be adopted to achieve the purpose of global optimization [8], and then the weight vector W can be got.

3.2 Decision steps based on grey correlation projection method

(1) Ideal solution

According to normalized method, optimum solution corresponding to different attributes is: $V^{+}=\underbrace{(1,1,\cdots,1)}_{n}$.

(2) Calculate grey correlation coefficient between i*th* alternative and ideal alternative with respect to j*th* attribute[7]

$$\xi_{ij}^{+} = \frac{N + \rho M}{d_{ij}^{+} + \rho M}, \ \rho \in (0,1) \tag{15}$$

In the above equation, $d_{ij}^{+} = E\left|R_{ij} - v_j^{+}\right| = \int_{-\infty}^{+\infty} \left|R_{ij} - 1\right| g_{ij}(R_{ij}) dR_{ij}$, $N = \underbrace{\min}_{i} \underbrace{\min}_{j} d_{ij}^{+}$, $M = \underbrace{\max}_{i} \underbrace{\max}_{j} d_{ij}^{+}$, ρ represents resolution coefficient, and generally is set to 0.5.

Then, grey correlation coefficient matrix between every alterative and ideal alternative is as follows:

$$\xi^{+} = \begin{bmatrix} \xi_{11}^{+} & \xi_{12}^{+} & \cdots & \xi_{1n}^{+} \\ \xi_{21}^{+} & \xi_{22}^{+} & \cdots & \xi_{2n}^{+} \\ \vdots & \vdots & \vdots & \vdots \\ \xi_{m1}^{+} & \xi_{m2}^{+} & \cdots & \xi_{mn}^{+} \end{bmatrix}$$

And correlation coefficient between ideal alternative and ideal alternative is:

$$\xi_0 = (\xi_{01}, \xi_{02}, \cdots, \xi_{0n}) = \underbrace{(1,1,\cdots,1)}_{n} \tag{16}$$

(3) Weighted grey correlation matrix

Suppose augmented matrix constructed by weighted vector W is the weighted grey correlation decision matrix Y, then:

$$Y = \begin{bmatrix} w_1\xi_{11}^{+} & w_2\xi_{12}^{+} & \cdots & w_n\xi_{1n}^{+} \\ w_1\xi_{21}^{+} & w_2\xi_{22}^{+} & \cdots & w_n\xi_{2n}^{+} \\ \vdots & \vdots & \vdots & \vdots \\ w_1\xi_{m1}^{+} & w_2\xi_{m2}^{+} & \cdots & w_n\xi_{mn}^{+} \end{bmatrix}$$

And weighted correlation coefficient between ideal alternative and ideal alternative is:

$$Y_0 = (w_1\xi_{01}, w_2\xi_{02}, \cdots, w_n\xi_{0n}) = \underbrace{(w_1, w_2, \cdots, w_n)}_{n} \tag{17}$$

(4) Projection method

Definition 1[12,13] Suppose $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ are two vectors, then cosine of included angle between vector α and β can be defined as:

$$\cos(\alpha,\beta)=\frac{\sum_{j=1}^{n}(\alpha_j\beta_j)}{\sqrt{\sum_{j=1}^{n}\alpha_j^2}\times\sqrt{\sum_{j=1}^{n}\beta_j^2}}$$

Obviously, cosine of included angle is in this range: $0<\cos(\alpha,\beta)\le 1$, and for its value, the bigger, the better. Because the value of $\cos(\alpha,\beta)$ is bigger, this directions of vector α and β are more accordant.

Definition 2[12,13]: Suppose $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$, then

$\|\alpha\|=\sqrt{\sum_{j=1}^{n}\alpha_j^2}$ is the norm of vector α.

A vector is constituted by direction and norm two parts. And cosine of included angle $\cos(\alpha,\beta)$ can only measure if their directions are accordant, but can not reflect the magnitude of norm. So considering norm magnitude and cosine of included angle together, the closeness degree of two vectors can be measured. In this case, the definition of projection is given as follows:

Definition 3[12,13]: Suppose $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ and $\beta=(\beta_1,\beta_2,\cdots,\beta_n)$ are two vectors, and the projection of vector α onto vector β can be defined:

$$p(\alpha)=\|\alpha\|\cos(\alpha,\beta)=\sqrt{\sum_{j=1}^{n}\alpha_j^2}\times\frac{\sum_{j=1}^{n}(\alpha_j\beta_j)}{\sqrt{\sum_{j=1}^{n}\alpha_j^2}\times\sqrt{\sum_{j=1}^{n}\beta_j^2}}=\frac{\sum_{j=1}^{n}(\alpha_j\beta_j)}{\sqrt{\sum_{j=1}^{n}\beta_j^2}}$$

Generally, the value of $p(\alpha)$ is bigger, vector α is more close to vector β.

For i*th* row Y_i in weighted grey correlation decision matrix Y, according to definition 1 to 3, the weighted grey correlation projection of alternative a_i onto optimum solution V^+ can be got:

$$\begin{aligned}p(Y_i)&=\|Y_i\|\cos(Y_i,Y_0)=\sqrt{\sum_{j=1}^{n}(w_j\xi_{ij}^+)^2}\times\frac{\sum_{j=1}^{n}((w_j\xi_{ij}^+)\times w_j)}{\sqrt{\sum_{j=1}^{n}(w_j\xi_{ij}^+)^2}\times\sqrt{\sum_{j=1}^{n}w_j^2}}\\&=\frac{\sum_{j=1}^{n}(w_j^2\xi_{ij}^+)}{\sqrt{\sum_{j=1}^{n}w_j^2}}=\sum_{j=1}^{n}\left(\frac{w_j^2}{\sqrt{\sum_{j=1}^{n}w_j^2}}\times\xi_{ij}^+\right)\end{aligned}\qquad(18)$$

Let the weight of grey correlation projection be $\overline{w}_j = \dfrac{w_j^2}{\sqrt{\sum_{j=1}^{n} w_j^2}}$, then

$$p(Y_i) == \sum_{j=1}^{n}\left(\overline{w}_j \times \xi_{ij}^{+}\right) \tag{19}$$

(5) Ranking

The alternatives will be ranked by the value of grey correlation projection. The bigger the value of projection is, the closer a certain alternative to ideal solution; the smaller the value of projection is, the more away a certain alternative to ideal alternative.

4. APPLICATION EXAMPLE

To develop new product, five investment alternatives $a_i (i = 1,2,3,4,5)$ are drafted. And decision attributes includes expected net present value, venture profit value, investment amount and risk loss value. Expected net present value and venture profit value are benefit attributes; investment amount and risk loss value are cost attribute. The attribute values of every alternative are showed in Table 1(unit: ten-thousand RMB). Suppose decision maker give subjective weights as follows: $\lambda = (0.2,0.25,0.25,0.3)$, please make investment decision.

Table 1 Product investment decision data[7]

investment amount	expected net present value	venture profit value	risk loss value
[5,7]	[4,5]	[4,6]	[0.4,0.6]
[10,11]	[6,7]	[5,6]	[1.5,2.0]
[5,6]	[4,5]	[3,4]	[0.4,0.7]
[9,11]	[5,6]	[5,7]	[1.3,1.5]
[6,8]	[3,5]	[3,4]	[0.8,1]

For the attributes of this decision problem are represented by interval number, and there is no more information about attribute values, the attribute value could be regarded as random variable with homogeneous distribution on this interval. And the decision process using the method proposed in this paper is as follows:

4.1 Decision data normalization

(1) Normalize interval $[x_{ij}^L, x_{ij}^U]$ into interval $[r_{ij}^L, r_{ij}^U]$, then get:

$$R = \begin{bmatrix} [0.6667,1.0000] & [0.2500,0.5000] & [0.2500,0.7500] & [0.8750,1.0000] \\ [0.0000,0.1667] & [0.7500,1.0000] & [0.5000,0.7500] & [0.0000,0.3125] \\ [0.8333,1.0000] & [0.2500,0.5000] & [0.0000,0.2500] & [0.8125,1.0000] \\ [0.0000,0.3333] & [0.5000,0.7500] & [0.5000,1.0000] & [0.3125,0.4375] \\ [0.5000,0.8333] & [0.0000,0.5000] & [0.0000,0.2500] & [0.6250,0.7500] \end{bmatrix}$$

(2) Transformed density function

The probability density is constant, so the matrix by it is:

$$g = \begin{bmatrix} 3.0000 & 4.0000 & 2.0000 & 8.0000 \\ 6.0000 & 4.0000 & 4.0000 & 3.2000 \\ 6.0000 & 4.0000 & 4.0000 & 5.3333 \\ 3.0000 & 4.0000 & 2.0000 & 8.0000 \\ 3.0000 & 2.0000 & 4.0000 & 8.0000 \end{bmatrix}$$

4.2 Calculate index weight

About genetic algorithm, this paper adopts real coding based accelerating genetic algorithm("RAGA" in abbreviation). When use genetic algorithm to optimize optimal projection direction, n group random variables are randomly generated on the value intervals of every decision variable. (Attention: n here has no relation with decision index number n), n represents father chromosome, and let n=400. Suppose crossover probability $P_c = 0.8$, mutation probability $P_m = 0.8$, $\alpha = 0.05$, and maximum iteration number is 10000, then by programming with MATLAB, index weight can be got:

$$W = \left(0.3112 \;\; 0.1218 \;\; 0.1656 \;\; 0.4014\right)$$

4.3 Grey correlation projection rank method

(1) Calculate correlation coefficient with ideal solutions:

$$\xi^+ = \begin{bmatrix} 0.8333 & 0.4808 & 0.5435 & 1.0000 \\ 0.3788 & 0.8929 & 0.6250 & 0.4000 \\ 0.9615 & 0.4808 & 0.3906 & 0.9434 \\ 0.4032 & 0.6250 & 0.7353 & 0.4808 \\ 0.6579 & 0.4310 & 0.3906 & 0.6757 \end{bmatrix}$$

(2) Calculate grey correlation projection weight:

$$\bar{W} = (0.1767 \;\; 0.0271 \;\; 0.0500 \;\; 0.2941)$$

(3) Calculate weighted grey correlation projection value with respect to every alternative:

$$p(Y_i) = (0.4816 \;\; 0.2400 \;\; 0.4799 \;\; 0.2664 \;\; 0.3462)$$

(4) Ranking:

According to the value of weighted grey correlation projection with respect to every alternative, the rank of alternatives will be:

$$a_1 \succ a_3 \succ a_5 \succ a_4 \succ a_2$$

4.4 Analysis

The rank result obtained by using the method proposed in this paper is accordant with that by using decision methods discussed in literature [6] and [7]. And this shows the validity of the method proposed in this paper enough.

5. CONCLUSION

Multiple attribute decision making under risk problem has wide application in practice. Aiming at multiple attribute decision making problem with continuous random variable on finite interval, this paper proposes a rank method based on projection model, and presents decision-making steps. On the whole, this method shows clear thought, and is easy to understand, and it can be regarded as the enrichment and development for risk decision theory and methods. In the meantime, this method also provides a new idea for multiple attribute decision making problem.

Acknowledgement-This paper is supported by Shandong Province Natural Science Foundation(No. Y2009H23). The authors also would like to express appreciation to the anonymous reviewers for their helpful comments on improving the paper.

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