Mathematical and Computational Applications, Vol. 15, No. 3, pp. 325-333, 2010.

NOETHER, PARTIAL NOETHER OPERATORS AND FIRST INTEGRALS FOR THE COUPLED LANE-EMDEN SYSTEM

Ben Muatjetjeja and Chaudry Masood Khalique

International Institute for Symmetry Analysis and Mathematical Modelling,
Department of Mathematical Sciences, North-West University, Mafikeng Campus,
Private Bag X2046, Mmabatho 2735, South Africa
Ben.Muatjetjeja@nwu.ac.za Masood.Khalique@nwu.ac.za

Abstract- Systems of Lane-Emden equations arise in the modelling of several physical phenomena, such as pattern formation, population evolution and chemical reactions. In this paper we construct Noether and partial Noether operators corresponding to a Lagrangian and a partial Lagrangian for a coupled Lane-Emden system. Then the first integrals with respect to Noether and partial Noether operators are obtained for the Lane-Emden system under consideration. We show that the first integrals for both the Noether and partial Noether operators are the same. However, the gauge function is different in certain cases.
Key Words- Lagrangian, Noether and partial Noether operators, First integrals, Lane-Emden system, Lie group methods

1. INTRODUCTION

The generalized Lane-Emden equation

$$\frac{d^2y}{dx^2}+\frac{n}{x}\frac{dy}{dx}+\phi(y)=0 \tag{1}$$

where n is a real constant and $\phi(y)$ is a real-valued continuous function of the variable y, is many problems arising in mathematical physics and astrophysics. For certain fixed values of n and $\phi(y)$, Eq. (1) has been used to model several phenomena such as the theory of stellar structure, the thermal behaviour of a spherical cloud of gas, isothermal gaseous sphere and the theory of thermionic currents [1-3].

Various methods (for example numerical, perturbation, Adomian's decomposition, homotopy analysis, power series, differential transformation, and variational approach) for the solution to the generalized Lane-Emden equation (1) have been widely studied in the literature. See for example [4-7] and the references therein.

Noether's theorem [8] reveals the general connection between symmetries and conservation laws. In fact, it provides the formula for construction of the conserved quantities (first integrals) for Euler-Lagrange differential equations once their symmetries are known. First integrals are of interest because they tell us something physically about the system [9] and also they reduce the order of the differential equations.

In [10] the authors studied the Noether symmetries of Eq. (1) and obtained exact solutions for various cases which admitted Noether point symmetries. Some other works on symmetries and solutions of Lane-Emden-type equations can be found in [11-

17]. For the applications of Lie group methods to differential equations the interested reader is referred to [18-21].

Systems of Lane-Emden equations arise in the modelling of several physical phenomena, such as: pattern formation; population evolution; chemical reactions; and so on (see for example [22]) and have attracted much attention in recent years. Several authors have established existence and uniqueness results for the Lane-Emden systems [23, 24] and other related systems (see e.g., [25-27] and references therein).

The purpose of this study is to investigate the Noether and partial Noether operators and construct first integrals for the coupled Lane-Emden system

$$\frac{d^2u}{dt^2}+\frac{n}{t}\frac{du}{dt}+\upsilon^q=0,\ \frac{d^2\nu}{dt^2}+\frac{n}{t}\frac{d\nu}{dt}+u^p=0, \tag{2}$$

where n, p and q are real constants, which is a natural extension of the celebrated Lane-Emden equation.

The paper is structured as follows. In Section 2 we briefly recall the preliminaries of the Noether and partial Noether symmetry approach. We obtain all Noether operators and the corresponding first integrals for the system (2) in Section 3. Then in Section 4 we determine all partial Noether operators and the associated first integrals for the same system and comparison is made with the Noether case. Concluding remarks are mentioned in Section 5.

2. PRELIMINARIES ON NOETHER, PARTIAL NOETHER OPERATORS AND FIRST INTEGRAL

In this section we present some definitions, which we utilize in Section 3. For details the reader is referred to [28-32].

Consider the vector field

$$X=\tau(t,u,v)\frac{\partial}{\partial t}+\xi(t,u,v)\frac{\partial}{\partial u}+\eta(t,u,v)\frac{\partial}{\partial v}, \tag{3}$$

which has first extension

$$X^{[1]}=X+(\dot{\xi}-\dot{u}\dot{\tau})\frac{\partial}{\partial \dot{u}}+(\dot{\eta}-\dot{v}\dot{\tau})\frac{\partial}{\partial \dot{v}}, \tag{4}$$

where $\dot{\tau}$, $\dot{\xi}$ and $\dot{\eta}$ denote total time derivatives of τ, ξ and η respectively.

Let us consider the second-order system of differential equations

$$\ddot{u}=E_1(t,u,v,\dot{u},\dot{v}),\ddot{v}=E_2(t,u,v,\dot{u},\dot{v}) \tag{5}$$

which has a Lagrangian $L(t,u,\nu,\dot{u},\dot{v})$, i.e., Eqs. (5) are equivalent to the Euler-Lagrange equations [32]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right)-\frac{\partial L}{\partial u}=0,\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{v}}\right)-\frac{\partial L}{\partial \nu}=0 \tag{6}$$

Definition 1
The vector field X of the form (5) is called a Noether point symmetry generator corresponding to a Lagrangian $L(t,u,\nu,\dot{u},\dot{\nu})$ of Eqs. (6) if there exists a gauge function $B(t,u,\nu)$ such that

$$X^{[1]}(L)+D(\tau)L=D(B). \tag{7}$$

Here D is the total differentiation operator defined by [28]

$$D=\frac{\partial}{\partial t}+\dot{u}\frac{\partial}{\partial u}+\dot{\nu}\frac{\partial}{\partial \nu}+\ddot{u}\frac{\partial}{\partial \dot{u}}+\ddot{\nu}\frac{\partial}{\partial \dot{\nu}}+...$$

(8)

Definition 2
The generator X as in (3) is called a partial Noether operator corresponding to a partial Lagrangian $L(t,u,\nu,\dot{u},\dot{\nu})$ [29] of Eqs. (5) if there exists a gauge function $B(t,u,\nu)$ such that

$$X^{[1]}(L)+D(L)=(\dot{\xi}-\tau\dot{u})\frac{\partial L}{\partial u}+(\dot{\eta}-\tau\dot{\nu})\frac{\partial L}{\partial \nu}+D(B)$$

(9)

The following theorems are taken from [8] and [29] respectively.

Theorem 1 (**Noether** [8])
If X as given in (3) is a Noether point symmetry generator corresponding to a Lagrangian $L(t,u,\nu,\dot{u},\dot{\nu})$ of Eqs. (5), then

$$I=\tau L+(\xi-\tau\dot{u})\frac{\partial L}{\partial \dot{u}}+(\eta-\tau\dot{\nu})\frac{\partial L}{\partial \dot{\nu}}-B, \tag{10}$$

is a Noether first integral of Eqs. (5) associated with the operator X.

Proof. See, e.g., [18, 31].

Theorem 2 (**Partial Noether** [29])
If X is a partial Noether operator corresponding to a partial Lagrangian $L(t,u,\nu,\dot{u},\dot{\nu})$ then I in (10) is a first integral of (5) associated with X.

Proof. See [29].

3. NOETHER POINT SYMMETRIES OF SYSTEM (2)

Consider the Lane-Emden system (2), viz.,

$$\frac{d^2u}{dt^2}+\frac{n}{t}\frac{du}{dt}+\nu^q=0,\qquad \frac{d^2\nu}{d^2u}+\frac{n}{t}\frac{d\nu}{dt}+\nu^p=0.$$

It can be verified that the natural Lagrangian of this system is

$$L=t^n\dot{u}\dot{\nu}-\frac{t^n u^{p+1}}{p+1}-\frac{t^n\nu^{q+1}}{q+1},\ \ p\neq -1, q\neq -1.$$

(11)

The insertion of L from (11) into Eq. (7) and separation with respect to powers of $\dot{u}$ and $\dot{\nu}$ yields linear overdetermined system of eight PDEs. These are

$$\tau_u=0 \tag{12}$$

$$\tau_\nu=0 \tag{13}$$

$$n\tau t^{n-1}+t^{n}\xi_{u}+t^{n}\eta_{\nu}-t^{n}\tau_{t}=0, \tag{14}$$

$$\xi_{\nu}=0, \tag{15}$$

$$\eta_{u}=0, \tag{16}$$

$$t^{n}\xi_{t}=B_{\nu} \tag{17}$$

$$t^{n}\eta_{t}=B_{u'} \tag{18}$$

$$-\frac{u^{p+1}}{p+1}n\tau t^{n-1}-\frac{\nu^{q+1}}{q+1}n\tau t^{n-1}-t^{n}u^{p}\xi-t^{n}\nu^{q}\eta-\frac{u^{p+1}}{p+1}\tau_{t}t^{n}-\frac{\nu^{q+1}}{q+1}\tau_{t}=B_{t'} \tag{19}$$

After some straightforward, albeit tedious and lengthy calculations, the above system gives

$$\tau=C_{1}t, \tag{20}$$

$$\xi=-\frac{C_{1}(1+n)}{p+1}u, \tag{21}$$

$$\eta=-\frac{C_{1}(1+n)}{q+1}\nu,\ C_{1}=\text{constant}, \tag{22}$$

$$B=0 \tag{23}$$

$$n=\frac{2p+2q+pq+3}{pq-1},\ \ pq\neq-1,\ p\neq-1,\ p\neq1\ \text{ and }\ q\neq-1,\ q\neq1. \tag{24}$$

Thus we obtain a single Noether point symmetry

$$X=t\frac{\partial}{\partial t}-\frac{(1+n)}{p+1}\frac{\partial}{\partial u}-\frac{(1+n)}{q+1}\frac{\partial}{\partial\nu'}$$

for the system (2) with condition (24). Using Theorem 1, due to Noether, we obtain the first integral

$$I=t^{n+1}\frac{u^{p+1}}{p+1}+t^{n+1}\frac{\nu^{q+1}}{q+1}+\frac{(n+1)}{p+1}t^{n}\nu\dot{u}+\frac{(n+1)}{q+1}t^{n}u\dot{\nu}+t^{n+1}\dot{u}\dot{\nu}.$$

We now consider the case when $p=-1$ and $q=-1$. In this case the Lane-Emden system (2) becomes

$$\frac{d^{2}u}{dt^{2}}+\frac{n}{t}\frac{du}{dt}+\frac{1}{\nu}=0,\ \frac{d^{2}\nu}{d^{2}u}+\frac{n}{t}\frac{d\nu}{dt}+\frac{1}{u}=0, \tag{25}$$

which has a standard Lagrangian

$$L=t^{n}\dot{u}\dot{\nu}-t^{n}\ln u-t^{n}\ln\nu. \tag{26}$$

The substitution of L from (25) into Eq. (7) and splitting with respect to the powers of $\dot{u}$ and $\dot{v}$ gives the following system of PDEs:

$\tau_u = 0$,
(27)

$\tau_v = 0$, (28)

$n\tau t^{n-1} + t^n\xi_u + t^n\eta_v - t^n\tau_t = 0,$ (29)

$\xi_v = 0$, (30)

$\eta_u = 0$, (31)

$t^n\xi_t = B_v$ (32)

$t^n\eta_t = B_{u'}$
(33)

$-n\tau t^{n-1}\ln u - n\tau t^{n-1}\ln v - t^n u^{-1}\xi - t^n v^{-1}\eta - \tau_t t^n \ln u - \tau_t t^n \ln v = B_{t'}$
(34)

The analysis of (34) yields the following two cases:

Case 1. n=0 $(p=-1, q=-1)$

This case provides us with two Noether point symmetries namely, $X_1 = u\partial/\partial u - v\partial/\partial v$ and $X_2 = \partial/\partial t$. For both cases B=0. Using Theorem 1, we obtain the Noether's first integrals corresponding to X_1 and X_2 as

$I_1 = \dot{u}v - u\dot{v}$ and $I_2 = \dot{u}\dot{v} + \ln u + \ln v$, respectively.

Case 2. n=-1 $(p=-1, q=-1)$

Here we also obtain two Noether operators; $X_1 = u\partial/\partial u - v\partial/\partial v$ with B=0 and $X_2 = t\partial/\partial t + 2u\partial/\partial u$ with B=$-2\ln t$. Invoking Theorem 1 due to Noether, we obtain the first integrals associated with X_1 and X_2 as

$I_1 = \dot{u}vt^{-1} - u\dot{v}t^{-1}$ and $I_2 = -2\ln t + \ln u + \ln v - 2u\dot{v}t^{-1} + \dot{u}\dot{v}$, respectively.

4. PARTIAL NOETHER OPERATORS OF THE LANE-EMDEN SYSTEM (2)

According to definition 2, the operator X in (3) is a partial Noether operator corresponding to a partial Lagrangian

$L = t^n\dot{u}\dot{v}$, (35)

of the Lane-Emden system (2) if there exists a function $B(t,u,v)$ such that it satisfies (9). The operator X satisfies the following system:

$\tau_u = 0,$ (36)

$\tau_v = 0,$ (37)

$n\tau t^{n-1} + t^n\xi_u + t^n\eta_v - t^n\tau_t = 0,$ (38)

$\xi_v = 0$ (39)

$$\eta_u = 0 \tag{40}$$
$$t^n \xi_t = B_\nu - t^n \nu^q \tau, \tag{41}$$
$$t^n \eta_t = B_u - t^n u^p \tau, \tag{42}$$
$$\xi t^n u^p + \eta t^n \nu^q + B_t = 0 \tag{43}$$

A routine calculation for the above system yields

$$\tau = C_1 t,\ \ \xi = -\frac{C_1(1+n)}{p+1} u\,,\ \ \eta = -\frac{C_1(1+n)}{q+1}\nu, \tag{44}$$

$$B = t^{n+1}\frac{u^{p+1}}{p+1} + t^{n+1}\frac{\nu^{q+1}}{q+1} + k,\ \ C_1, k\text{=constant}, \tag{45}$$

$$n = \frac{2p+2q+pq+3}{pq-1},\ \ pq \neq -1\,, p \neq -1,\ \ p \neq 1 \ \text{and}\ q \neq -1,\ q \neq 1. \tag{46}$$

Hence we obtain a single partial Noether operator viz.,

$$X = t\frac{\partial}{\partial t} - \frac{(1+n)}{p+1}\frac{\partial}{\partial u} - \frac{(1+n)}{q+1}\frac{\partial}{\partial \nu}$$

for the system (2) with condition (46). Thus we see that in this case the partial Noether operator is the same as in the Noether case. The difference resides in the value of the gauge function B. Using Theorem 2, and the partial Noether operator X we obtain the first integral

$$I = t^{n+1}\frac{u^{p+1}}{p+1} + t^{n+1}\frac{\nu^{q+1}}{q+1} + \frac{(n+1)}{p+1}t^n \nu\dot{u} + \frac{(n+1)}{q+1}t^n u\dot{\nu} + t^{n+1}\dot{u}\dot{\nu}.$$

which also coincides with the Noether case.

We now consider the case when $p = -1$ and $q = -1$, viz., (25).

Using the partial Lagrangian $L = t^n \dot{u}\dot{\nu}$ and following the above procedure we arrive at the following two cases.

Case 1. n=0 $(p = -1, q = -1)$

Here we obtain two partial Noether symmetry generators namely, $X_1 = u\partial/\partial u - \nu\partial/\partial\nu$ with B=0 and $X_2 = \partial/\partial t$ with $B = \ln u + \ln \nu$. We note that these partial Noether operators are identical to the Noether operators. However, the gauge function B is different for X_2. Invoking Theorem 2, we obtain the first integrals associated with X_1 and X_2 as

$I_1 = \dot{u}\nu - u\dot{\nu}$ and $I_2 = \dot{u}\dot{\nu} + \ln u + \ln \nu$, respectively.

Case 2. n=-1 $(p = -1, q = -1)$

In this case we obtain two partial Noether operators, which are identical to the Noether operators. However, we note that the gauge function $B = \ln u + \ln v - 2\ln t$ corresponding to X_2 is different. Using Theorem 2, we obtain the first integrals corresponding to X_1 and X_2 as

$I_1 = \dot{u}vt^{-1} - u\dot{v}t^{-1}$ and $I_2 = -2\ln t + \ln u + \ln v - 2u\dot{v}t^{-1} + \dot{u}\dot{v}$, respectively.

5. CONCLUDING REMARKS

We have studied Noether and partial Noether operators with respect to the Lagrangian and partial Lagrangian of the Lane-Emden system (2). We obtained three cases, which resulted in Noether and partial Noether operators. For each of these three cases we obtained the first integrals corresponding to the Noether and partial Noether operators. We have seen that the first integrals associated with the Noether and partial Noether operators are the same for the Lane-Emden system. However, the gauge function B is different for some cases. This is due to the fact that different Lagrangian was used for the respective approaches. Further work on systems from a partial Lagrangian viewpoint can be done as systems in general do not admit Lagrangians.

6. REFERENCES

1. S. Chandrasekhar, An Introduction to the Study of Stellar Structure. Dover Publications Inc., New York, 1957.
2. H.T. Davis, Introduction to Nonlinear Differential and Integral Equations. Dover Publications Inc., New York, 1962.
3. O.W. Richardson, The Emission of Electricity from Hot Bodies. 2nd edition, Longmans, Green & Co., London, 1921.
4. M. Dehghan, F. Shakeri, Approximate solution of a differential equation arising in astrophysics using the variational iteration method, *New Astronomy* **13,** 53-59 2008.
5. J.I. Ramos, Series approach to the Lane–Emden equation and comparison with the homotopy perturbation method, *Chaos, Solitons and Fractals* **38,** 400-408, 2008.
6. H.R. Marzban, H.R. Tabrizidooz, M. Razzaghi, Hybrid functions for nonlinear initial-value problems with applications to Lane-Emden type equations, *Physics Letters A* **372,** 5883-5886, 2008.
7. V. S Ertürk, Differential transformation method for solving differential equations of Lane-Emden type, *Mathematical and Computational Applications* **12,** 135-139, 2007.
8. E. Noether, Invariante Variationsprobleme. König Gesell Wissen Göttingen, *Math-Phys Kl. Heft* **2,** 235-257, 1918.
9. H. Goldstein, C. Poole, J. Safko, Classical mechanics. 3rd edition, Pearson Education Inc., Singapore, 2004.
10. C.M. Khalique, F.M. Mahomed, B. Muatjetjeja, Lagrangian formulation of a generalized Lane-Emden equation and double reduction, *Journal of Nonlinear Mathematical Physics* **15,** 152-161, 2008.
11. C.M. Mellin, F.M. Mahomed, P.G.L. Leach, Solution of generalized

Emden-Fowler equations with two symmetries, *International Journal of Non-linear Mechanics* **29,** 529-538,1994.
12. P.G.L. Leach, First integrals for the modified Emden equation
$\ddot{q} + \alpha(t)\dot{q} + q^n = 0$, *Journal of Mathamatical Physics* 26**,** 2510-2514, 1985.
13. A.H. Kara, F.M. Mahomed, Equivalent Langrangians and solutions of some classes of nonlinear equations $\ddot{q} + p(t)\dot{q} + r(t)q = \mu\dot{q}^2 q^{-1} + f(t)q^n$, *International Journal of Non-linear Mechanics* **27,** 919-927, 1992.
14. A.H. Kara, F.M. Mahomed, A note on the solutions of the Emden-Fowler equation, *International Journal of Non-linear Mechanics* **28**, 379-384, 1993.
15. Y. Bozhkov, A.C.G. Martins, Lie point symmetries of the Lane-Emden systems, *Journal of Mathematical Analysis and Applications* 294**,** 334-344, 2004.
16. Y. Bozhkov, A.C.G. Martins, Lie point symmetries and exact solutions of quasilinear differential equations with critical exponents, *Nonlinear Analysis* 57**,** 773-793, 2004.
17. C.M. Khalique, P. Ntsime, Exact solutions of the Lane-Emden-type equation, *New Astronomy* **13,** 476-480, 2008.
18. L.V. Ovsiannikov, Group Analysis of Differential Equations. Academic Press, New York,(English translation by WF Ames) 1982.
19. G.W. Bluman, S. Kumei, Symmetries and Differential Equations. Springer-Verlag, New York, 1989.
20. P.J. Olver, Applications of Lie groups to differential equations. Springer-Verlag. New York, 1993.
21. N.H. Ibragimov, Elementary Lie Group Analysis and Ordinary Differential Equations. Wiley. Chichester, 1999.
22.H. Zou, A priori estimates for a semilinear elliptic system without variational structure and their applications, *Math. Ann* **323,** 713-735, 2002.
23. J. Serrin, H. Zou, Non-existence of positive solutions of the Lane-Emden system, *Differential Integral Equations* **9,** 635-653, 1996.
24. J. Serrin, H. Zou, Existence of positive solutions of Lane-Emden system. *Atti Sem. Mat. Fis. Univ. Modena* **46**, Suppl., 369-380, 1998.
25. Y. Qi, The existence of ground states to a weakly coupled elliptic system, *Nonlinear Analysis* **48,** 905-925, 2002.
26. R. Dalmasso, Existence and uniqueness of solutions for a semilinear elliptic system, *International Journal of Mathematics and Mathematical Sciences* **10,** 1507-1523, 2005.
27. Q. Dai, C.C. Tisdell, Non-degeneracy of positive solutions to homogeneous second order differential systems and its applications, *Acta Math. Sci. Ser. B Engl. Ed.* To appear.
28. V.M. Gorringe, P.G.L. Leach, Lie point symmetries for systems of second order linear ordinary differential equations, *Queastiones Mathematicae* **11,** 95-117, 1998.
29. A.H. Kara, F.M. Mahomed, I. Naeem, C. Wafo Soh, Partial Noether operators and first integrals via partial Lagrangians, *Mathematical Methods in the Applied Sciences* **30,** 2079-2089, 2007.
30. I. Naeem, F.M. Mahomed, First integrals for a general linear system of two second-order ODEs via a partial Lagrangian, *Journal of Physics. A: Mathematicals and Theoretical* **41,** 355207, 2008.

31. N.H. Ibragimov, A.H. Kara, F.M. Mahomed, Lie-Bäcklund and Noether Symmetries with Applications, *Nonlinear Dynamics* **15,** 115-136, 1998.
32. B. Van Brunt, The calculus of variations, Springer, New York, 2004.