Mathematical and Computational Applications, Vol. 15, No. 3, pp. 309-317, 2010.

MODIFIED VARIATIONAL ITERATION METHOD FOR SCHRODINGER EQUATIONS

Syed Tauseef Mohyud-Din[1*], Muhammad Aslam Noor[2] and Khalida Inayat Noor[3]
Department of Mathematics
COMSATS Institute of Information Technology, 44000
Islamabad, Pakistan.
[1]syedtauseefs@hotmail.com, [2] noormaslam@hotmail.com, [3] khalidanoor@hotmail.com

Abstract- In this paper, we apply the modified variational iteration method (MVIM) for solving Schrödinger equations. The proposed modification is made by introducing He's polynomials in the correction functional of variational iteration method (VIM). The suggested iterative scheme finds the solution without any discretization, linearization or restrictive assumptions. The use of Lagrange multiplier coupled with He's polynomials are the clear advantages of this technique over the decomposition method. Several examples are given to verify the reliability and efficiency of the proposed algorithm.

Key words- Variational iteration method, partial differential equations, Schrödinger equations, He's polynomials.

1. INTRODUCTION

Many problems in natural and engineering sciences are modeled by partial differential equations. These equations arise in number of scientific models such as the propagation of shallow water waves, fluid mechanics, long wave and chemical reaction-diffusion models, (see Abbasbandy [1, 2] Abdou and Soliman [4, 5], Abassy et. al. [6], Baitha et. al. [7], Bizar and Ghazvini [8], Wakil et. al. [9], Ganji et. al.[10], Ghorbani and Nadifi [11, 12], Golbabi and Javidi [13], He [14-25], Inokuti et. al. [27], Lu [28], Ma [29], Momani and Odibat [30], Noor and Mohyud-Din [31, 32, 34-37], Rafi and Danili [38], Sweilman [39], Sadighi and Ganji [40]). A substantial amount of work has been invested for solving such models. Several techniques including method of characteristic, Riemann invariants, combination of waveform relaxation and multi grid, periodic multi grid wave form, variational iteration, homotopy perturbation and Adomian's decomposition, (see Abbasbandy [1, 2] Abdou and Soliman [4, 5], Abassy et. al. [6], Baitha et. al. [7], Bizar and Ghazvini [8], Wakil et. al. [9], Ganji et. al.[10], Ghorbani and Nadifi [11, 12], Golbabi and Javidi [13], He [14-25], Inokuti et. al. [27], Lu [28], Ma [29], Momani and Odibat [30], Noor and Mohyud-Din [31, 32, 34-37], Rafi and Danili [38], Sweilman [39], Sadighi and Ganji [40]) have been used for the solutions of such problems. Most of these techniques encounter the inbuilt deficiencies and involve huge computational work. He developed the variational iteration and homotopy perturbation methods for solving linear, nonlinear, initial and boundary value problems, (see He [14-26]). These methods are fully synchronized with the versatile nature of the problems and have been applied to solve a wide class of initial and boundary value problems, (see Abbasbandy [1, 2] Abdou and Soliman [4, 5], Abassy et. al. [6], Baitha et. al. [7], Bizar and Ghazvini [8], Wakil et. al. [9], Ganji et. al.[10], Ghorbani and Nadifi [11, 12], Golbabi and Javidi [13], He [14-25], Inokuti et. al. [27], Lu [28], Ma [29], Momani and Odibat [30], Noor and Mohyud-Din [31, 32, 34-37], Rafi and Danili [38], Sweilman [39], Sadighi and Ganji [39]). Recently, Ghorbani et. al. introduced He's polynomials by splitting the nonlinear term and also proved that He's polynomials are fully compatible with Adomian's polynomials but are easier to calculate and are more user friendly (see Ghorbani et. al. [11, 12]). More recently, Noor and Mohyud-Din combined He's polynomials and correction functional of the variational iteration method (VIM) and applied this reliable version to a number of physical problems; (see Noor and Mohyud-Din [34-36]). The paper is devoted to the study of an important type of partial differential equation which is called the Schrödinger equation, is of the form

$$u_t + i\,u_{xx} = 0, \quad u(x,0) = f(x), \quad i^2 = -1,$$

or

$$iu_t + u_{xx} + \gamma |u|^2 u = 0, \quad u(x,0) = f(x), \quad i^2 = -1,$$

and arises in various areas of applied sciences including nonlinear optics, plasma physics, super conductivity and quantum mechanics, (see, Sadighi and Ganji. [39] and the reference therein). Several techniques including decomposition and homotopy perturbation have been employed for the solution of such problems, (see, Mohyud-Din and Noor [32], Sadighi and Ganji [39]). In this paper, we apply the modified variational iteration method (MVIM) which is formulated by the elegant coupling of variational iteration method (VIM) and He's polynomials for solving Schrödinger equations. It is shown that the MVIM provides the solution in a rapid convergent series. We write the correction functional for the Schrödinger equations and calculate Lagrange multiplier optimally via variational theory. The He's polynomials are introduced in the correction functional. The use of Lagrange multiplier reduces the successive application of the integral operator and minimizes the computational work. Moreover, the selection of the initial value is done very carefully because the approximants are heavily dependent on it. Several examples are given to illustrate the reliability and performance of the proposed method. It is to be highlighted that the modified variational iteration method (MVIM) has certain advantages as compare to the decomposition method. Firstly, the use of Lagrange multiplier reduces the successive applications of the integral operator and hence minimizes the computational work to a tangible level while still maintaining a very high level of accuracy. Moreover, He's polynomials are easier to calculate as compare to Adomian's polynomials and this gives it a clear edge over the traditional decomposition method. The MVIM is also independent of the small parameter assumption (which is either not there in the physical problems or difficult to locate) and hence is more convenient to apply as compare to the traditional perturbation method. It is worth mentioning that the MVIM is applied without any discretization, restrictive assumption or transformation and is free from round off errors. We apply the proposed MVIM for all the nonlinear terms in the problem without discretizing either by finite difference or spline techniques at the nodes, involves laborious calculations coupled with a strong possibility of the ill-conditioned resultant equations which is a complicated problem to solve. Moreover, unlike the method of separation of variables that requires initial and boundary conditions, the VIMHP provides the solution by using the initial conditions only, (see Noor and Mohyud-Din [34-36]).

2. VARIATIONAL ITERATION METHOD (VIM)

To illustrate the basic concept of the He's VIM, we consider the following general differential equation

$$Lu + Nu = g(x), \tag{1}$$

where L is a linear operator, N a nonlinear operator and g(x) is the inhomogeneous term. According to variational iteration method (see Abbasbandy [1, 2] Abdou and Soliman [4, 5], Abassy et. al. [6], Baitha et. al. [7], Bizar and Ghazvini [8], Wakil et. al. [9], Ganji et. al.[10], Golbabi and Javidi [13], He [14, 16, 21-26], Inokuti et. al. [27], Lu [28], Momani and Odibat [30], Noor and Mohyud-Din [31, 33-37], Rafi and Danili [38], Sweilman [39]), we can construct a correction functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda (Lu_n(s) + N\tilde{u}_n(s) - g(s))\,ds, \tag{2}$$

where λ is a Lagrange multiplier (see He [14, 16, 21-26]). which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. i.e. $\delta \tilde{u}_n = 0$; (2) is called a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in (see He [14, 16, 21-26]). In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation u_{n+1}, $n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by $u = \lim_{n\to\infty} u_n$.

3. HOMOTOPY PERTURBATION METHOD (HPM)

To explain the He's homotopy perturbation method, we consider a general equation of the type,

$$L(u) = 0, \tag{3}$$

where L is any integral or differential operator. We define a convex homotopy H (u, p) by

$$H(u,p) = (1-p)F(u) + pL(u), \tag{4}$$

where F (u) is a functional operator with known solutions v_0, which can be obtained easily. It is clear that, for

$$H(u,p) = 0, \tag{5}$$

we have

$$H(u,0) = F(u), \qquad H(u,1) = L(u).$$

This shows that $H(u,p)$ continuously traces an implicitly defined curve from a starting point H (v_0, 0) to a solution function H (*f*, 1). The embedding parameter monotonically increases from zero to unit as the trivial problem F (u) = 0 is continuously deforms the original problem L (u) = 0. The embedding parameter $p \in (0, 1]$ can be considered as an expanding parameter (see Ghorbani and Nadifi [11, 12], He [14-20], Noor and Mohyud-Din [31, 32, 34-37], Xu [42]). The homotopy perturbation method uses the homotopy parameter p as an expanding parameter (see He [14-20]) to obtain

$$u=\sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \cdots, \tag{6}$$

if p $\to$ 1, then (6) corresponds to (4) and becomes the approximate solution of the form,

$$f=\lim_{p\to 1} u = \sum_{i=0}^{\infty} u_i. \tag{7}$$

It is well known that series (7) is convergent for most of the cases and also the rate of convergence is dependent on L (u); (see He [14-20]). We assume that (7) has a unique solution. The comparisons of like powers of p give solutions of various orders. In sum, according to (Ghorbani and Nadifi [11, 12]), He's HPM considers the solution, $u(x)$, of the homotopy equation in a series of p as follows:

$$u(x)=\sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + \ldots,$$

and the method considers the nonlinear term $N(u)$ as

$$N(u)=\sum_{i=0}^{\infty} p^i H_i = H_0 + p H_1 + p^2 H_2 + \ldots,$$

where H_n's are the so-called He's polynomials (Ghorbani and Nadifi [11, 12]), which can be calculated by using the formula

$$H_n(u_0,\ldots,u_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(N\left(\sum_{i=0}^{n} p^i u_i\right)\right)_{p=0}, \qquad n=0,1,2,\ldots.$$

4. MODIFIED VARIATIONAL ITERATION METHOD (MVIM)

The modified variational iteration method (MVIM) is obtained by the elegant coupling of correction functional (2) of variational iteration method (VIM) with He's polynomials and is given by

$$\sum_{n=0}^{\infty} p^{(n)} u_n = u_0(x) + p\int_0^x \lambda(s)\left(\sum_{n=0}^{\infty} p^{(n)} L(u_n) + \sum_{n=0}^{\infty} p^{(n)} N(\tilde{u}_n)\right)ds - \int_0^x \lambda(s)\, g(s)\, ds.$$

(8)

Comparisons of like powers of p give solutions of various orders (see Noor and Mohyud-Din [34-37]).

5. NUMERCICAL APPLICATOIONS

In this section, we apply the modified variational iteration (MVIM) for solving Schrödinger equations. The results are very encouraging indicating the reliability and efficiency of the proposed method.

Example 5.1 Consider the following linear Schrödinger equation

$$u_t + i u_{xx} = 0,$$

with initial conditions

$$u(x,0) = 1 + 2\cos h(2x).$$

The correction functional is given as

$$u_{n+1}(x,t) = 1 + 2\cos h(2x) + \int_0^t \lambda(s)\left(\frac{\partial u_n}{\partial s} + i(\tilde{u}_n)_{xx}\right)ds.$$

Making the correction functional stationary, the Lagrange multipliers can be identified as $\lambda(s) = -1$, consequently

$$u_{n+1}(x,t) = 1 + 2\cos h(2x) - \int_0^t \left(\frac{\partial u_n}{\partial s} + i(u_n)_{xx}\right)ds.$$

Applying the modified variational iteration method (MVIM)

$$u_0 + pu_1 + \cdots = 1 + 2\cos h(2x) - \int_0^t \left(\left(\frac{\partial u_0}{\partial s} + p\frac{\partial u_1}{\partial s} + p^2\frac{\partial u_2}{\partial s} + \cdots\right) + i(u_0 + pu_1 + \cdots)_{xx}\right)ds.$$

Comparing the co-efficient of like powers of p, approximants are obtained

$p^{(0)} : u_0(x,t) = 1 + 2\cos h(2x),$

$p^{(1)} : u_1(x,t) = 1 + 2\cos h(2x)(1 - 4it),$

$p^{(2)} : u_2(x,t) = 1 + 2\cos h(2x)(1 - 4it - 8t^2),$

$p^{(3)} : u_3(x,t) = 1 + 2\cos h(2x)\left(1 - 4it - 8t^2 + \frac{32}{3}it^3\right),$

$\vdots$

The solution in a series form is given by

$$u(x,t) = 1 + 2\cos h(2x)\left(1 - 4it + \frac{(4it)^2}{2!}t^2 - \frac{(4it)^3}{3!}t^3 + \frac{(4it)^4}{4!}t^4 + \cdots\right),$$

and in a closed form by

$$u(x,t) = 1 + 2\cos h(2x)e^{-4it}.$$

Example 5.2 Consider the following linear Schrödinger equation

$$u_t + iu_{xx} = 0,$$

with initial conditions

$$u(x,0) = e^{3ix}.$$

The correction functional is given as

$$u_{n+1}(x,t) = e^{3ix} + \int_0^t \lambda(s)\left(\frac{\partial u_n}{\partial s} + i(\tilde{u}_n)_{xx}\right)ds.$$

Making the correction functional stationary, the Lagrange multipliers can be identified as $\lambda(s) = -1$, consequently

$$u_{n+1}(x,t) = e^{3ix} - \int_0^t \left(\frac{\partial u_n}{\partial s} + i(u_n)_{xx}\right)ds.$$

Applying the modified variational iteration method (MVIM)

$$u_0 + pu_1 + \cdots = e^{3ix} - \int_0^t \left(\left(\frac{\partial u_0}{\partial s} + p\frac{\partial u_1}{\partial s} + p^2\frac{\partial u_2}{\partial s} + \cdots\right) + i(u_0 + pu_1 + \cdots)_{xx}\right)ds.$$

Comparing the co-efficient of like powers of p, approximants are obtained

$p^{(0)}: u_0(x,t) = e^{3ix},$

$p^{(1)}: u_1(x,t) = e^{3ix}(1+9it),$

$p^{(2)}: u_2(x,t) = e^{3ix}\left(1+9it-\frac{81}{2}t^2\right),$

$p^{(3)}: u_3(x,t) = e^{3ix}\left(1+9it-\frac{81}{2}t^2-\frac{243}{2}it^3\right),$

$\vdots$

The solution in a series form is given by

$$u(x,t) = e^{3ix}\left(1+9it+\frac{(9it)^2}{2!}t^2+\frac{(9it)^3}{3!}t^3+\frac{(9it)^4}{4!}t^4+\cdots\right),$$

and in a closed form by

$$u(x,t) = e^{3i(x+3t)}.$$

Example 5.3 Consider the following nonlinear Schrödinger equation

$$iu_t + u_{xx} - 2u|u|^2 = 0,$$

with initial conditions

$$u(x,0) = e^{ix}.$$

The correction functional is given as

$$u_{n+1}(x,t) = e^{ix} + \int_0^t \lambda(s)\left(\frac{\partial u_n}{\partial s} - i\left((\tilde{u}_n)_{xx} - 2\tilde{u}_n|\tilde{u}_n|^2\right)\right)ds.$$

Making the correction functional stationary, the Lagrange multipliers can be identified as $\lambda(s) = -1$, consequently

$$u_{n+1}(x,t) = e^{ix} - \int_0^t \left(\frac{\partial u_n}{\partial s} - i\left((u_n)_{xx} - 2u_n|u_n|^2\right)\right)ds.$$

Applying the modified variational iteration method (MVIM)

$$u_0 + pu_1 + \cdots = e^{ix} - \int_0^t \left(\left(\frac{\partial u_0}{\partial s} + p\frac{\partial u_1}{\partial s} + \cdots\right) - i(u_0 + pu_1 + \cdots)_{xx} - 2(u_0 + pu_1 + \cdots)|u_0 + pu_1 + \cdots|^2\right)ds.$$

Comparing the co-efficient of like powers of p, approximants are obtained

$p^{(0)}: u_0(x,t) = e^{ix},$

$p^{(1)}: u_1(x,t) = e^{ix}(1-3it),$

$p^{(2)}: u_2(x,t) = e^{ix}\left(1-3it-\frac{9}{2!}t^2\right),$

$p^{(3)}: u_3(x,t) = e^{ix}\left(1-3it-\frac{9}{2!}t^2+\frac{9}{2!}it^3\right),$

$\vdots$

The solution in a series form is given by

$$u(x,t) = e^{ix}\left(1 - 3it + \frac{(3it)^2}{2!}t^2 - \frac{(3it)^3}{3!}t^3 + \frac{(3it)^4}{4!}t^4 - \cdots\right),$$

and in a closed form by

$$u(x,t) = e^{i(x-3t)}.$$

6. CONCLUSION

In this paper, we applied modified variational iteration method (MVIM) for solving Schrödinger equations. The method is applied in a direct way without using linearization, transformation, discretization or restrictive assumptions. It may be concluded that MVIM is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compare to the classical methods while still maintaining the high accuracy of the numerical result.

Acknowledgment-The authors are highly grateful to Dr S. M. Junaid Zaidi for the provision of excellent research facilities coupled with the appropriate environment.

7. REFERENCES

1. S. Abbasbandy, A new application of He’s variational iteration method for quadratic Riccati differential equation by using Adomian’s polynomials, J. Comput. Appl. Math. **207**, 59-63, 2007.
2. S. Abbasbandy, Numerical solutions of nonlinear Klein-Gordon equation by variational iteration method, Internat. J. Numer. Meth. Engrg. **70**, 876-881, 2007.
3. G. Adomian, R. Rach and N. T. Shawagfeh, On the analytic solution of Lane-Emden equation, Found. Phys. Lett. **8**, 161, 1995.
4. M. A. Abdou and A. A. Soliman, Variational iteration method for solving Burger’s and coupled Burger’s equations, J. Comput. Appl. Math. **181**, 245-251, 2005.
5. M. A. Abdou and A. A. Soliman, New applications of variational iteration method, Phys. D **211**, 1-8, 2005.
6. T. A. Abassy, M. A. El-Tawil and H. El-Zoheiry, Solving nonlinear partial differential equations using the modified variational iteration Pade´ technique, J. Comput. Appl. Math **207**, 73-91, 2007.
7. B. Batiha, M. S. M. Noorani and I. Hashim, Variational iteration method for solving multi species Lotka-Volterra equations, Comput. Math. Appl. **54**, 903-909, 2007.
8. J. Biazar and H. Ghazvini, He’s variational iteration method for fourth-order parabolic equations, Comput. Math. Appl. **54**, 1047-1054, 2007.
9. S. A. El-Wakil, M. A. Madkour and M. A. Abdou, Application of exp-function method for nonlinear evolution equations with variable co-efficient, Phys. Lett. A, **369,** 62-69, 2007.
10. D. D. Ganji, H. Tari, M. B. Jooybari, Variational iteration method and homotopy perturbation method for evolution equations, Comput. Math. Appl. **54** 1018-1027, 2007.

11. A. Ghorbani and J. S. Nadjfi, He's homotopy perturbation method for calculating Adomian's polynomials, Int. J. Nonlin. Sci. Num. Simul. **8,** 229-332, 2007.
12. A. Ghorbani, Beyond Adomian's polynomials: He polynomials, Chaos, Solitons & Fractals, 2007, in press.
13. A. Golbabai and M. Javidi, A variational iteration method for solving parabolic partial differential equations, Comput. Math. Appl. **54,** 1003-1009, 2007.
14. J. H. He, An elementary introduction of recently developed asymptotic methods and nanomechanics in textile engineering, Int. J. Mod. Phys. B, **22**, 3487-4578, 2008.
15. J. H. He, Recent developments of the homotopy perturbation method, Top. Meth. Nonlin. Anal. **31** 205-209, 2008.
16. J. H. He, Some asymptotic methods for strongly nonlinear equation, Int. J. Mod. Phys. **(20)10**, 1144-1199, 2006.
17. J. H. He, Comparison of homotopy perturbation method and homotopy analysis method, Appl. Math. Comput. **156**, 527-539, 2004.
18. J. H. He, Homotopy perturbation method for bifurcation of nonlinear problems, Int. J. Nonlin. Sci. Numer. Simul. **6 (2),** 207-208, 2005.
19. J. H. He, The homotopy perturbation method for nonlinear oscillators with discontinuities, Appl. Math. Comput. **151,** 287-292, 2004.
20. J. H. He, A coupling method of homotopy technique and perturbation technique for nonlinear problems, Int. J. Nonlin. Mech. **35,** 115-123, 2000.
21. J. H. He, Variational iteration method- Some recent results and new interpretations, J. Comput. Appl. Math. **207**, 3-17, 2007.
22. J. H. He and X. Wu, Variational iteration method: New developments and applications, Comput. Math. Appl. **54,** 881-894, 2007.
23. J. H. He, Variational iteration method, A kind of non-linear analytical technique, some examples, Internat. J. Nonlinear Mech. **34 (4),** 699-708, 1999.
24. J. H. He, Variational iteration method for autonomous ordinary differential systems, Appl. Math. Comput. **114 (2-3),** 115-123, 2000.
25. J. H. He and X. H. Wu, Construction of solitary solution and compaction-like solution by variational iteration method, Chaos, Solitons & Fractals**, 29 (1)**, 108-113, 2006.
26. J. H. He, The variational iteration method for eighth-order initial boundary value problems, Phys. Scr. **76 (6),** 680-682, 2007.
27. M. Inokuti, H. Sekine and T. Mura, General use of the Lagrange multiplier in nonlinear mathematical physics, in: S. Nemat-Naseer (Ed.), Variational Method in the Mechanics of Solids, Pergamon press, New York, 156-162, 1978.
28. J. Lu, Variational iteration method for solving two-point boundary value problems, J. Comput. Appl. Math. **207,** 92-95, 2007.
29. W. X. Ma and Y. You, Rational solutions of the Toda lattice equation in Casoratian form, Chaos, Solitons & Fractals, **22**, 395-406, 2004.
30. S. Momani and Z. Odibat, Application of He's variational iteration method to Helmholtz equations, Chaos, Solitons & Fractals, **27 (5),** 1119-1123, 2006.
31. S. T. Mohyud-Din, M. A. Noor and K. I. Noor, Travelling wave solutions of seventh-order generalized KdV equations using He's polynomials, Int. J. Nonlin. Sci. Num. Sim. **10 (2)**, 223-229, 2009.

32. S. T. Mohyud-Din and M. A. Noor, Homotopy perturbation method for solving partial differential equations, Zeitschrift für Naturforschung A, **64a,** 1-14, 2008
33. S. T. Mohyud-Din, M. A. Noor and K. I. Noor, Solution of singular equations by He's variational iteration method, Int. J. Nonlin. Sci. Num. Sim. **10-121**, 2009.
34. M. A. Noor and S. T. Mohyud-Din, Variational iteration method for solving higher-order nonlinear boundary value problems using He's polynomials, Int. J. Nonlin. Sci. Num. Simul. **9 (2)**, 141-157, 2008.
35. M. A. Noor and S. T. Mohyud-Din, Modified variational iteration method for heat and wave-like equations, Acta Applnda. Mathmtce. 2008, DOI: 10.1007/s10440-008-9255-x.
36. M. A. Noor and S. T. Mohyud-Din, Variational homotopy perturbation method for solving higher dimensional initial boundary value problems, Math. Prob. Engg. **2008,** 2008, Article ID 696734, doi:10.1155/2008/696734.
37. M. A. Noor and S. T. Mohyud-Din, Solution of singular and nonsingular initial and boundary value problems by modified variational iteration method, Math. Prob. Engg. **2008,** 2008, Article ID 917407, 23 pages, doi:10.1155/2008/917407.
38. M. Rafei and H. Daniali, Application of the variational iteration method to the Whitham-Broer-Kaup equations, Comput. Math. Appl. **54**, 1079-1085, 2007.
39. N. H. Sweilam, Harmonic wave solutions in nonlinear thermo elasticity by variational iteration method and Adomian's method, J. Comput. Appl. Math. **207,** 64-72, 2007.
40. A. Sadighi and D. D. Ganji, Analytic treatment of linear and nonlinear Schrödinger equations: A study with homotopy perturbation and Adomian's decomposition method, Phys. Lett. A, **372,** 465-469, 2008.
41. M. Tatari and M. Dehghan, On the convergence of He's variational iteration method, J. Comput. Appl. Math**. 207,** 121-128, 2007.
42. L. Xu, He's homotopy perturbation method for a boundary layer equation in unbounded domain, Comput. Math. Appl. **54,** 1067-1070, 2007.