APPROXIMATE SOLUTION OF HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS BY MEANS OF A NEW RATIONAL CHEBYSHEV COLLOCATION METHOD

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Abstract- In this paper, a new approximate method for solving higher-order linear ordinary differential equations with variable coefficients under the mixed conditions is presented. The method is based on the rational Chebyshev (RC) Tau, Chebyshev and Taylor collocation methods. The solution is obtained in terms of rational Chebyshev (RC) functions. Also, illustrative examples are given to demonstrate the validity and applicability of the method.

Keywords- Rational Chebyshev Functions, Higher-order Ordinary Differential Equations, Taylor and Chebyshev Collocation Methods.

1. INTRODUCTION

Many problems arising in science and engineering are formulated in bounded and unbounded domains. Recently a number of different methods associated with orthogonal systems for solving higher-order differential equations, which are the Hermite spectral method [1,2], the Laguerre method [3,4], the Jacoby polynomials method [5], the methods based on rational Chebhshev (RC) functions [6,7], the Laguerre tau method [8] and the rational Chebyshev tau method [9], have been studied.

On the other hand, Chebyshev and Taylor (matrix and collocation) methods for the approximate solution of high-order differential and difference equations have been presented in many paper by Sezer et.al. [10-16].

In this paper, the Chebyshev tau [9], the Taylor collocation [14,15] methods are developed and applied to the mth-order linear nonhomogenous differential equation

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = g(x), \ \ 0 \le x < \infty, \tag{1}$$

with the mixed conditions

$$\sum_{k=0}^{m-1} \sum_{j=0}^{J} c_{ij}^{k} y^{(k)}(c_j) = \lambda_i, \tag{2}$$

$0 \le c_j < \infty$, $j = 0, 1, ..., J$; $i = 0, 1, ..., m-1$,

and the solution is expressed in terms of the rational Chebyshev functions [9] as follows:

$$y(x) = \sum_{n=0}^{N} a_n R_n(x) , \; 0 \le x < \infty, \tag{3}$$

here $P_k(x)$ and $g(x)$ are continuous functions on $[0, \infty)$; a_n, $n = 0, 1, ..., N$ are the coefficients to be determined; $R_n(x)$, $n = 0, 1, ..., N$ is the rational Chebyshev functions; c_{ij}^{k} , c_j and λ_i are appropriate constants.

2. PROPERTIES OF THE RATIONAL CHEBYSHEV (RC) FUNCTIONS [9]

In cases when errors near the ends of an interval $[a, b]$ are particular importance, a weighting function which is the form $1/\sqrt{(x-a)(b-x)}$ is often useful. It is supposed again that a linear change in variables has tranformed the given interval into the interval $[-1, 1]$, so that the weighting function becomes $w(x) = 1/\sqrt{1-x^2}$. In other words a great variety of other types of least-square polynomial approximation can be formulated in terms of other weighting functions. In particular, for the weighting function $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, ($\alpha > -1$, $\beta > -1$) over $[-1, 1]$, which reduces to Legendre case when $\alpha = \beta = 0$ and to the Chebyshev case when $\alpha = \beta = -1/2$. The well-known Chebyshev polynomials are orthogonal in the interval $[-1, 1]$ with respect to the weight function $w(x) = 1/\sqrt{1-x^2}$ and can be determined with the aid of the recurrence formulae

$$T_0(x) = 1, \; T_1(x) = x , \; T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) , \; n \ge 1.$$

The RC functions are defined by

$$R_n(x) = T_n\left(\frac{x-1}{x+1}\right),$$

or clearly

$$R_0(x) = 1, \; R_1(x) = \frac{x-1}{x+1},$$

$$R_{n+1}(x) = 2\left(\frac{x-1}{x+1}\right)R_n(x) - R_{n-1}(x) , \; n \ge 1. \tag{4}$$

These functions are orthogonal with respect to the weight function $w(x) = 1/((x+1)\sqrt{x})$ in the interval $[0, \infty)$.

3. FUNDAMENTAL MATRIX RELATIONS

Let us first assume that the solution of Eq.(1) can be expressed in the form (3), which is a truncated Chebyshev series in terms of RC functions. Then $y(x)$ and its derivative $y^{(k)}(x)$ can be put in the matrix forms

$$[y(x)] = \mathbf{R}(x)\,\mathbf{A}, \tag{5}$$

and

$$[y^{(k)}(x)] = \mathbf{R}^{(k)}(x)\mathbf{A}, \quad k = 0, 1, 2, ..., m \le N,$$

so that

$$\mathbf{R}^{(k)}(x) = [R_0^{(k)}(x)\ R_1^{(k)}(x)\ ...\ R_N^{(k)}(x)\,],$$
$$\mathbf{A} = [a_0\ a_1\ ...\ a_N]^T$$

where $y^{(0)}(x) \equiv y(x)$, $\mathbf{R}^{(0)}(x) \equiv \mathbf{R}(x)$; $R_0(x), R_1(x), ..., R_N(x)$ are the RC functions defined in Eq.(4); $a_0, a_1, ..., a_N$ are coefficients defined in Eq.(3).

If we use the expression $v(x) = \dfrac{x-1}{x+1}$ in the RC function (4), then the matrix $\mathbf{R}(x)$ becomes

$$\mathbf{R}^T(x) = \mathbf{C}\mathbf{V}^T(x) \text{ or } \mathbf{R}(x) = \mathbf{V}(x)\mathbf{C}^T \tag{6}$$

so that

$$\mathbf{R}(x) = [R_0(x)\quad R_1(x)\quad ...\quad R_N(x)]$$
$$\mathbf{V}(x) = [v^0(x)\quad v^1(x)\quad ...\quad v^N(x)]$$

$$\mathbf{C}^{-1} = \left[\begin{array}{cccccc|c}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
\tfrac{1}{2} & 0 & \tfrac{1}{2} & 0 & \cdots & 0 & 0 \\
0 & \tfrac{3}{4} & 0 & \tfrac{1}{4} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\dfrac{1}{2^N}\dbinom{N}{\frac{N-0}{2}} & 0 & \dfrac{1}{2^{N-1}}\dbinom{N}{\frac{N-2}{2}} & 0 & \cdots & \dfrac{1}{2^{N-1}}\dbinom{N}{0} & 0 \\
\hline
0 & \dfrac{1}{2^{N-1}}\dbinom{N}{\frac{N-1}{2}} & 0 & \dfrac{1}{2^{N-1}}\dbinom{N}{\frac{N-3}{2}} & \cdots & 0 & \dfrac{1}{2^{N-1}}\dbinom{N}{0}
\end{array}\right]$$

In this case, we are going to use the last row for odd values of N, and othervise previous one as the last row of matrix.

For example, in the cases $N = 3$ and $N = 4$, the matrix $\mathbf{C}$ becomes

$$
\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & -3 & 0 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ 0 & -3 & 0 & 4 & 0 \\ 1 & 0 & -8 & 0 & 8 \end{bmatrix}
$$

Consequently, the kth derivative of the matrix $\mathbf{R}(x)$ defined in (5), from Eq (6), can be obtained as

$$
\mathbf{R}^{(k)}(x) = \mathbf{V}^{(k)}(x)\mathbf{C}^{T}
$$

and thereby, from the expression (5)

$$
\left[y^{(k)}(x) \right] = \mathbf{V}^{(k)}(x)\mathbf{C}^{T}\mathbf{A}, \tag{7}
$$

where

$$
\mathbf{V}^{(k)}(x) = \left[(v^{0}(x))^{(k)} \quad (v^{1}(x))^{(k)} \quad (v^{2}(x))^{(k)} \quad \ldots \quad (v^{N}(x))^{(k)} \right],
$$

and

$$
v^{0}(x) = \left(\frac{x-1}{x+1} \right)^{0} = 1, \quad v^{1}(x) = \left(\frac{x-1}{x+1} \right), \quad v^{2}(x) = \left(\frac{x-1}{x+1} \right)^{2}, \quad \ldots, \quad v^{N}(x) = \left(\frac{x-1}{x+1} \right)^{N}.
$$

4. MATRIX RELATION BASED ON COLLOCATION POINTS

Now, let us define the collocation points as

$$
x_{r} = \frac{c}{N}r, \quad r = 0, 1, 2, ..., N, \tag{8}
$$

so that $0 \leq x_{r} \leq c < \infty$; $c \in IR^{+}$.

Then we substitute the collocation points (8) into Eq.(1) to obtain the system

$$
\sum_{k=0}^{m} P_{k}(x_{r})y^{(k)}(x_{r}) = g(x_{r}); \quad r = 0, 1, 2, ..., N. \tag{9}
$$

The system (9) can be written in the matrix form

$$
\sum_{k=0}^{m} \mathbf{P}_{k}\mathbf{Y}^{(k)} = \mathbf{G}, \tag{10}
$$

where

$$
\mathbf{P}_{k} = \begin{bmatrix} p_{k}(x_{0}) & 0 & \ldots & 0 \\ 0 & p_{k}(x_{1}) & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & p_{k}(x_{N}) \end{bmatrix}, \quad \mathbf{Y}^{(k)} = \begin{bmatrix} y^{(k)}(x_{0}) \\ y^{(k)}(x_{1}) \\ \vdots \\ y^{(k)}(x_{N}) \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} g(x_{0}) \\ g(x_{1}) \\ \vdots \\ g(x_{N}) \end{bmatrix}.
$$

By putting the collocation points x_r, $r = 0, 1, 2, ..., N$ in the relation (7) we have the matrix system

$$\left[y^{(k)}(x_r)\right] = \mathbf{V}^{(k)}(x_r)\mathbf{C}^T\mathbf{A} \; ; r = 0, 1, 2, ..., N,$$

or briefly

$$\mathbf{Y}^{(k)} = \mathbf{V}^{(k)}\mathbf{C}^T\mathbf{A}, \tag{11}$$

where

$$\mathbf{V}^{(k)} = \begin{bmatrix} v^{(k)}(x_0) \\ v^{(k)}(x_1) \\ \vdots \\ v^{(k)}(x_N) \end{bmatrix} = \begin{bmatrix} (v^0(x))^{(k)}_{x=x_0} & (v^1(x))^{(k)}_{x=x_0} & \cdots & (v^N(x))^{(k)}_{x=x_0} \\ (v^0(x))^{(k)}_{x=x_1} & (v^1(x))^{(k)}_{x=x_1} & \cdots & (v^N(x))^{(k)}_{x=x_1} \\ \vdots & \vdots & \cdots & \vdots \\ (v^0(x))^{(k)}_{x=x_N} & (v^1(x))^{(k)}_{x=x_N} & \cdots & (v^N(x))^{(k)}_{x=x_N} \end{bmatrix}$$

Consequently, from the matrix forms (10) and (11), we obtain the fundamental matrix equation for Eq.(1) as

$$\sum_{k=0}^{m}\mathbf{P}_k\mathbf{V}^{(k)}\mathbf{C}^T\mathbf{A} = \mathbf{G} \; . \tag{12}$$

Next, we can obtain the corresponding matrix forms for the conditions (2) as follows: Using the relation (7) for $x = c_j$, we have the fundemental matrix equation corresponding to the mixed conditions (2):

$$\sum_{k=0}^{m-1}\sum_{j=0}^{J} c_{ij}^k \, \mathbf{V}^{(k)}(c_j)\mathbf{C}^T\mathbf{A} = [\lambda_i] \; ; \; i = 0, 1, 2, ..., m-1, \tag{13}$$

so that $0 \le c_j \le c < \infty \; ; \; j = 0, 1, 2, ..., J$.

5. METHOD OF SOLUTION

The fundamental matrix equation (12) for Eq.(1) corresponds to a system of $(N+1)$ algebraic equations for the $(N+1)$ unknown coefficients $a_o, a_1, ..., a_N$. Briefly we can write Eq.(12) as

$$\mathbf{WA} = \mathbf{G} \; \text{ or } \; [\mathbf{W};\mathbf{G}] \tag{14}$$

so that

$$\mathbf{W} = [W_{pq}] = \sum_{k=0}^{m}\mathbf{P}_k\mathbf{V}^{(k)}\mathbf{C}^T \; ; p,q = 0, 1, ..., N.$$

We can obtain the matrix form for the mixed conditions (2), by means of Eq.(13), briefly, as

$$\mathbf{U}_i\mathbf{A} = [\lambda_i] \; ; \; \text{ or } [\mathbf{U}_i;\lambda_i] \quad i = 0, 1, ..., m-1, \tag{15}$$

where

$$\mathbf{U}_i = \sum_{k=0}^{m-1}\sum_{j=0}^{J} c_{ij}^k \mathbf{V}^{(k)}(c_j)\mathbf{C}^T \equiv [u_{i0} \; u_{i1} \; ... \; u_{iN}].$$

To obtain the solution of Eq.(1) under the conditions (2), by replacing the rows of matrices (15) by the last m rows of the matrix (14), we have the required augmented matrix

$$\left[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}\right] = \begin{bmatrix} w_{00} & w_{01} & \cdot & \cdot & \cdot & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & \cdot & \cdot & \cdot & w_{1N} & ; & g(x_1) \\ \ldots & \ldots & & & & \ldots & ; & \ldots \\ w_{N-m,0} & w_{N-m,1} & \cdot & \cdot & \cdot & w_{N-m,N} & ; & g(x_{N-m}) \\ u_{00} & u_{01} & \cdot & \cdot & \cdot & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdot & \cdot & \cdot & u_{1N} & ; & \lambda_1 \\ \ldots & \ldots & & & & \ldots & ; & \ldots \\ u_{m-1,0} & u_{m-1,1} & \cdot & \cdot & \cdot & u_{m-1,N} & ; & \lambda_{m-1} \end{bmatrix}. \tag{16}$$

If rank $\widetilde{\mathbf{W}}$ = rank $\left[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}\right] = N+1$, then we can write

$$\mathbf{A} = \left(\widetilde{\mathbf{W}}\right)^{-1}\widetilde{\mathbf{G}}. \tag{17}$$

Thus the coefficients a_n ; $n = 0, 1, ..., N$ are uniquely determined by Eq.(16).

Also we can easily check the accuracy of the obtained solutions as follows [13,15]: Since the obtained rational Chebyshev function expansion is an approximate solution of Eq.(1), the resulting equation must be satisfied approximately; that is, for $x = x_i \in [0,b]$, $i = 0, 1, 2, ...$

$$E(x_i) = \left| \sum_{k=0}^{m} P_k(x_i)y^{(k)}(x_i) - g(x_i) \right| \cong 0,$$

or $E(x_i) \le 10^{-k_i}$ (k_i is any positive integer).

If max $(10^{-k_i}) = 10^{-k}$ (k is any positive integer) is prescribed, then the truncation limit N is increased until the difference $E(x_i)$ at each of the points x_i becomes smaller than the prescribed 10^{-k}.

6. ILLUSTRATIVE EXAMPLE

In this section, several numerical examples are given to illustrate the accuracy and effectiveness of properties of the method. All of them were performed on the computer using a program written in MATHEMATICA 5.2. The absolute errors in Tables are the values of $|y(x) - y_N(x)|$ at selected points.

Example 1.([9], Example 1) Let us consider the following two point boundary value problem

$$y''(x) - \frac{1-x}{(1+x)^2} y(x) = \frac{1}{(x+1)^2}, \quad x \in [0,1] \qquad (18)$$

with $y(0) = 1$, $y(1) = \frac{1}{2}$ and approximate the solution $y(x)$ by the rational Chebyshev functions

$$y(x) = \sum_{n=0}^{N} a_n R_n(x)$$

where $P_0(x) = \frac{-(1-x)}{(1+x)^2}$, $P_1(x) = 0$, $P_2(x) = 1$, $g(x) = \frac{1}{(x+1)^2}$. Then, for $N = 4$, the collocation points are

$$x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}, x_4 = 1$$

and the fundamental matrix equation of problem is

$$\left\{ P_0 V^{(0)} C^T + P_1 V^{(1)} C^T + P_2 V^{(2)} C^T \right\} A = G$$

where P_0, P_1, P_2, $V^{(0)}$, $V^{(1)}$, $V^{(2)}$, C are matrices of order (5×5) defined by

$$P_0 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & \frac{-12}{25} & 0 & 0 & 0 \\ 0 & 0 & \frac{-2}{9} & 0 & 0 \\ 0 & 0 & 0 & \frac{-4}{49} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$V^{(0)} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & -\frac{3}{5} & \frac{9}{25} & -\frac{27}{125} & \frac{81}{625} \\ 1 & -\frac{1}{3} & \frac{1}{9} & -\frac{1}{27} & \frac{1}{81} \\ 1 & -\frac{1}{7} & \frac{1}{49} & -\frac{1}{343} & \frac{1}{2401} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad V^{(1)} = \begin{bmatrix} 0 & 2 & -4 & 6 & -8 \\ 0 & \frac{32}{25} & -\frac{192}{125} & \frac{864}{625} & -\frac{3456}{3125} \\ 0 & \frac{8}{9} & -\frac{16}{27} & \frac{8}{27} & -\frac{32}{243} \\ 0 & \frac{32}{49} & -\frac{64}{343} & \frac{96}{2401} & -\frac{128}{16807} \\ 0 & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{V}^{(2)} = \begin{bmatrix} 0 & -4 & 16 & -36 & 64 \\ 0 & -\dfrac{256}{125} & \dfrac{3584}{625} & -\dfrac{25344}{3125} & \dfrac{27648}{3125} \\ 0 & -\dfrac{32}{27} & \dfrac{64}{27} & -\dfrac{160}{81} & \dfrac{896}{729} \\ 0 & -\dfrac{256}{343} & \dfrac{2560}{2401} & -\dfrac{6912}{16807} & \dfrac{13312}{117649} \\ 0 & -\dfrac{1}{2} & \dfrac{1}{2} & 0 & 0 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ 0 & -3 & 0 & 4 & 0 \\ 1 & 0 & -8 & 0 & 8 \end{bmatrix}.$$

The augmented matrix forms of the conditions for $N = 4$ are

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & ; & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 & ; & \dfrac{1}{2} \end{bmatrix}.$$

Then, we obtain the augmented matrix (16) as

$$[\widetilde{\mathbf{W}} ; \widetilde{\mathbf{G}}] = \begin{bmatrix} -1 & -3 & 31 & -131 & 383 & ; & 1 \\ -\dfrac{12}{25} & -\dfrac{44}{25} & \dfrac{7252}{625} & -\dfrac{16716}{625} & \dfrac{395444}{15625} & ; & \dfrac{16}{25} \\ -\dfrac{2}{9} & -\dfrac{10}{9} & \dfrac{398}{81} & -\dfrac{1102}{243} & -\dfrac{2230}{243} & ; & \dfrac{4}{9} \\ 1 & -1 & 1 & -1 & 1 & ; & 1 \\ 1 & 0 & -1 & 0 & 1 & ; & \dfrac{1}{2} \end{bmatrix}.$$

We obtain the solution

$$\mathbf{A} = \begin{bmatrix} \dfrac{1}{2} & -\dfrac{1}{2} & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}.$$

Therefore, we find the solution

$$y(x) = \frac{1}{2} R_0(x) - \frac{1}{2} R_1(x)$$

or in the form

$$y(x) = \frac{1}{x+1}$$

which is exact solution of two-point boundary value problem [9].

Example 2.([9], Example 2) Consider the differential equation

$$y''(x) + 2xy'(x) = 0, \ x \in [0,1) \tag{19}$$
$$y(0) = 0, \ y'(0) = \frac{2}{\sqrt{\pi}}.$$

We applied the RC collocation method and solved this problem. In Table 1, the resulting values for $N = 4$ and $N = 8$ using the present method together with the rationalized Haar and RC Tau method with $N = 8$ and also the exact values of $y(x)$ i.e

$$y(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt$$

are tabulated. The present method is also very effective and convenient. The errors in numerical solution of Example 2 are seen in Figure 1. The error decreases when the integer N is increased.

Table 1. Approximates and exact values for Example 2

x	Exact Solution	Rationalized Haar(N=8)	Chebyshev Tau Met(N=8)	Present Method(N=4)	Present Method(N=8)
0.0	0.0000000	0.00000	0.0000000	0.0000000	0.0000000
0.1	0.1124629	0.11244	0.1124630	0.1159364	0.1124629
0.2	0.2227025	0.22268	0.2227026	0.2313501	0.2227025
0.3	0.3286267	0.32861	0.3286269	0.3428859	0.3286267
0.4	0.4283923	0.42837	0.4283925	0.4476373	0.4283923
0.5	0.5204998	0.52047	0.5204998	0.5439239	0.5205003
0.5	0.6038560	0.60384	0.6038561	0.6310895	0.6038590
0.7	0.6778011	0.67779	0.6778012	0.7091601	0.6778169
0.8	0.7421009	0.74208	0.7421011	0.7785628	0.7421682
0.9	0.7969082	0.79689	0.7969085	0.8399335	0.7971486

Figure 1. Exact and other method solutions of the Example 2

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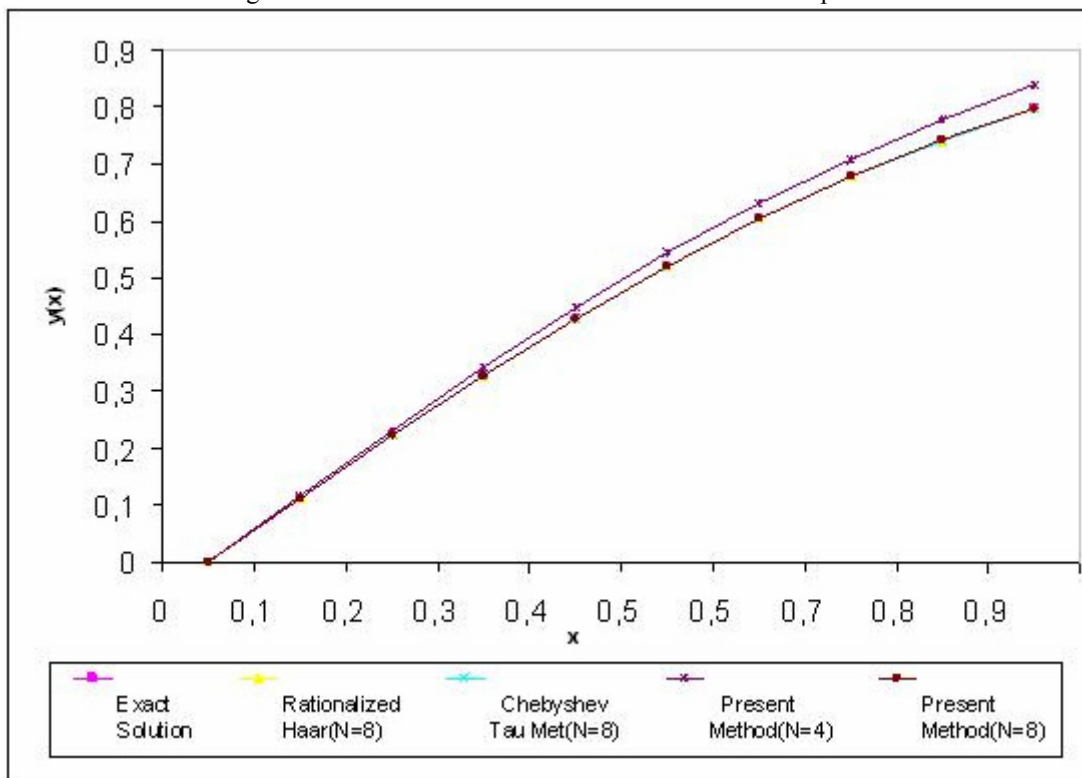
Example 3.([17],p.153) Consider the first order linear initial value problem

$$(x+1)y'(x) + y(x) = 1 \ , \ y(0) = 0 \ , \ x \in [0,1] \tag{20}$$

Following the procedures in the previous examples, we obtain the augmented matrix in the form:

$$[\tilde{\mathbf{W}};\tilde{\mathbf{G}}] = \begin{bmatrix} 1 & 1 & -7 & 17 & -31 & ; & 1 \\ 1 & 1 & -\dfrac{103}{25} & \dfrac{381}{125} & \dfrac{2161}{625} & ; & 1 \\ 1 & 1 & -\dfrac{23}{9} & -\dfrac{37}{27} & \dfrac{155}{27} & ; & 1 \\ 1 & 1 & -\dfrac{79}{49} & -\dfrac{937}{343} & \dfrac{8033}{2401} & ; & 1 \\ 1 & -1 & 1 & -1 & 1 & ; & 0 \end{bmatrix}$$

This system has the solution

$$\mathbf{A} = \begin{bmatrix} \dfrac{1}{2} & \dfrac{1}{2} & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}.$$

Therefore, we find the solution

$$y(x) = \frac{1}{2}R_0(x) + \frac{1}{2}R_1(x)$$

or

$$y(x) = \frac{x}{x+1}$$

which is the exact solution of Example 3.

Example 4. Our simple example is the linear initial value problem as follows

$$(1+x)y'(x) + (1+x+x^2)y(x) = 1-x, \ y(0) - \frac{3}{4}y(1) = 1, \ 0 \le x \le 1 \tag{21}$$

Using (17) to determine the individual terms of the RC collocation method, we find

$a_0 = -0.1864452793, \quad a_1 = -2.306641207, \quad a_2 = -0.7977329645, \quad a_3 = -0.138068253,$
$a_4 = -0.0826180758.$

Using (3) leads immediately to the solution of problem given by

$y(x) = -0.1864452793\,R_0(x) - 2.306641207\,R_1(x)\,R_1 - 0.7977329645\,R_2(x) -$
$0.138068253\,R_3(x) - 0.0826180758\,R_4(x)$

This expansion is approximate solution , that is, the first five terms of the Taylor series expansions of the Chebyshev solution given by Fox and Parker [17,p.137]. In Figure 2, the results obtained by our method are compared with the results of Fox and Parker [17,p.137]. The present method is also very effective and convenient. The errors in numerical solution of Example 4 are seen in Figure 3.

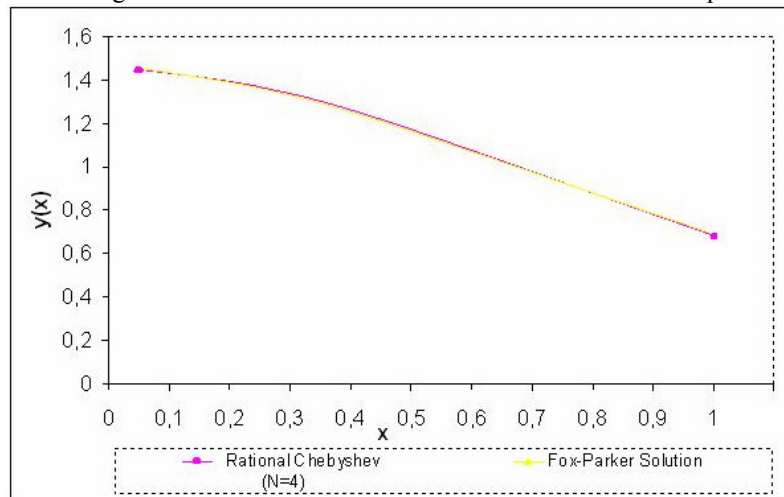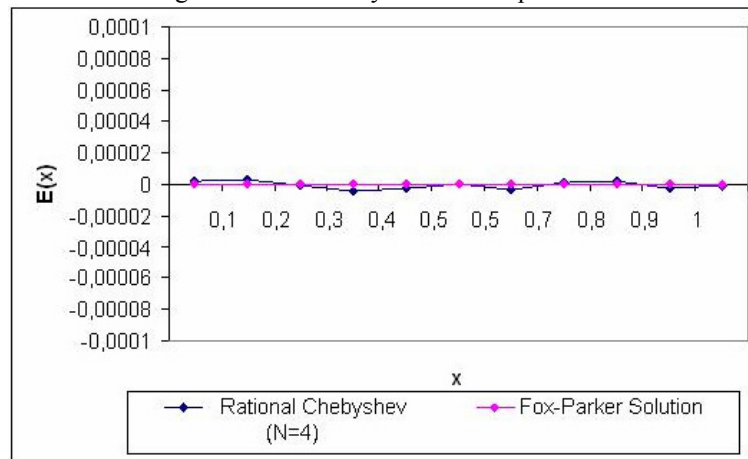Figure 2. Numerical and Fox-Parker solution of the Example 4

Figure 3. Error analysis for Example 4

7. CONCLUSION

The rational Chebyshev collocation method based on the rational Chebyshev Tau and Chebyshev-Taylor collocation methods are used to solve the higher-order ordinary differential equations numerically. A considerable advantage of the method is that the rational Chebyshev coefficients of the solution are found very easily by using computer programs. For this reason, this process is much faster than the other methods. In addition, an interesting feature of this method is to find the analytical solutions if the equation has an exact solution that is a rational functions. Illustrative examples with the satisfactory results are used to demonstrate the application of this method.

The method can also be extended to the system of linear differential equations with variable coefficients, but some modifications are required.

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