Mathematical and Computational Applications, Vol. 15, No. 2, pp. 187-198, 2010.

COMPARISON AND COUPLING OF POLYNOMIALS FOR FLIERL-PETVIASHIVILI EQUATION

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Abstract- This paper outlines a comparison of the couplings of He's and Adomian's polynomials with correction functional of variational iteration method (VIM) to investigate a solution of Flierl-Petviashivili (FP) equation which plays a very important role in mathematical physics, engineering and applied sciences. These elegant couplings give rise to two modified versions of VIM which are very efficient in solving initial and boundary value problems of diversified nature. Moreover, we also introduces a new transformation which is required for the conversion of the Flierl-Petviashivili equation to a first order initial value problem and a reliable framework designed to overcome the difficulty of the singular point at $x = 0$. The proposed modified versions are applied to the reformulated first order initial value problem which gives the solution in terms of transformed variable. The desired series of solution is obtained by making use of the inverse transformation. It is observed that the modification based on He's polynomials is much easier to implement and is more user friendly.

Key words- Flierl-Petviashivili equation, variational iteration method, He's polynomials, Adomian's polynomials, Pade´ approximants.

1. INTRODUCTION

The Flierl-Petviashivili (FP) equation is used to model several phenomena in mathematical physics, astrophysics, theory of stellar structure, thermal behavior of a spherical cloud of gas, isothermal gas spheres and theory of thermionic currents, (see Adomian [5], Russell and Shampine [41], Shawagfeh [44] Wazwaz [48]). Several techniques including decomposition and homotopy perturbation have been applied for solving FP equation, (see Adomian [5], Russell and Shampine [41, Shawagfeh [44] Wazwaz [48]). Most of the developed techniques have their limitations like limited convergence, divergent results, linearization, discretization unrealistic assumptions and non-compatibility with the physical problems. He foresaw the potential and compatibility of variational iteration and homotopy perturbation methods and exploited this reliable technique for solving physical problems of diversified nature, (see He [17-29]). These methods are fully synchronized with the versatile nature of the problems and have been applied to solve a wide class of initial and boundary value problems, (see Abbasbandy [1, 2] Abdou and Soliman [6, 7], Abassy et. al. [8], Baitha et. al. [10],

Bizar and Ghazvini [11], Wakil et. al. [12], Ganji et. al.[13], Ghorbani and Nadifi [14, 15], Golbabi and Javidi [16], He [17-29], Inokuti et. al. [30], Lu [31], Ma and You [32], Momani and Odibat [33], Noor and Mohyud-Din [34-40], Russel and Shampine [41], Rafi and Danili [42], Sweilman [43]). Abbasbandy introduced the coupling of Adomian's polynomials with the correction functional of the VIM and applied this reliable version for solving Riccati differential and Klein Gordon equations (see Abbasbandy [1, 2]). In a later work, Noor and Mohyud-Din exploited this concept for solving various singular and non singular boundary and initial value problems (see Noor and Mohyud-Din [35, 40]). Recently, Ghorbani et. al. introduced He's polynomials by splitting the nonlinear term and also proved that He's polynomials are fully compatible with Adomian's polynomials but are easier to calculate and are more user friendly (see Ghorbani et. al. [14, 15]). More recently, Noor and Mohyud-Din combined He's polynomials and correction functional of the variational iteration method (VIM) and applied this reliable version to a number of physical problems; (see Noor and Mohyud-Din [37-39]). The basic motivation of the present study is the implementation and comparison of these two modified versions of VIM for solving FP equation. The singularity behavior at $x = 0$ is a difficult element in this type of equations which has been tackled by transforming the Flierl-Petviashivili (FP) equation to a first order initial value problem. The proposed modified versions are applied to the reformulated first order initial value problem which leads the solution in terms of transformed variable. The desired series of solutions is obtained by implementing the inverse transformation. To make the work more concise and for the better understanding of the solution behavior the diagonal Pade´ approximants are applied. It is observed that the modification based on He's polynomials (VIMHP) is much easier to implement as compare to the one (VIMAP) where the so-called Adomian's polynomials along with their complexities are used. It is to be highlighted that the variational iteration method using He's polynomials (VIMHP) has certain advantages as compare to the decomposition method. Firstly, the use of Lagrange multiplier reduces the successive applications of the integral operator and hence minimizes the computational work to a tangible level while still maintaining a very high level of accuracy. Moreover, He's polynomials are easier to calculate as compare to Adomian's polynomials and this gives it a clear edge over the traditional decomposition method. The VIMHP is also independent of the small parameter assumption (which is either not there in the physical problems or difficult to locate) and hence is more convenient to apply as compare to the traditional perturbation method. It is worth mentioning that the VIMHP is applied without any discretization, restrictive assumption or transformation and is free from round off errors. We apply the proposed VIMHP for all the nonlinear terms in the problem without discretizing either by finite difference or spline techniques at the nodes, involves laborious calculations coupled with a strong possibility of the ill-conditioned resultant equations which is a complicated problem to solve. Moreover, unlike the method of separation of variables that requires initial and boundary conditions, the VIMHP provides the solution by using the initial conditions only. Finally, the variational iteration method using Adomian's polynomials (VIMAP) is also easier to implement as compare to the traditional decomposition method due to the fact that it involves Lagrange multiplier which reduces the successive application of integral operator and hence minimizes the computational work. Moreover, the VIMAP is also

independent of the small parameter assumption, discretization, linearization or transformation and so may be considered as a more efficient and convenient algorithm as compare to the traditional techniques which involve these deficiencies. Moreover, the use of Lagrange multiplier in VIMHP gives it a clear advantage over the traditional homotopy perturbation method (HPM) since it avoids the successive application of the integral operator. The proposed modified versions (VIMHP and VIMAP) can be applied to a number of physical problems related to fluid mechanics including Blasius' viscous flow, boundary layer flow with exponential or algebraic properties, Von Karman swirling viscous flow, nonlinear progressive waves in deep water, porous medium, financial mathematics, deep shallow water waves, electrical signals along a telegraph line, digital image processing, telecommunication, signals and systems, beam deflection theory, quantum field theory, relativistic physics, dispersive wave-phenomena, plasma physics, astrophysics, nonlinear optics, engineering and applied sciences, (see Noor and Mohyud-Din [37-39]).

2. VARIATIONAL ITERATION METHOD (VIM)

To illustrate the basic concept of the He's VIM, we consider the following general differential equation

$$Lu + Nu = g(x), \tag{1}$$

where L is a linear operator, N a nonlinear operator and g(x) is the inhomogeneous term. According to variational iteration method (see Abbasbandy [1, 2] Abdou and Soliman [6, 8], Abassy et. al. [8], Baitha et. al. [10], Bizar and Ghazvini [11], Wakil et. al. [12], Ganji et. al.[13], Ghorbani and Nadifi [14, 15], Golbabi and Javidi [16], He [17, 24-29], Inokuti et. al. [30], Lu [31], Momani and Odibat [33], Noor and Mohyud-Din [34-40], Russel and Shampine [41], Rafi and Danili [42], Sweilman [43]), we can construct a correction functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\,(L u_n(s) + N\widetilde{u}_n(s) - g(s))\,ds, \tag{2}$$

where λ is a Lagrange multiplier (see He [17, 24-29]). which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\widetilde{u}_n$ is considered as a restricted variation. i.e. $\delta\widetilde{u}_n = 0$; (2) is called a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in (see He [17, 24-29]). In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation u_{n+1}, $n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by $u = \lim_{n\to\infty} u_n$.

3. HOMOTOPY PERTURBATION METHOD (HPM)

To explain the He's homotopy perturbation method, we consider a general equation of the type,

$$L(u) = 0, \tag{3}$$

where L is any integral or differential operator. We define a convex homotopy H (u, p) by

$$H(u, p) = (1 - p)F(u) + pL(u), \tag{4}$$

where F (u) is a functional operator with known solutions v_0, which can be obtained easily. It is clear that, for

$$H(u, p) = 0, \tag{5}$$

we have

$$H(u,0) = F(u), \qquad H(u,1) = L(u).$$

This shows that $H(u, p)$ continuously traces an implicitly defined curve from a starting point H (v_0, 0) to a solution function H (f, 1). The embedding parameter monotonically increases from zero to unit as the trivial problem F (u) = 0 is continuously deforms the original problem L (u) = 0. The embedding parameter $p \in (0, 1]$ can be considered as an expanding parameter (see Ghorbani and Nadifi [14, 15], He [17-23], Noor and Mohyud-Din [34-40]). The homotopy perturbation method uses the homotopy parameter p as an expanding parameter (see He [17-23]) to obtain

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \cdots, \tag{6}$$

if p $\to$ 1, then (6) corresponds to (4) and becomes the approximate solution of the form,

$$f = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i. \tag{7}$$

It is well known that series (7) is convergent for most of the cases and also the rate of convergence is dependent on L (u); (see He [17-23]). We assume that (7) has a unique solution. The comparisons of like powers of p give solutions of various orders. In sum, according to (Ghorbani and Nadifi [14, 15]), He's HPM considers the solution, $u(x)$, of the homotopy equation in a series of p as follows:

$$u(x) = \sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + \ldots,$$

and the method considers the nonlinear term $N(u)$ as

$$N(u) = \sum_{i=0}^{\infty} p^i H_i = H_0 + p H_1 + p^2 H_2 + \ldots,$$

where H_n's are the so-called He's polynomials (Ghorbani and Nadifi [14, 15]), which can be calculated by using the formula

$$H_n(u_0, \ldots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left(N\left(\sum_{i=0}^{n} p^i u_i\right) \right)_{p=0}, \qquad n = 0,1,2, \ldots .$$

4. MODIFIED VARIATIONAL ITERATION METHODS

The modified variational iteration techniques are obtained by the elegant coupling of correction functional of VIM with He's and Adomian's polynomials.

4.1 Variational Iteration Method Using He's Polynomials (VIMHP)

This modified version of variational iteration method is obtained by the elegant coupling of correction functional (2) of variational iteration method (VIM) with He's polynomials and is given by

$$\sum_{n=0}^{\infty} p^{(n)} u_n = u_0(x) + p\int_0^x \lambda(s)\left(\sum_{n=0}^{\infty} p^{(n)} L(u_n) + \sum_{n=0}^{\infty} p^{(n)} N(\widetilde{u}_n)\right) ds - \int_0^x \lambda(s)\, g(s)\, ds. \quad (8)$$

Comparisons of like powers of p give solutions of various orders (see Noor and Mohyud-Din [37-39]).

4.2 Variational Iteration Method Using Adomian's Polynomials (VIMAP)

This modified version of VIM is obtained by the coupling of correction functional (2) of variational iteration method with Adomian's polynomials and is given by

$$u_{n+1}(x) = u_n(x) + \int_0^t \lambda\,(\, L\, u_n(x) + \sum_{n=0}^{\infty} A_n - g(x)\,)\, dx, \quad (9)$$

where A_n are the so-called Adomian's polynomials and are calculated for various classes of nonlinearities by using the specific algorithm developed in (see Abbasbandy [1, 2], Noor and Mohyud-Din [35, 40]).

5. NUMERICAL APPLICATION

In this section, we apply and compare both the modified versions of VIM for solving Flierl-Petviashivili (FP) equation.

Consider the generalized variant of the Flierl-Petviashivili equation

$$y'' + \frac{1}{x} y' - y^n - y^{n+1} = 0, \quad (10)$$

with boundary conditions

$$y(0) = \alpha, \qquad y'(0) = 0, \qquad y(\infty) = 0. \quad (11)$$

For $n = 1$, above equation reduces to the standard Flierl-Petviashivili equation. The general series solution for the equation is to be constructed for all possible values of $n \geq 1$. Using the transformation $u(x) = x\, y'(x)$, the generalized Flierl-Petviashivili equation (10, 11) can be converted to the following first order initial value problem

$$u'(x) = x\left(\int_0^x \left(\frac{u(x)}{x}\right)^n + \left(\frac{u(x)}{x}\right)^{n+1} dx\right), \quad (12)$$

with initial conditions

$$u(0) = 0, \quad u'(0) = 0. \quad (13)$$

The correction functional is given as

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left(\frac{\partial u_n}{\partial x} - s\left(\int_0^s \frac{\widetilde{u}_n(\xi)}{\xi} + \left(\frac{\widetilde{u}_n(\xi)}{\xi} \right)^{n+1} \right) d\xi \right) ds.$$

Making the correction functional stationary, the Lagrange multiplier can easily be identified as $\lambda(s) = -1$.

$$u_{n+1}(x) = -\int_0^x \left(\frac{\partial u_n}{\partial x} - \left(s \left(\int_0^s \left(\left(\frac{u_n(\xi)}{\xi} \right)^n \right) + \left(\frac{u_n(\xi)}{\xi} \right)^{n+1} d\xi \right) \right) \right) ds.$$

Applying the variational iteration method using He's polynomials (VIMHP), we get

$$u_0 + pu_1 + \cdots = -p\int_0^s \left(\left(\frac{\partial u_0}{\partial x} + p\frac{\partial u_1}{\partial x} + p^2 \frac{\partial u_2}{\partial x} + \cdots \right) dx - s\left(\int_0^s \left(\frac{1}{\xi}\left(u_0 + pu_1 + \cdots \right) \right)^n d\xi \right) \right) ds.$$

$$+ p\int_0^x s\left(\frac{1}{\xi}\left(u_0 + pu_1 + p^2 u_2 + \cdots \right) \right)^{n+1} d\xi \Bigg) ds.$$

Comparing the co-efficient of like powers of p, following approximants are obtained

$$p^{(0)} : u_0(x) = 0,$$

$$p^{(1)} : u_1(x) = \left(\frac{\alpha^n + \alpha^{n+1}}{2} \right) x^2,$$

$$p^{(2)} : u_2(x) = \left(\frac{\alpha^n + \alpha^{n+1}}{2} \right) x^2 + \frac{(\alpha^n + \alpha^{n+1})(n\alpha^n + (n+1)\alpha^{n+1})}{16\alpha} x^4,$$

$$p^{(3)} : u_3(x) = \left(\frac{\alpha^n + \alpha^{n+1}}{2} \right) x^2 + \frac{(\alpha^n + \alpha^{n+1})(n\alpha^n + (n+1)\alpha^{n+1})}{16\alpha} x^4$$

$$+ \frac{(\alpha^n + \alpha^{n+1})(2n(3n-1)\alpha^{2n} + 2n(3n+1)\alpha^{2n+1} + (3n+1)(n+1)\alpha^{2n+2})}{384\alpha^2} x^6,$$

$$p^{(4)} : u_4(x) = \frac{(\alpha^n + \alpha^{n+1})}{2} x^2 + \frac{(\alpha^n + \alpha^{n+1})(n\alpha^n + (n+1)\alpha^{n+1})}{16\alpha} x^4$$

$$+ \frac{(\alpha^n + \alpha^{n+1})(2n(3n-1)\alpha^{2n} + 2n(3n+1)\alpha^{2n+1} + (3n+1)(n+1)\alpha^{2n+2})}{384\alpha^2} x^6$$

$$+ \frac{(\alpha^n + \alpha^{n+1})(n(18n^2 - 29n + 12)\alpha^{3n} + n(54n^2 - 33n + 7)\alpha^{3n+1}}{18432\alpha^3} x^8$$

$$+ \frac{(18n^2 + 7n + 1)(3n\alpha^{3n+2} + (n+1)\alpha^{3n+3}))}{18432\alpha^3} x^8,$$

$\vdots$.

The solution in a series form is given by

$$u(x)=\frac{(\alpha^n+\alpha^{n+1})}{2}x^2+\frac{(\alpha^n+\alpha^{n+1})(n\alpha^n+(n+1)\alpha^{n+1})}{16\alpha}x^4$$
$$+\frac{(\alpha^n+\alpha^{n+1})(2n(3n-1)\alpha^{2n}+2n(3n+1)\alpha^{2n+1}+(3n+1)(n+1)\alpha^{2n+2})}{384\alpha^2}x^6$$
$$+\frac{(\alpha^n+\alpha^{n+1})(n(18n^2-29n+12)\alpha^{3n}+n(54n^2-33n+7)\alpha^{3n+1}}{18432\alpha^3}x^8$$
$$+\frac{(18n^2+7n+1)(3n\alpha^{3n+2}+(n+1)\alpha^{3n+3}))}{18432\alpha^3}x^8+\cdots,$$

the inverse transformation will yield

$$y(x)=\alpha+\frac{(\alpha^n+\alpha^{n+1})}{4}x^2+\frac{(\alpha^n+\alpha^{n+1})(n\alpha^n+(n+1)\alpha^{n+1})}{64\alpha}x^4$$
$$+\frac{(\alpha^n+\alpha^{n+1})(2n(3n-1)\alpha^{2n}+2n(3n+1)\alpha^{2n+1}+(3n+1)(n+1)\alpha^{2n+2})}{2304\alpha^2}x^6 \quad (14)$$
$$+\frac{(\alpha^n+\alpha^{n+1})(n(18n^2-29n+12)\alpha^{3n}+n(54n^2-33n+7)\alpha^{3n+1})}{147456\alpha^3}x^8$$
$$+\frac{(\alpha^n+\alpha^{n+1})(18n^2+7n+1)(3n\alpha^{3n+2}+(n+1)\alpha^{3n+3})}{147456\alpha^3}x^8+\cdots.$$

Now, we apply the diagonal Pade´ approximants to the obtained series solution to handle the boundary conditions at infinity because power series in isolation are never useful in boundary value problems because mostly radius of convergence is not sufficiently large, (see Noor and Mohyud-Din [36], Wazwaz [48]). This makes the use of Pade´ approximants very essential in unbounded domain. The series solution is used to obtain various Pade´ approximants [2/2], [4/4], [6/6], [8/8]. Roots of the Pade´ approximants to the Flierl-Petviashivili monopole α were obtained by using the limit of the Pade´ approximant [m/m] as $x\to\infty$ is $\frac{a_8}{b_8}$, where a_m and b_m are the leading coefficients of the numerator and the denominator, respectively. For $n=2$, the complex roots along with other real roots are discarded since these do not meet the physical requirements.

TABLE 5.1 Roots of the Pade´ approximants monopole (Noor and Mohyud-Din [36], Wazwaz [48]) $\alpha,\ n=1$

Degree	Roots
[2/2]	-1.5
[4/4]	-2.50746
[6/6]	-2.390278
[8/8]	-2.392214

TABLE 5.2 Roots of the Pade´ approximants monopole (Noor and Mohyud-Din [36], Wazwaz [48]) $\alpha,\ n=2$

Degree	Roots
[2/2]	-2.0
[4/4]	-2.0
[6/6]	-2.0
[8/8]	-2.0

TABLE 5.3 Roots of the Pade´ approximants monopole (Noor and Mohyud-Din [36], Wazwaz [48]) $\alpha,\ n=3$

Degree	Roots
[2/2]	0.0
[4/4]	-2.197575908
[6/6]	-1.1918424398
[8/8]	-1.848997181

TABLE 5.4 Roots of the Pade´ approximants [8/8] monopole α for several values of *n* (Noor and Mohyud-Din [36], Wazwaz [48])

n	[8/8] Roots	n	[8/8] Roots
1	-2.392213866	7	-1.000708285
2	-2.0	8	-1.000601615
3	-1.848997181	9	-1.000523005
4	-1.286025892	10	-1.000462636
5	-1.001101141	11	-1.000262137
6	-1.000861533	$n\to\infty$	-1.0

Table 5.4 shows that the roots of the monopole α converge to -1 as *n* increases.
Now applying the modified version 4.2 (VIMAP) on (10, 11) and applying the same transformation, we get the following iterative scheme

$$u_{n+1}(x) = -\int_0^x \left(\frac{\partial u_n}{\partial x} - \sum_{n=0}^{\infty} A_n \right) ds, \tag{15}$$

where A_n are the so-called Adomian's polynomials and can be generated for all types of nonlinearities according to the algorithms developed in (Wazwaz [47, 48]). First few Adomian's polynomials are as follows

$A_0 = u_0^n + u_0^{n+1}$,

$A_1 = n u_1 u_0^{n-1} + (n+1) u_1 u_0^n$,

$$A_2 = n u_2 u_0^{n-1} + n(n-1) \frac{u_1^2}{2!} u_0^{n-2} + (n+1) u_2 u_0^n + n(n+1) \frac{u_1^2}{2!} u_0^{n-1},$$

$\vdots$.

Employing these polynomials coupled with the iterative scheme (15) and using the inverse transformation, we obtained the series solution which is in full agreement with (14) where the same problem has been solved by using the modified version 4.1 (VIMHP).

REMARK 5.1: It is observed that the solution based upon He's polynomials (VIMHP) is much easier to calculate as compare to the modified version 4.2 (VIMAP) where Adomian's polynomials coupled with their complexities have been applied.

6. CONCLUSION

In this paper, we applied and compared two modified versions of variational iteration method (VIM) Flierl-Petviashivili (FP) equation by converting the Flierl-Petviashivili equation to a first order initial value problem. The proposed methods are applied to the reformulated first-order initial value problem which gives the solution in terms of transformed variable. The desired series of solutions are obtained by making use of inverse transformation. The difficulty in this type of equation, due to the existence of singular point at x = 0, is overcome here. To make the work more concise and for the better understanding of the solution behavior the Pade´ approximants were employed. It is concluded that the solution based upon He's polynomials (VIMHP) is much easier to calculate as compare to the modified version 4.2 (VIMAP) where Adomian's polynomials coupled with their complexities have been applied.

Acknowledgment- The authors are highly grateful to the referee for his very constructive comments. We would like to thank Dr S. M. Junaid Zaidi, Rector CIIT for providing excellent research environment and facilities. The first author is also thankful to Brig (R) Qamar Zaman, Vice Chancellor HITEC University for the provision of very conducive environs for research.

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