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LEGENDRE SERIES SOLUTIONS OF FREDHOLM INTEGRAL EQUATIONS

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Abstract- A matrix method for approximately solving linear Fredholm integral equations of the second kind is presented. The solution involves a truncated Legendre series approximation. The method is based on first taking the truncated Legendre series expansions of the functions in equation and then substituting their matrix forms into the equation. Thereby the equation reduces to a matrix equation, which corresponds to a linear system of algebraic equations with unknown Legendre coefficients. In addition, some equations considered by other authors are solved in terms of Legendre polynomials and the results are compared.

Keywords- Legendre series, Fredholm integral equations.

1. INTRODUCTION

In this paper we consider the Fredholm integral equations of the second kind

$$y(x) = f(x) + \lambda \int_{-1}^{1} K(x,t)y(t)dt \qquad (1)$$

where $y(x)$ is the function to be determined. The constant λ, the kernel function $K(x,t)$ and the function $f(x)$ are given. We assume that the range of the variables is $-1 \le x,t \le 1$.

The solution of equation (1) is expressed as the truncated Legendre series

$$y(x) = \sum_{r=0}^{N} a_r P_r(x) \qquad (2)$$

where $P_r(x)$ is the Legendre polynomial and of degree r [6], or in the matrix form

$$[y(x)] = \mathbf{P_x A} \qquad (3)$$

where

$$\mathbf{P_x} = [P_0(x)P_1(x)P_2(x)\dots P_N(x)]$$
$$\mathbf{A} = [a_0 a_1 a_2 \dots a_N]^T$$

and a_r, $r = 0,1,\dots,N$ are coefficients to be determined.

2. METHOD FOR SOLUTION

To obtain the solution of equation (1) in the form of expression (2) we can first deduce the following matrix approximations corresponding to the Legendre series expansions of the functions $f(x)$, $K(x,t)$ and $y(t)$.

Let the function $f(x)$ be approximated by a truncated Legendre series

$$f(x)=\sum_{r=0}^{N} f_r P_r(x)\,. \tag{4}$$

Then we can put series (4) in the matrix form

$$[f(x)]=\mathbf{P_x F} \tag{5}$$

where

$$\mathbf{F}=[f_0 f_1 \dots f_N]^T\,.$$

We now consider the kernel function $K(x,t)$. If the function $K(x,t)$ can be approximated by double Legendre series of degree N in both x and t of the form [2,7]

$$K(x,t)=\sum_{r=0}^{N}\sum_{s=0}^{N} k_{r,s} P_r(x) P_s(t) \tag{6}$$

then we can put series (6) in the matrix form

$$[K(x,t)]=\mathbf{P_x K P_t^T}$$

where

$$\mathbf{P_t}=[P_0(t)P_1(t)\dots P_N(t)] \tag{7}$$

$$\mathbf{K}=\begin{bmatrix} k_{00} & k_{01} & \cdots & k_{0N} \\ k_{10} & k_{11} & \cdots & k_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ k_{N0} & k_{N1} & \cdots & k_{NN} \end{bmatrix}.$$

On the other hand, for the unknown function $y(t)$ in integrand, we write from expressions (2) and (3)

$$[y(t)]=\mathbf{P_t A}\,. \tag{8}$$

Substituting the matrix forms (3), (5), (7) and (8) corresponding to the functions $y(x)$, $f(x)$, $K(x,t)$ and $y(t)$, respectively, into equation (1), and then simplifying the result equation, we have the matrix equation

$$\mathbf{A} = \mathbf{F} + \lambda\mathbf{K}\left\{\int_{-1}^{1} \mathbf{P}_{\mathbf{t}}^{\mathbf{T}} \mathbf{P}_t dt\right\}\mathbf{A}$$

or briefly

$$(\mathbf{I} - \lambda\mathbf{KQ})\mathbf{A} = \mathbf{F} \tag{9}$$

where

$$\mathbf{Q} = \int_{-1}^{1} P_t^T P_t dt = [q_{rs}] \quad , \quad r,s = 0,1,\ldots,N \tag{10}$$

and $\mathbf{I}$ is the unit matrix; the elements of the fixed matrix $\mathbf{Q}$ are given by [1,2]

$$q_{rs} = \int_{-1}^{1} P_r(t)P_s(t)dt = \begin{cases} 0 & , \ r \neq s \\ \dfrac{2}{2r+1} & , \ r = s \end{cases}.$$

In equation (9), if $D(\lambda) = |\mathbf{I} - \lambda\mathbf{KQ}| \neq 0$ we get

$$\mathbf{A} = (\mathbf{I} - \lambda\mathbf{KQ})^{-1}\mathbf{F} \quad , \quad \lambda \neq 0 . \tag{11}$$

Thus the unknown coefficients $a_r, \quad r = 0,1,\ldots,N$ are uniquely determined by equation (11) and thereby the integral equation (1) has a unique solution. This solution is given by the truncated Legendre series (2).

3. ACCURACY OF SOLUTION

We can easily check the accuracy of the method. Since the truncated Legendre series in (2) is an approximate solution of Eq.(1), it must be approximately satisfied this equation.

Then for each $x_i \in [-1,1]$

$$E(x_i) = \left| y(x_i) - f(x_i) - \lambda \int_{-1}^{1} K(x_i,t)y(t)dt \right| \cong 0$$

or

$$E(x_i) \leq 10^{-k_i} \qquad (\ k_i \text{ is any positive integer}) .$$

If

$$\max\left(10^{-k_i}\right) = 10^{-k} \quad (\ k \text{ is any positive integer})$$

is prescribed, then the truncation limit N is increased until the difference $E(x_i)$ at each of the points x_i becomes smaller than the prescribed 10^{-k} .

On the other hand, the error function can be estimated by

$$E(x) = y(x) - f(x) - \lambda \int_{-1}^{1} K(x,t)y(t)dt \ \ [6].$$

4. NUMERICAL ILLUSTRATIONS

We show the efficiency of the presented method using the following examples (In all figures different line shows the exact solution and the numerical solution).

Example 1. Let us first consider the linear Fredholm integral equation [2,3]

$$y(x) = (x+1)^2 + \int_{-1}^{1} (xt + x^2t^2)y(t)dt \tag{12}$$

and seek the solution $y(x)$ in Legendre series

$$y(x) = \sum_{r=0}^{N} a_r P_r(x)$$

so that

$$f(x) = x^2 + 2x + 1 \quad K(x,t) = (xt + x^2t^2) \quad \lambda = 1 \quad N = 2.$$

By using the expansions for the powers x^r in terms of the Legendre polynomials $P_r(x)$ [6], we easily find the representations

$$f(x) = x^2 + 2x + 1 = \frac{4}{3}P_0(x) + 2P_1(x) + \frac{2}{3}P_2(x)$$

and

$$K(x,t) = (xt + x^2t^2) = \frac{1}{9}P_0(x)P_0(t) + \frac{2}{9}P_2(x)P_0(t) + P_1(x)P_1(t) + \frac{2}{9}P_0(x)P_2(t) + \frac{4}{9}P_2(x)P_2(t)$$

and hence, from relations (5) and (7), the matrices

$$\mathbf{F} = \begin{bmatrix} 4/3 \\ 2 \\ 2/3 \end{bmatrix} \quad , \quad \mathbf{K} = \frac{1}{9}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 9 & 0 \\ 2 & 0 & 4 \end{bmatrix}.$$

If we use expression (10) for $r, s = 0,1,2$, we obtain the fixed matrix

$$\mathbf{Q} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 2/5 \end{bmatrix}.$$

Next, we substitute these matrices in equation (11) and then simplify to obtain

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 37/27 & 0 & 4/27 \\ 0 & 3 & 0 \\ 20/27 & 0 & 35/27 \end{bmatrix} \begin{bmatrix} 4/3 \\ 2 \\ 2/3 \end{bmatrix}.$$

The solution of this equation is

$$a_0 = 52/27 \quad a_1 = 6 \quad a_2 = 50/27.$$

By substituting the obtained coefficients in (2) the solution of (12) becomes

$$y(x) = \frac{52}{27}P_0(x) + 6P_1(x) + \frac{50}{27}P_2(x) \quad \text{or} \quad y(x) = \frac{25}{9}x^2 + 6x + 1$$

which is the exact solution [2,3].

Example 2. Second we can study the following linear Fredholm integral equation [5,8]

$$y(x) = 0.9x^2 + 0.5\int_0^1 x^2 t^2 y(t)dt$$

and seek the solution $y(x)$ in Legendre series

$$y(x) = \sum_{r=0}^{N} a_r P_r(x)$$

so that

$$f(x) = 0.9x^2 \;,\; K(x,t) = 0.5(x^2 t^2) \;,\; \lambda = 1 \;,\; N = 3 .$$

By using the expansions for the powers x^r in terms of the Legendre polynomials $P_r(x)$ [1], we easily find the representations

$$f(x) = 0.9x^2 = \frac{3}{10}P_0(x) + \frac{6}{10}P_2(x)$$

and

$$K(x,t) = 0{,}5(x^2t^2) = \frac{1}{18}P_0(x)P_0(t) + \frac{1}{9}P_2(x)P_0(t) + \frac{1}{9}P_0(x)P_2(t) + \frac{2}{9}P_2(x)P_2(t)$$

and hence, from relations (5) and (7), the matrices

$$\mathbf{F} = \frac{1}{10}\begin{bmatrix} 3 \\ 0 \\ 6 \\ 0 \end{bmatrix} \quad , \quad \mathbf{K} = \frac{1}{18}\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

If we use expression (10) for $r,s = 0,1,2$, we obtain the fixed matrix

$$\mathbf{Q} = \begin{bmatrix} 1 & 1/2 & 0 & -1/8 \\ 1/2 & 1/3 & 1/8 & 0 \\ 0 & 1/8 & 1/5 & 1/8 \\ -1/8 & 0 & 1/8 & 1/7 \end{bmatrix} .$$

Next, we substitute these matrices in equation (11) and then simplify to obtain

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 86/81 & 5/108 & 2/81 & 5/648 \\ 0 & 1 & 0 & 0 \\ 10/81 & 5/54 & 85/81 & 5/324 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3/10 \\ 0 \\ 6/10 \\ 0 \end{bmatrix} .$$

The solution of this equation is

$$a_0 = 1/3 \;,\; a_1 = 0 \;,\; a_2 = 2/3 \;,\; a_3 = 0.$$

Substituting these values in equation (2) we obtain

$$y(x) = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x) \quad \text{or} \quad y(x) = x^2$$

which is the exact solution [5,8].

Example 3. Let us now take the equation

$$y(x) = x^3 - \frac{2}{7} + 5 + \int_{-1}^{1} (x^2 t^3 + 1) y(t) dt .$$

Following the previous procedures, we get the exact solution of linear Fredholm integral equation for $N = 3$ as

$$y(x) = x^3 - 5$$

which is the exact solution.

Example 4. We consider the problem

$$y(x) = e^{2x} - \frac{3}{4}\frac{x}{e^2} - \frac{1}{4} e^2 x \int_{-1}^{1} xty(t) dt .$$

We give numerical analysis for $N = 9, 10$ in Table 1 and Fig. 1.

Table 1. Comparing the solutions and error analysis which has been found for $N = 9, 10$ at Example 4.

			Present method: Legendre Method			
i	x_i	Exact Solution	$N=9$		$N=10$	
		$y(x_i) = e^{2x}$	$y(x_i)$	$E(x_i)$	$y(x_i)$	$E(x_i)$
0	-1	0.135335283	0.135121217	2.14066 E-04	0.135403404	6.81207E-05
1	-0.8	0.201896518	0.201889475	7.043E-06	0.201919775	2.32566E-05
2	-0.6	0.301194212	0.301207204	1.2992E-05	0.30120891	1.46982E-05
3	-0.4	0.449328964	0.449338623	9.6585E-06	0.449338652	9.6881E-06
4	-0.2	0.670320046	0.670324889	4.843E-06	0.670324889	4.8431E-06
5	0	1	1	0	1	0
6	0.2	1.491824698	1.491819855	4.843E-06	1.491819855	4.843E-06
7	0.4	2.225540928	2.225531211	9.717E-06	2.22553124	9.688E-06
8	0.6	3.320116923	3.320100481	1.6442E-05	3.320102187	1.4736E-05
9	0.8	4.953032424	4.952977676	5.4748E-05	4.953007976	2.4448E-05
10	1	7.389056099	7.388688307	0.000367792	7.388970494	8.5605E-05

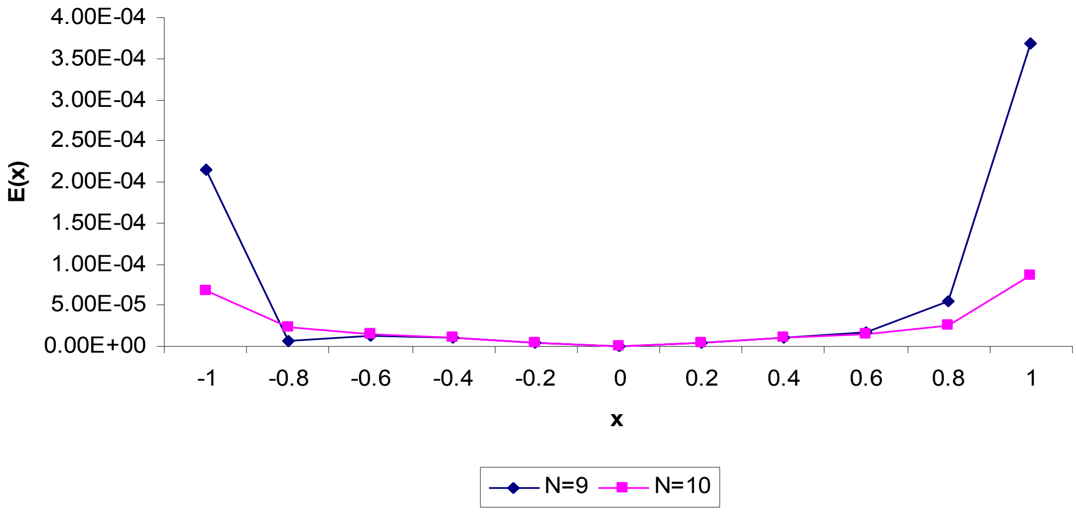

Figure 1. Numerical results of Example 4 for $N = 9, 10$.

Example 5. Let us consider the Fredholm integral equation of second kind

$$y(x) = e^{2x} + \frac{1}{2}x - \frac{1}{2}e^{2}x + \int_0^1 xy(t)dt .$$

The results obtained for $y(x)$ with various N values are presented in Table 2 and Fig. 2.

Table 2. Error analysis of Example 5 for the x values.

		Present method : Legendre Method				
i	x_i	Exact Solution $y(x_i)=e^{2x}$	$N=7$ $y(x_i)$	$N=8$ $y(x_i)$	$N=9$ $y(x_i)$	$N=10$ $y(x_i)$
0	0	1	1	1	1	1
1	0.1	1.221402758	1.221227307	1.2213684	1.221396619	1.221401750
2	0.2	1.491824698	1.491473778	1.491755982	1.491812420	1.491822681
3	0.3	1.8221188	1.821592001	1.822015698	1.822100382	1.822115775
4	0.4	2.225540928	2.224834561	2.225403096	2.225516341	2.225536893
5	0.5	2.718281828	2.717376712	2.718106981	2.718250831	2.718276760
6	0.6	3.320116923	3.318941442	3.319894645	3.320078176	3.320110666
7	0.7	4.055199967	4.053539735	4.054893409	4.055147876	4.055191762
8	0.8	4.953032424	4.950338847	4.952532814	4.952947936	4.953019281
9	0.9	6.049647464	6.044671362	6.048674329	6.049474922	6.049619491
10	1	7.389056099	7.379197869	7.38695801	7.388651132	7.388984626

i	x_i	$N=7$ $E(x_i)$	$N=8$ $E(x_i)$	$N=9$ $E(x_i)$	$N=10$ $E(x_i)$
0	0	0	0	0	0
1	0.1	1.754512278 E-04	3.435768992 E-05	6.138993623 E-05	1.008321797 E-06
2	0.2	3.509193322 E-04	6.871612934 E-05	1.227801717 E-05	2.01664468 E-06
3	0.3	5.267995935 E-04	1.031025988 E-04	1.841874272 E-05	3.025061042 E -06
4	0.4	7.063674298 E-04	1.378325162 E-04	2.458786861 E-05	4.035591972 E-06
5	0.5	9.051160196 E-04	1.748470602 E-04	3.099785382 E-05	5.068921645 E-06
6	0.6	1.175481143 E-03	2.222780103 E-04	3.874686954 E-05	6.256562185 E-06
7	0.7	1.660231652 E-03	3.065582196 E-04	5.209104922 E-05	8.205263771 E-06
8	0.8	2.693577825 E-03	4.996099664 E-04	8.448795163 E-05	1.314298433 E-05
9	0.9	4.976102022 E-03	9.731356093 E-04	1.725423267 E-04	2.79737754 E-05
10	1	9.858229607 E-03	2.098088514 E-03	4.049668207 E-04	7.147315391 E-05

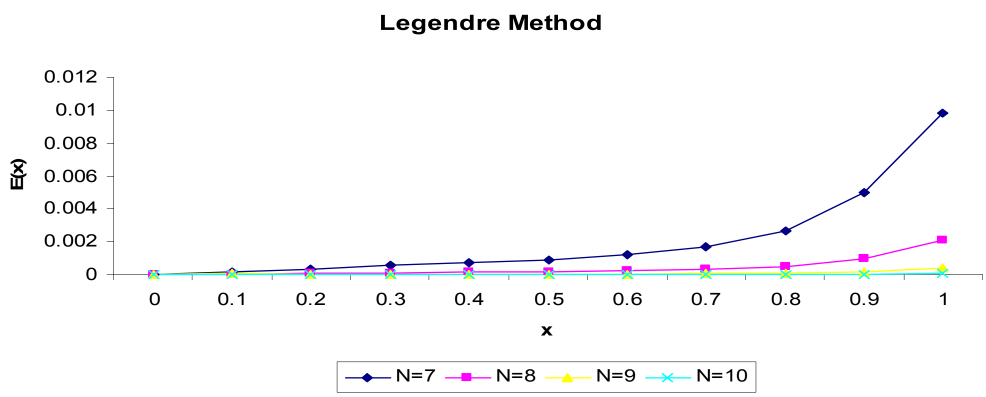

Figure 2. Numerical results of Example 5 for $N = 7-10$.

Example 6. Consider the following the integral equation in [4]

$$y(x)=e^{2x+\frac{1}{3}}-\frac{1}{3}\int_0^1 e^{2x-\frac{5}{3}t}y(t)dt\,.$$

We give numerical analysis for various N values in Table 3 and Fig. 3, 4.

Table 3. Error analysis of Example 6 for the x values and comparison of present method, exact and the methods in [4] $(N=6,7,8)$.

i	x_i	Present method : Legendre Method						
		Exact Solution $y(x_i)=e^{2x}$	$N=6$		$N=7$		$N=8$	
			$y(x_i)$	$E(x_i)$	$y(x_i)$	$E(x_i)$	$y(x_i)$	$E(x_i)$
0	0	1	0.99916	8.43545 E-04	1.00024	2.52450 E-04	0.99999	2.61806 E-05
1	0.1	1.22140	1.22037	1.03031 E-03	1.22171	3.08343 E-04	1.22137	3.19771 E-05
2	0.2	1.49182	1.49057	1.25876 E-03	1.49220	3.76595 E-04	1.49180	3.90577 E-05
3	0.3	1.82212	1.82058	1.54303 E-03	1.82257	4.59549 E-04	1.82210	4.77338 E-05
4	0.4	2.22554	2.22362	1.92347 E-03	2.22609	5.57275 E-04	2.22550	5.86679 E-05
5	0.5	2.71828	2.71576	2.51907 E-03	2.71894	6.58365 E-04	2.71821	7.42250 E-05
6	0.6	3.32012	3.31648	3.63368 E-03	3.32083	7.15360 E-04	3.32001	1.03054 E-05
7	0.7	4.05520	4.04926	5.94222 E-03	4.05579	5.91555 E-04	4.05502	1.72220 E-05
8	0.8	4.95303	4.94224	1.07885 E-02	4.95300	3.98974 E-05	4.95266	3.54416 E-05
9	0.9	6.04965	6.02901	2.06343 E-02	6.04778	1.87066 E-03	6.04882	8.22282 E-05
10	1	7.38906	7.34935	3.97052 E-02	7.38282	6.24039 E-03	7.38710	1.94791 E-05

i	x_i	BPF Method in [4] (Block Pulse Fnc.)	TF Method in [4] (Triangular Fnc.)
		$m=32$	
0	0	1.031832	0.999844
1	0.1	1.244627	1.221598
2	0.2	1.501307	1.492294
3	0.3	1.810922	1.822684
4	0.4	2.184388	2.225880
5	0.5	2.804810	2.717857
6	0.6	3.383247	3.320648
7	0.7	4.080975	4.056474
8	0.8	4.922595	4.954570
9	0.9	5.937783	6.050568
10	1	7.162334	7.387901

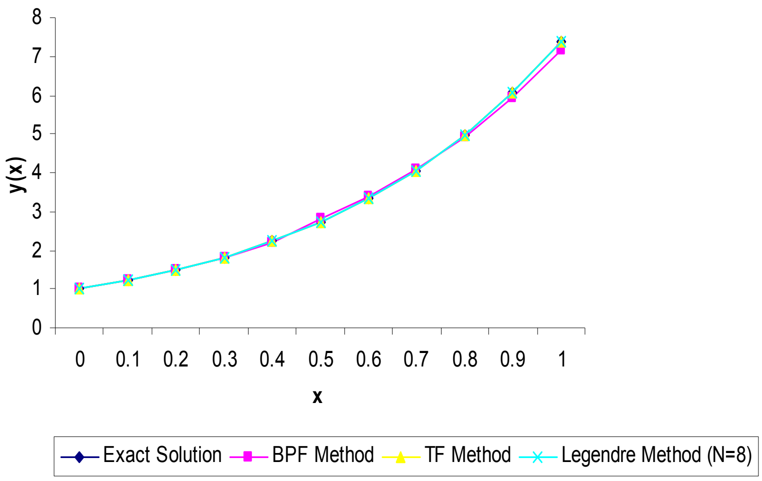

Figure 3. Comparing the solutions with the other methods which has been found for $N = 8$ at Example 6.

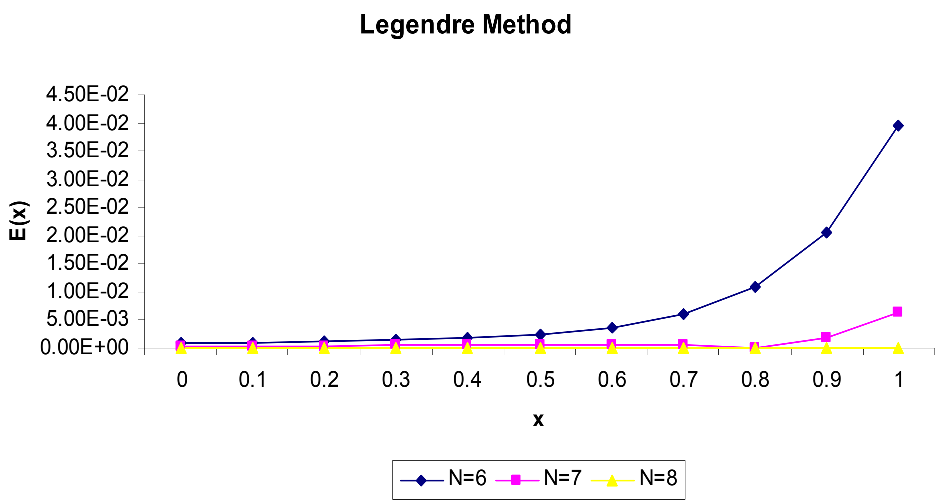

Figure 4. Numerical results of Example 6 for $N = 6-8$.

Example 7. Our last example is the equation

$$y(x) = e^{2x} - x + \int_0^1 xe^{-2t} y(t)dt .$$

The comparison of solutions (for $N = 7-10$) with exact solution e^{2x} is given in Table 4 and Fig. 5.

Table 4. Error analysis of Example 7 for the x values.

			Present method: Legendre Method			
i	x_i	Exact Solution $y(x_i)=e^{2x}$	$N=7$ $y(x_i)$	$N=8$ $y(x_i)$	$N=9$ $y(x_i)$	$N=10$ $y(x_i)$
0	0	1	1	1	1	1
1	0.1	1.221402758	1.220964379	1.221486556	1.221385903	1.221405440
2	0.2	1.491824698	1.490947923	1.491992292	1.491790987	1.49183006
3	0.3	1.8221188	1.820803218	1.822370164	1.822068232	1.822126844
4	0.4	2.225540928	2.223782851	2.225875718	2.225473475	2.225551652
5	0.5	2.718281828	2.716062075	2.718697759	2.718197248	2.718295208
6	0.6	3.320116923	3.317363876	3.320603577	3.320013877	3.320132804
7	0.7	4.055199967	4.051699242	4.055720497	4.05507286	4.055217590
8	0.8	4.953032424	4.948235426	4.953478058	4.952862204	4.953048799
9	0.9	6.049647464	6.042305015	6.049737728	6.049378474	6.049652698
10	1	7.389056099	7.376568594	7.388139565	7.388543967	7.389021523

i	x_i	$N=7$ $E(x_i)$	$N=8$ $E(x_i)$	$N=9$ $E(x_i)$	$N=10$ $E(x_i)$
0	0	0	0	0	0
1	0.1	4.383787703 E-04	8.379776899 E-04	1.685548769 E-05	2.68139265 E-06
2	0.2	8.767744172 E-04	1.675947885 E-04	3.371100531 E-05	5.362784215 E-06
3	0.3	1.315582221 E-03	2.513637779 E-04	5.056822493 E-05	8.044082301 E-06
4	0.4	1.758077600 E-03	3.347893194 E-04	6.745384489 E-05	1.072326582 E-05
5	0.5	2.219753732 E-03	4.159302344 E-04	8.458032418 E-05	1.337965059 E-05
6	0.6	2.753046398 E-03	4.866547432 E-04	1.030458340 E-04	1.58817245 E-05
7	0.7	3.500724449 E-03	5.205299928 E-04	1.271065077 E-04	1.762273736 E-05
8	0.8	4.796998165 E-03	4.456337049 E-04	1.702199042 E-04	1.637473125 E-05
9	0.9	7.342449904 E-03	9.026352097 E-05	2.689907734 E-04	5.233654626 E-06
10	1	1.248750503 E-02	9.165339247 E-04	5.121317614 E-04	3.457600943 E-05

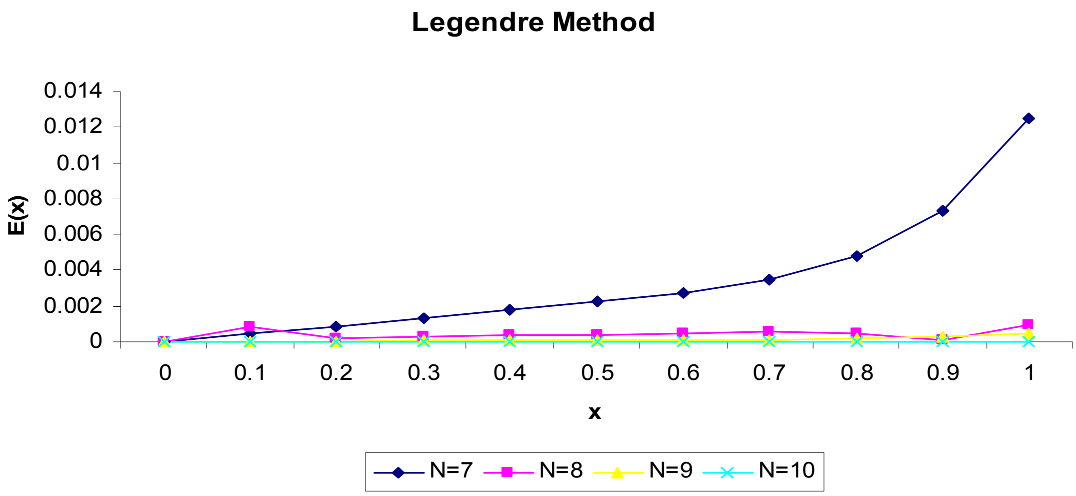

Figure 5. Numerical results of Example 7 for $N=7-10$.

5. CONCLUSIONS AND DISCUSSIONS

In this paper, the usefulness of the method presented for the approximate solution of Fredholm integral equation (1) is demonstrated. To show the accuracy of the method, five integral equations are chosen. A considerable advantage of the method is that the solution is expressed as a truncated Legendre series. This means that, after calculation of the Legendre coefficients, the solution $y(x)$ can be easily evaluated for arbitrary values of x at low computation effort. If the functions $f(x)$ and $K(x,t)$ can be expanded to the Legendre series in $-1 \leq x,t \leq 1$, then there exists the solution $y(x)$; otherwise, the method cannot be used in. On the other hand, it would appear that our method shows to best advantage when the known functions $f(x)$ and $K(x,t)$ have Taylor series about the origin which converge rapidly. To get the best approximating solution of the equation, we must take more terms from the Legendre expansions of functions, especially when they converge slowly. Briefly, for computational efficiency, the truncation limit N must be chosen sufficiently large.

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