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A NEW PERTURBATION ALGORITHM WITH BETTER CONVERGENCE PROPERTIES: MULTIPLE SCALES LINDSTEDT POINCARE METHOD

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Abstract- A new perturbation algorithm combining the Method of Multiple Scales and Lindstedt-Poincare techniques is proposed for the first time. The algorithm combines the advantages of both methods. Convergence to real solutions with large perturbation parameters can be achieved for both constant amplitude and variable amplitude cases. Three problems are solved: Linear damped vibration equation, classical duffing equation and damped cubic nonlinear equation. Results of Multiple Scales, new method and numerical solutions are contrasted. The proposed new method produces better results for strong nonlinearities.

Keywords- Perturbation Methods, Lindstedt Poincare method, Multiple Scales method, Numerical Solutions

1. INTRODUCTION

Perturbation methods are well established and used for over a century to determine approximate analytical solutions for mathematical models. Algebraic equations, integrals, differential equations, difference equations and integro-differential equations can be solved approximately with these techniques. The direct expansion method (pedestrian expansion) does not produce physically valid solutions for most of the cases and depending on the nature of the equation, many different perturbation techniques such as Lindstedt-Poincare technique, Renormalization method, Method of Multiple Scales, Averaging methods, Method of Matched Asymptotic Expansions etc. are developed within time.

One of the deficiencies in applying perturbation methods is that a small parameter is needed in the equations or the small parameter should be introduced artificially to the equations. Nevertheless, the problem solved is a weak nonlinear problem and it becomes hard to obtain an approximate solution valid for strongly nonlinear systems.

There have been a number of attempts recently to validate perturbation solutions for strongly nonlinear systems also. Hu and Xiong [1] contrasted two different approaches of Lindstedt-Poincare methods using the duffing equation. First, they solved the equation with classical method and then they made a slight modification in the expansions. Instead of expanding the transformation frequency, they expanded the natural frequency and obtained solutions with excellent convergence properties for the duffing equation. The time histories of solutions agree with the numerical solutions for arbitrarily large perturbation parameters. In a similar paper, the approximate and exact frequencies are contrasted for the duffing equation [2]. The case of vanishing restoring force was also treated for the same equation [3]. The periods obtained are contrasted with the exact period with good convergence properties for large parameters.

While a complete review of the attempts to validate perturbation solutions for strongly nonlinear oscillators is beyond the scope of this work, a partial list will be given. Among the many developed methods, Linearized perturbation method [4-6], parameter expanding method [7, 8], new time transformations as modifications of Lindstedt-Poincare method [9-11], iteration methods [12, 13] are some examples.

In this work, Multiple Scales method is modified by incorporating the time transformation of Lindstedt Poincare method. One of the major advantages of Multiple Scales method over the Lindstedt Poincare method is that transient solutions can be found using the former whereas it is impossible to retrieve such solutions using the latter. However, by expanding the natural frequency in modified Lindstedt Poincare method as in [1-3], convergent solutions for large parameters are possible. Combining both methods would augment the advantages of them in a single method so that transient solutions with improved convergence properties can be retrieved. The method will be outlined in the next section and will be applied to three models in the subsequent sections. Comparisons of both methods with numerical results will be displayed. For the models considered, the new method is in good agreement with the numerical simulations. The method may find applications in a variety of problems.

2. MULTIPLE SCALES LINDSTEDT POINCARE (MSLP) METHOD

The outline of the method and the guidelines will be given in this section. Consider the nonlinear equation

$$\ddot{u} + \omega_0^2 u + \varepsilon f(u) = 0$$

where ε is the perturbation parameter. The essential steps in the method are

1) Make the time transformation as in Lindstedt-Poincare method.

$$\tau = \omega t \tag{1}$$

$$\omega^2 u'' + \omega_0^2 u + \varepsilon f(u) = 0 \tag{2}$$

where prime denotes differentiation with respect to new time variable τ.

2) Define the fast and slow time scales

$$T_0 = \tau = \omega t, \qquad T_1 = \varepsilon\tau = \varepsilon\omega t, \qquad T_2 = \varepsilon^2\tau = \varepsilon^2\omega t \tag{3}$$

3) The transformed time derivatives and dependent variable are expanded as in classical Multiple Scales method

$$\frac{d}{d\tau} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots, \qquad \frac{d^2}{d\tau^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) + \ldots \tag{4}$$

$$u = u_0(T_0, T_1, T_2) + \varepsilon u_1(T_0, T_1, T_2) + \varepsilon^2 u_2(T_0, T_1, T_2) + \ldots \tag{5}$$

4) As in classical Lindstedt-Poincare method ω^2 is expanded

$$\omega^2 = \omega_0^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots \tag{6}$$

5) Following [1-3], instead of ω^2, as usual in Lindstedt-Poincare method, ω_0^2 is substituted

$$\omega_0^2 = \omega^2 - \varepsilon\omega_1 - \varepsilon^2\omega_2 + \ldots \tag{7}$$

6) Equations (4), (5) and (7) are substituted into (2)

$$\omega^2(D_0^2+2\varepsilon D_0D_1+\varepsilon^2(D_1^2+2D_0D_2))(u_0+\varepsilon u_1+\varepsilon^2u_2) + (\omega^2-\varepsilon\omega_1-\varepsilon^2\omega_2)(u_0+\varepsilon u_1+\varepsilon^2u_2)+\varepsilon f(u_0+\varepsilon u_1+\varepsilon^2u_2)=0 \quad (8)$$

7) The equations at each order are separated

O(1): $\omega^2D_0^2u_0+\omega^2u_0=0$ (9)

O(ε): $\omega^2D_0^2u_1+\omega^2u_1=-2\omega^2D_0D_1u_0+\omega_1u_0-f(u_0)$ (10)

O(ε^2): $\omega^2D_0^2u_2+\omega^2u_2=-2\omega^2D_0D_1u_1-\omega^2(D_1^2+2D_0D_2)u_0 + \omega_1u_1+\omega_2u_0-f'(u_0)u_1$ (11)

The solution at the first order is

$$u_0=A(T_1,T_2)e^{iT_0}+cc \quad (12)$$

where cc stands for complex conjugates of the preceding terms.

8) The problem at the second and third order is that one has two mechanisms for eliminating secular terms. Either select D_1A or ω_1 in equation (10) and select D_2A or ω_2 in equation (11). One should first try $D_1A=0$ and solve ω_1. If ω_1 is a real number then the selection is fine. If ω_1 is complex, this choice will provide unphysical solutions as also pointed by Nayfeh [14]. In that case, select $\omega_1=0$ and eliminate secularities by choosing D_1A. The same reasoning is also valid at the next order of approximation. If ω_2 is real when $D_2A=0$, the selection is fine. If not, then select $\omega_2=0$ and choose D_2A to eliminate secularities. The algorithm will be applied to three different problems in the forthcoming sections.

3.LINEAR DAMPED OSCILLATOR

Consider the linear damped oscillator

$$\ddot{u}+\omega_0^2u+2\varepsilon\mu\dot{u}=0 \quad (13)$$

The new method will be applied to this equation. First, the time transformation is made

$$\tau=\omega t \quad (14)$$

$$\omega^2u''+\omega_0^2u+2\varepsilon\mu\omega u'=0 \quad (15)$$

where prime denotes differentiation with respect to new time variable τ. Fast and slow time scales are

$$T_0=\tau=\omega t,\qquad T_1=\varepsilon\tau=\varepsilon\omega t,\qquad T_2=\varepsilon^2\tau=\varepsilon^2\omega t \quad (16)$$

The time derivatives, dependent variable and frequency are expanded

$$\frac{d}{d\tau}=D_0+\varepsilon D_1+\varepsilon^2D_2+\ldots,\qquad \frac{d^2}{d\tau^2}=D_0^2+2\varepsilon D_0D_1+\varepsilon^2(D_1^2+2D_0D_2)+\ldots \quad (17)$$

$$u=u_0(T_0,T_1,T_2)+\varepsilon u_1(T_0,T_1,T_2)+\varepsilon^2u_2(T_0,T_1,T_2)+\ldots \quad (18)$$

$$\omega^2=\omega_0^2+\varepsilon\omega_1+\varepsilon^2\omega_2+\ldots \quad (19)$$

and substituted into (15)

$$\omega^2(D_0^2+2\varepsilon D_0D_1+\varepsilon^2(D_1^2+2D_0D_2))(u_0+\varepsilon u_1+\varepsilon^2u_2) + (\omega^2-\varepsilon\omega_1-\varepsilon^2\omega_2)(u_0+\varepsilon u_1+\varepsilon^2u_2)+2\varepsilon\mu\omega(D_0+\varepsilon D_1)(u_0+\varepsilon u_1)=0 \quad (20)$$

Note that instead of ω^2, ω_0^2 is expanded and substituted. The equations at each order are

O(1): $\omega^2D_0^2u_0+\omega^2u_0=0$ (21)

$O(\varepsilon)$: $\omega^2 D_0^2 u_1 + \omega^2 u_1 = -2\omega^2 D_0 D_1 u_0 + \omega_1 u_0 - 2\mu\omega D_0 u_0$ (22)

$O(\varepsilon^2)$: $$\begin{aligned}\omega^2 D_0^2 u_2 + \omega^2 u_2 &= -2\omega^2 D_0 D_1 u_1 - \omega^2 (D_1^2 + 2D_0 D_2) u_0 \\ &\quad + \omega_1 u_1 + \omega_2 u_0 - 2\mu\omega (D_0 u_1 + D_1 u_0)\end{aligned} \quad (23)$$

The solution at the first order is

$$u_0 = A(T_1, T_2) e^{iT_0} + cc \quad (24)$$

Substituting the solution to (22) and rearranging yields

$$\omega^2 D_0^2 u_1 + \omega^2 u_1 = e^{iT_0} (-2\omega^2 i D_1 A + \omega_1 A - 2\mu\omega i A) + cc \quad (25)$$

Elimination of secular terms requires

$$-2\omega^2 i D_1 A + \omega_1 A - 2\mu\omega i A = 0 \quad (26)$$

As outlined, one first tries $D_1 A = 0$ and solve $\omega_1 = 2\mu\omega i$. Since ω_1 is complex, this is not a suitable choice. Therefore, the appropriate selection is

$$\omega_1 = 0, \qquad D_1 A = -\frac{\mu}{\omega} A \quad (27)$$

Substituting the polar form

$$A = \frac{1}{2} a e^{i\beta} \quad (28)$$

to (27), separating real and imaginary parts yield

$$a = a(T_2) e^{-\frac{\mu}{\omega} T_1}, \qquad \beta = \beta(T_2) \quad (29)$$

Since the right hand side of (25) is annihilated, u_1 can be selected as zero

$$u_1 = 0 \quad (30)$$

At the last order, substitute (24), (27) and (30) to the right hand side of (23) and eliminate the secular terms

$$2\omega^2 i D_2 A - \omega_2 A - \mu^2 A = 0 \quad (31)$$

If $D_2 A = 0$, ω_2 is a real number, so this choice is appropriate

$$D_2 A = 0, \qquad \omega_2 = -\mu^2 \quad (32)$$

From (32) A and hence a and β does not depend on T_2. Equation (29) reads

$$a = a_0 e^{-\frac{\mu}{\omega} T_1}, \quad \beta = \beta_0 \quad (33)$$

The frequency expansion is

$$\omega^2 = \omega_0^2 - \varepsilon^2 \mu^2 \quad (34)$$

Using (34), (33), (30), (28), (24), (18) and (16), the final solution in terms of original time variable is

$$u(t) = a_0 e^{-\varepsilon\mu t} \cos\left(\sqrt{\omega_0^2 - \varepsilon^2 \mu^2}\, t + \beta_0\right) + O(\varepsilon^2) \quad (35)$$

In fact, this solution is the exact solution of the original equation. Application of classical multiple scales produces the following solution

$$u(t) = a_0 e^{-\varepsilon\mu t} \cos\left(\omega_0 \left(1 - \frac{\varepsilon^2 \mu^2}{2\omega_0^2}\right) t + \beta_0\right) + O(\varepsilon^2) \quad (36)$$

As can be verified easily, the frequency is a Taylor expansion of the exact frequency. Solution (36) is valid only when $\varepsilon\mu \ll \omega_0$ whereas one does not have this restriction for (35) obtained by MSLP method.

4. DUFFING EQUATION

Duffing equation will be solved by classical Multiple Scales (MS) and Multiple Scales Lindstedt Poincare (MSLP) method. Solutions will be contrasted with the numerical simulations. Consider the equation

$$\ddot{u} + \omega_0^2 u + \varepsilon u^3 = 0 \tag{37}$$

with initial conditions

$$u(0)=a_0, \qquad \dot{u}_0(0) = 0 \tag{38}$$

4.1. Multiple Scales (MS) Method

Fast and slow time scales are

$$T_0 = t, \qquad T_1 = \varepsilon t, \qquad T_2 = \varepsilon^2 t \tag{39}$$

Using

$$\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) + \ldots \tag{40}$$

and substituting the expansion

$$u = u_0(T_0, T_1, T_2) + \varepsilon u_1(T_0, T_1, T_2) + \varepsilon^2 u_2(T_0, T_1, T_2) + \ldots \tag{41}$$

into the original equation and initial conditions yields after separation

$$O(1){:}\ D_0^2 u_0 + \omega_0^2 u_0 = 0 \qquad u_0(0) = a_0, \quad D_0 u_0(0) = 0 \tag{42}$$

$$O(\varepsilon){:}\ D_0^2 u_1 + \omega_0^2 u_1 = -2D_0 D_1 u_0 - u_0^3 \qquad u_1(0) = 0, \quad (D_0 u_1 + D_1 u_0)(0) = 0 \tag{43}$$

$$O(\varepsilon^2){:}\ D_0^2 u_2 + \omega_0^2 u_2 = -2D_0 D_1 u_1 - (D_1^2 + 2D_0 D_2) u_0 - 3u_0^2 u_1 \tag{44}$$

The solution at the first order is

$$u_0 = A(T_1, T_2) e^{i\omega_0 T_0} + cc \tag{45}$$

where

$$A = \frac{1}{2} a e^{i\beta} \tag{46}$$

In terms of real amplitude and phase, the first order solution is

$$u_0 = a(T_1, T_2) \cos(\omega_0 t + \beta(T_1, T_2)) \tag{47}$$

Applying the initial conditions yield

$$a(0)=a_0, \qquad \beta(0)=0 \tag{48}$$

Equation (45) is substituted into (43) and secular terms are eliminated

$$2i\omega_0 D_1 A + 3A^2 \overline{A} = 0 \tag{49}$$

The polar form is substituted to above with the following results

$$a = a_0(T_2), \qquad \beta = \frac{3}{8\omega_0} a_0^2 T_1 + \beta_0(T_2) \qquad \beta_0(0)=0 \tag{50}$$

The solution at order ε is

$$u_1 = B e^{i\omega_0 T_0} + \frac{1}{8\omega_0^2} A^3 e^{3i\omega_0 T_0} + cc = b\cos(\omega_0 T_0 + \gamma) + \frac{1}{32\omega_0^2} a^3 \cos(3\omega_0 T_0 + 3\beta) \tag{51}$$

where

$$B = \frac{1}{2} b e^{i\gamma} \tag{52}$$

The initial conditions at this order imply

$$b(0) = -\frac{1}{32\omega_0^2} a_0^3, \quad \gamma(0) = \beta_0(0) = 0 \tag{53}$$

At the last order, equations (51) and (45) are inserted into (44) and secular terms are eliminated

$$2i\omega_0 D_1 B + D_1^2 A + 2i\omega_0 D_2 A + 3A^2\overline{B} + 6A\overline{A}B + \frac{3}{8\omega_0^2} A^3\overline{A}^2 = 0 \tag{54}$$

If (46), (50), (52) and (53) are used above, one finally has

$$a = a_0, \quad b = -\frac{1}{32\omega_0^2} a_0^3, \quad \beta = \gamma = \frac{3}{8\omega_0} a_0^2 T_1 - \frac{21}{256\omega_0^3} a_0^4 T_2 \tag{55}$$

The final solution is

$$u = a_0 \cos(\omega t) + \frac{\varepsilon a_0^3}{32\omega_0^2}\left[\cos(3\omega t) - \cos(\omega t)\right] + O(\varepsilon^2) \tag{56}$$

where

$$\omega = \omega_0 + \varepsilon \frac{3}{8\omega_0} a_0^2 - \varepsilon^2 \frac{21}{256\omega_0^3} a_0^4 \tag{57}$$

Invoking the well known perturbation criteria that correction terms are much smaller than the leading terms for valid solutions requires

$$\frac{\varepsilon a_0^2}{32\omega_0^2} << 1 \tag{58}$$

4.2. Multiple Scales Lindstedt Poincare (MSLP)Method

First the time transformation is applied to (37)

$$\omega^2 u'' + \omega_0^2 u + \varepsilon u^3 = 0 \tag{59}$$

where prime is derivative with respect to time variable $\tau=\omega t$. Fast and slow time scales are

$$T_0 = \tau, \qquad T_1 = \varepsilon\tau, \qquad T_2 = \varepsilon^2\tau \tag{60}$$

Using

$$\frac{d^2}{d\tau^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) + \dots \tag{61}$$

and substituting the expansions

$$u = u_0(T_0, T_1, T_2) + \varepsilon u_1(T_0, T_1, T_2) + \varepsilon^2 u_2(T_0, T_1, T_2) + \dots \tag{62}$$

$$\omega_0^2 = \omega^2 - \varepsilon\omega_1 - \varepsilon^2\omega_2 \tag{63}$$

into (59) and initial conditions (38) yield after separation

$$O(1):\ \omega^2 D_0^2 u_0 + \omega^2 u_0 = 0 \qquad u_0(0) = a_0, \quad D_0 u_0(0) = 0 \tag{64}$$

$$O(\varepsilon):\ \omega^2 D_0^2 u_1 + \omega^2 u_1 = -2\omega^2 D_0 D_1 u_0 + \omega_1 u_0 - u_0^3 \quad u_1(0) = 0, (D_0 u_1 + D_1 u_0)(0) = 0 \tag{65}$$

$$O(\varepsilon^2):\ \omega^2 D_0^2 u_2 + \omega^2 u_2 = -2\omega^2 D_0 D_1 u_1 - \omega^2 (D_1^2 + 2D_0 D_2) u_0 + \omega_1 u_1 + \omega_2 u_0 - 3u_0^2 u_1 \tag{66}$$

The solution at the first order is

$$u_0 = Ae^{iT_0} + cc = a\cos(T_0 + \beta) \tag{67}$$

Applying the initial conditions yield

$$a(0) = a_0, \qquad \beta(0) = 0 \tag{68}$$

Equation (67) is substituted into (65) and secular terms are eliminated

$$-2i\omega^2 D_1 A + \omega_1 A - 3A^2\overline{A} = 0 \tag{69}$$

$D_1A=0$ is selected, which implies $a=a(T_2)$, $\beta=\beta(T_2)$ and ω_1 is solved

$$\omega_1 = 3A\overline{A} = \frac{3}{4}a^2 \tag{70}$$

which is suitable because ω_1 is real. The solution at order ε is

$$u_1 = Be^{iT_0} + \frac{1}{8\omega^2}A^3e^{3iT_0} + cc = b\cos(T_0+\gamma) + \frac{1}{32\omega^2}a^3\cos(3T_0+3\beta) \tag{71}$$

The initial conditions at this order imply

$$b(0) = -\frac{1}{32\omega^2}a_0^3, \quad \gamma(0) = 0 \tag{72}$$

At the last order, equations (71) and (67) are inserted into (66) and secular terms are eliminated

$$-2i\omega^2 D_1 B - 2i\omega^2 D_2 A + \omega_1 B + \omega_2 A - 3A^2\overline{B} - 6A\overline{A}B - \frac{3}{8\omega^2}A^3\overline{A}^2 = 0 \tag{73}$$

After algebraic manipulations, equation (73) yields

$$a = a_0, \quad b = -\frac{1}{32\omega^2}a_0^3, \quad \beta = \gamma = 0, \quad \omega_2 = -\frac{3}{128\omega^2}a_0^4 \tag{74}$$

The frequency is

$$\omega^2 = \omega_0^2 + \varepsilon\frac{3}{4}a_0^2 - \varepsilon^2\frac{3}{128\omega^2}a_0^4 \tag{75}$$

ω^2 appear both at the left and right hand sides. Frequency is solved

$$\omega = \frac{1}{4}\sqrt{8\omega_0^2 + 6\varepsilon a_0^2 + \sqrt{64\omega_0^2 + 96\varepsilon\omega_0^2 a_0^2 + 30\varepsilon^2 a_0^4}} \tag{76}$$

which is exactly the same frequency given by Hu and Xiong [1] obtained by modified Lindstedt Poincare method. The final solution in terms of this frequency is

$$u = a_0\cos(\omega t) + \frac{\varepsilon a_0^3}{32\omega^2}\left[\cos(3\omega t) - \cos(\omega t)\right] + O(\varepsilon^2) \tag{77}$$

For valid solutions, the criterion is

$$\frac{\varepsilon a_0^2}{32\omega^2} << 1 \tag{78}$$

4.3. Comparisons with the Numerical Solutions

Instead of criterion (58) in the classical method, convergence criterion is given in (78) in the new method. The major difference is that while ε tends to infinity, criterion (58) would not be valid whereas (78) is valid for arbitrarily large ε. To test, one takes the limit

$$\lim_{\varepsilon\to\infty}\frac{\varepsilon a_0^2}{32\omega^2} = \frac{\varepsilon a_0^2}{32\frac{1}{16}(8\omega_0^2 + 6\varepsilon a_0^2 + \sqrt{64\omega_0^2 + 96\varepsilon\omega_0^2 a_0^2 + 30\varepsilon^2 a_0^4})} \cong 0.044 << 1 \tag{79}$$

Therefore, the correction term in solution (77) is always a small quantity compared to the leading term which assures convergence. This cannot be said for solution (56). ω_0 being a constant in the denominator makes the correction term much larger than the leading term for large ε.

To verify the results, time histories of both solutions are contrasted with the numerical solutions obtained by directly integrating the original Duffing equation. In all simulations ω_0=1 and a_0=1 are selected. In Figure 1, results are compared for ε=1. Although numerical and MSLP solutions are indistinguishable, MS solution is in reasonable agreement. This is because the convergence criterion (58)

$$\frac{\varepsilon a_0^2}{32\omega_0^2} = \frac{1}{32} << 1 \tag{80}$$

is still valid. In Figure 2, ε=10 and criterion (58) ceases to be valid and a qualitative and quantitative difference between the results can readily be seen. Note that numerical and MSLP solutions are still indistinguishable. Finally, in Figure 3, ε=100 is selected. For this strongly nonlinear case, MSLP and numerical solutions have an excellent agreement, while MS solutions are not realistic at all.

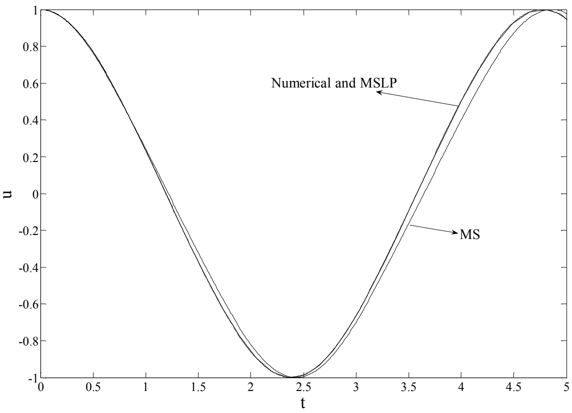

Figure 1- Comparison of approximate and numerical solutions for duffing equation for $\varepsilon = 1$ $(a_0 = 1,\ \omega_0 = 1)$

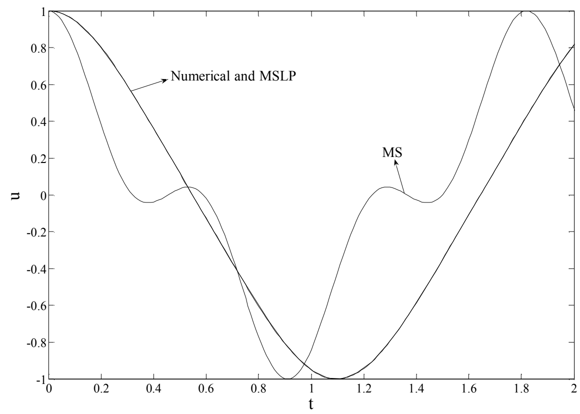

Figure 2- Comparison of approximate and numerical solutions for duffing equation for $\varepsilon = 10$ $(a_0 = 1,\ \omega_0 = 1)$

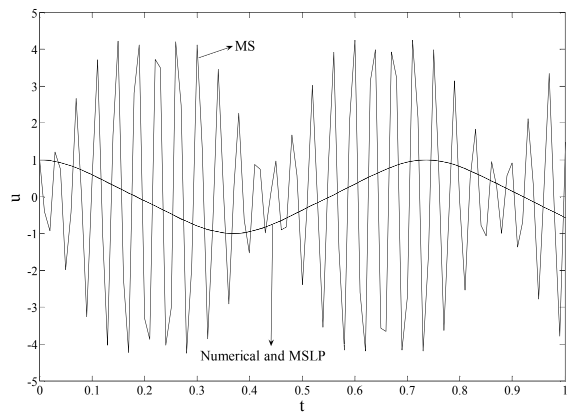

Figure 3- Comparison of approximate and numerical solutions for duffing equation for $\varepsilon = 100$ $(a_0 = 1,\ \omega_0 = 1)$

5. DAMPED CUBIC NONLINEAR SYSTEM

In the previous section, a problem with constant amplitude solution was treated. For these problems, modified Lindstedt Poincare method also produces results convergent for large ε as was presented by Hu and Xiong [1]. For variable amplitude solutions, however, Lindstedt Poincare method does not produce realistic solutions. The aim in this section is to show that the new method proposed here is effective in producing convergent solutions for strongly nonlinear variable amplitude problems. Consider the equation

$$\ddot{u}+\omega_0^2 u+2\varepsilon^2\mu\dot{u}+\varepsilon\alpha u^3=0 \tag{81}$$

with initial conditions

$$u(0)=a_0, \qquad \dot{u}_0(0)=0 \tag{82}$$

5.1. Multiple Scales (MS) Method

Similar expansions are selected as in Section 4.1. The equations at each order are

$$O(1):\ D_0^2u_0+\omega_0^2u_0=0 \qquad u_0(0)=a_0,\quad D_0u_0(0)=0 \tag{83}$$

$$O(\varepsilon):\ D_0^2u_1+\omega_0^2u_1=-2D_0D_1u_0-\alpha u_0^3 \quad u_1(0)=0,\quad (D_0u_1+D_1u_0)(0)=0 \tag{84}$$

$$O(\varepsilon^2):\ D_0^2u_2+\omega_0^2u_2=-2D_0D_1u_1-(D_1^2+2D_0D_2)u_0-2\mu D_0u_0-3\alpha u_0^2u_1 \tag{85}$$

The solution at the first order is

$$u_0=A(T_1,T_2)e^{i\omega_0T_0}+cc=a\cos(\omega_0T_0+\beta) \tag{86}$$

Applying the initial conditions yield

$$a(0)=a_0, \quad \beta(0)=0 \tag{87}$$

Solution (86) is substituted into (84) and secular terms are eliminated

$$2i\omega_0D_1A+3\alpha A^2\overline{A}=0 \tag{88}$$

The polar form is substituted to above with the following results

$$a=a_0(T_2)\,, \quad \beta=\frac{3\alpha}{8\omega_0}a_0^2T_1+\beta_1(T_2) \tag{89}$$

The solution at order ε is

$$u_1=Be^{i\omega_0T_0}+\frac{\alpha}{8\omega_0^2}A^3e^{3i\omega_0T_0}+cc=b\cos(\omega_0T_0+\gamma)+\frac{\alpha}{32\omega_0^2}a^3\cos(3\omega_0T_0+3\beta) \tag{90}$$

where

$$B=\frac{1}{2}be^{i\gamma} \tag{91}$$

The initial conditions at this order imply

$$b(0)=-\frac{\alpha}{32\omega_0^2}a_0^3, \quad \gamma(0)=0 \tag{92}$$

At the last order, equations (90) and (86) are inserted into (85) and secular terms are eliminated

$$2i\omega_0D_1B+D_1^2A+2i\omega_0D_2A+2\mu i\omega_0A+3\alpha\left(\frac{\alpha}{8\omega_0^2}A^3\overline{A}^2+A^2\overline{B}+2A\overline{A}B\right)=0 \tag{93}$$

If (88), (89), (91) and (92) are used above, one finally has

$$a = a_0 e^{-\mu T_2}, \quad b = -\frac{\alpha}{32\omega_0^2} a^3, \quad \beta = \gamma = \frac{21\alpha^2}{1024\mu\omega_0^3} a_0^4 \left(e^{-4\mu T_2} - 1\right) + \frac{3\alpha}{8\omega_0} a_0^2 e^{-2\mu T_2} T_1 \tag{94}$$

The final solution is

$$u = a\cos(\omega_0 t + \beta) + \frac{\varepsilon\alpha}{32\omega_0^2} a^3 \left[\cos(3\omega_0 t + 3\beta) - \cos(\omega_0 t + \beta)\right] + O(\varepsilon^2) \tag{95}$$

where

$$a = a_0 e^{-\varepsilon^2 \mu t}, \quad \beta = \frac{21\alpha^2}{1024\mu\omega_0^3} a_0^4 \left(e^{-4\varepsilon^2 \mu t} - 1\right) + \varepsilon \frac{3\alpha}{8\omega_0} a_0^2 e^{-2\varepsilon^2 \mu t} t \tag{96}$$

The convergence criterion is

$$\frac{\varepsilon\alpha a^2}{32\omega_0^2} << 1 \tag{97}$$

5.2. Multiple Scales Lindstedt Poincare (MSLP) Method

First the time transformation is applied to (81)

$$\omega^2 u'' + \omega_0^2 u + 2\varepsilon^2 \mu\omega u' + \varepsilon\alpha u^3 = 0 \tag{98}$$

with time variable $\tau=\omega t$. Selecting the fast and slow time variables and expansions as in Section 4.2, the equations at each order are

$$O(1): \omega^2 D_0^2 u_0 + \omega^2 u_0 = 0 \qquad u_0(0) = a_0, \quad D_0 u_0(0) = 0 \tag{99}$$

$$O(\varepsilon): \omega^2 D_0^2 u_1 + \omega^2 u_1 = -2\omega^2 D_0 D_1 u_0 + \omega_1 u_0 - \alpha u_0^3$$

$$u_1(0) = 0, \quad (D_0 u_1 + D_1 u_0)(0) = 0 \tag{100}$$

$$O(\varepsilon^2): \begin{aligned} \omega^2 D_0^2 u_2 + \omega^2 u_2 &= -2\omega^2 D_0 D_1 u_1 - \omega^2 (D_1^2 + 2D_0 D_2) u_0 + \omega_1 u_1 + \omega_2 u_0 \\ &\quad - 2\mu\omega D_0 u_0 - 3\alpha u_0^2 u_1 \end{aligned} \tag{101}$$

The solution at the first order is

$$u_0 = A e^{iT_0} + cc = a\cos(T_0 + \beta) \tag{102}$$

Applying the initial conditions yield

$$a(0)=a_0, \quad \beta(0)=0 \tag{103}$$

The first order solution is substituted into (100) and secular terms are eliminated

$$-2i\omega^2 D_1 A + \omega_1 A - 3\alpha A^2 \overline{A} = 0 \tag{104}$$

$D_1A=0$ is selected, which implies $a=a(T_2)$, $\beta=\beta(T_2)$ and ω_1 is solved

$$\omega_1 = 3\alpha A\overline{A} = \frac{3}{4}\alpha a^2 \tag{105}$$

which is suitable because ω_1 is real. The solution at order ε is

$$u_1 = B e^{iT_0} + \frac{\alpha}{8\omega^2} A^3 e^{3iT_0} + cc = b\cos(T_0 + \gamma) + \frac{\alpha}{32\omega^2} a^3 \cos(3T_0 + 3\beta) \tag{106}$$

The initial conditions at this order imply

$$b(0) = -\frac{\alpha}{32\omega^2} a_0^3, \quad \gamma(0) = 0 \tag{107}$$

At the last order, equations (106) and (102) are inserted into (101) and secular terms are eliminated

$$-2i\omega^2 D_1 B - 2i\omega^2 D_2 A + \omega_1 B + \omega_2 A - 2\mu i\omega A - 3\alpha\left(\frac{\alpha}{8\omega^2}A^3\overline{A}^2 + A^2\overline{B} + 2A\overline{A}B\right) = 0 \tag{108}$$

If $D_2A=0$, ω_2 will be a complex number which is not a suitable option. Therefore

$$\omega_2 = 0 \tag{109}$$

is selected. After algebraic manipulations, equation (108) yields

$$a = a_0 e^{-\frac{\mu}{\omega}T_2}, \quad b = -\frac{\alpha}{32\omega^2}a^3, \quad \beta = \gamma = \frac{3\alpha^2 a_0^4}{1024\omega^3\mu}\left(e^{-\frac{4\mu}{\omega}T_2} - 1\right) \tag{110}$$

The final solution in terms of original variables is

$$u = a\cos(\omega t + \beta) + \frac{\varepsilon\alpha}{32\omega^2}a^3\left[\cos(3\omega t + 3\beta) - \cos(\omega t + \beta)\right] + O(\varepsilon^2) \tag{111}$$

where

$$a = a_0 e^{-\varepsilon^2\mu t}, \quad \beta = \frac{3\alpha^2 a_0^4}{1024\omega^3\mu}\left(e^{-4\varepsilon^2\mu t} - 1\right), \quad \omega^2 = \omega_0^2 + \varepsilon\frac{3\alpha}{4}a_0^2 e^{-2\varepsilon^2\mu t} \tag{112}$$

For valid solutions, the criterion is

$$\frac{\varepsilon\alpha}{32\omega^2}a^2 << 1 \tag{113}$$

5.3. Comparisons with the Numerical Solutions

In criteria (97) and (113), amplitude is a function of time. For sufficiently large t, convergence is assured because the amplitude decays in time. Letting ε tending to infinity would not spoil both convergence criteria also since increasing ε increases the amplitude decay rate. Instead, for strongly nonlinear systems, one should inspect the system behavior for large α. Adjusting α, nonlinearity can be increased without increasing the damping in the system. For large α and small times, criterion (97) can not be satisfied.

Instead of criterion (97) in the classical method, convergence criterion is given in (113) in the new method. In case of large α, since ω is a function of α, it serves as a stabilizing factor in the denominator. The criterion is largest for t=0 which means $a=a_0$. For this choice

$$\lim_{\alpha\to\infty}\frac{\varepsilon\alpha}{32\omega^2}a_0^2 = \frac{\varepsilon\alpha a_0^2}{32\left(\omega_0^2 + \varepsilon\frac{3\alpha}{4}a_0^2\right)} = \frac{1}{24} << 1 \tag{114}$$

Therefore, for large α, agreement with numerical solutions is expected in MSLP method. Time histories of both solutions are contrasted with the numerical solutions obtained by directly integrating the original damped equation with cubic nonlinearity in Figures 4-7. In all simulations $\omega_0=1$, $a_0=1$, $\mu=1$ and $\varepsilon=0.1$ are selected. In Figure 4, results are compared for $\alpha=1$. Since criteria (97) and (113) are both satisfied, both approximate solutions are indistinguishable from the numerical ones. With increasing α in Figure 5 (i.e. $\alpha=10$), separation starts, MSLP being in better agreement with the numerical solutions. For α=50 in Figure 6, the qualitative behavior can also be distinguished for numerical and MS solutions but a good agreement can be seen with MSLP and numerical results. Same qualitative and quantitative behavior can be observed for α=100 in Figure 7.

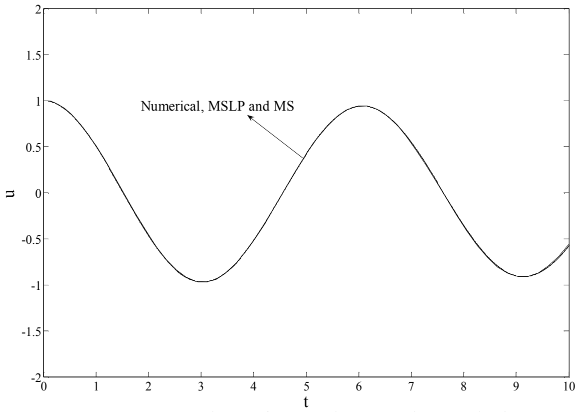

Figure 4- Comparison of approximate and numerical Solutions for damped cubic nonlinear equation for $\alpha = 1(\varepsilon = 0.1, a_0 = 1, \omega_0 = 1, \mu = 1)$

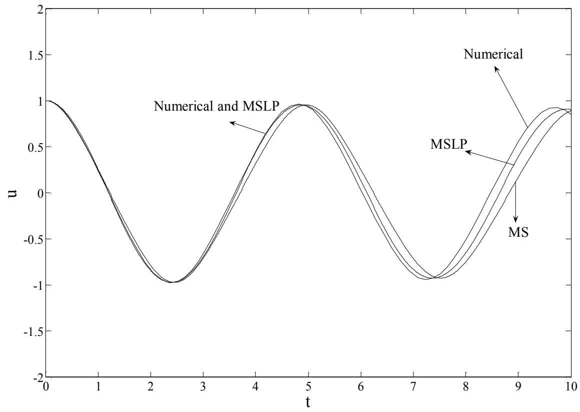

Figure 5- Comparison of approximate and numerical solutions for damped cubic nonlinear equation for $\alpha = 10(\varepsilon = 0.1, a_0 = 1, \omega_0 = 1, \mu = 1)$

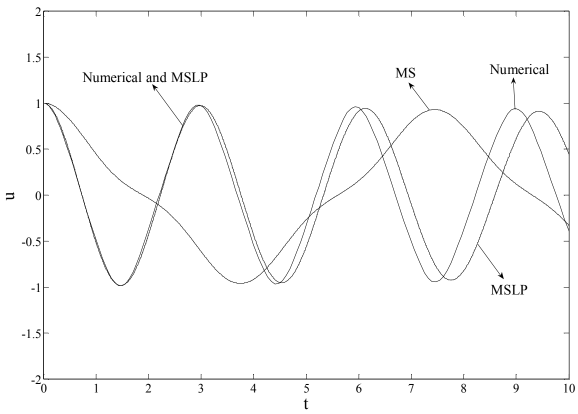

Figure 6- Comparison of approximate and numerical solutions for damped cubic nonlinear equation for $\alpha = 50(\varepsilon = 0.1, a_0 = 1, \omega_0 = 1, \mu = 1)$

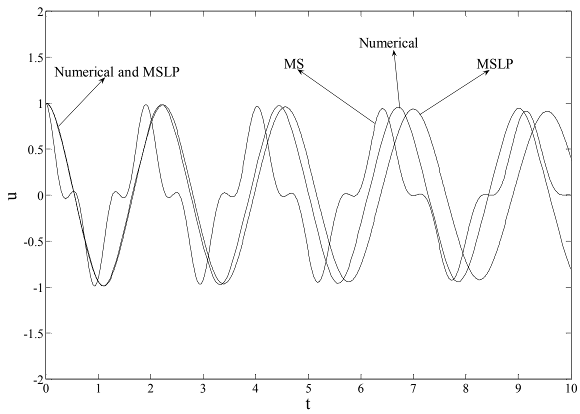

Figure 7- Comparison of approximate and numerical solutions for damped cubic nonlinear equation for $\alpha = 100(\varepsilon = 0.1, a_0 = 1, \omega_0 = 1, \mu = 1)$

6. CONCLUDING REMARKS

A new perturbation technique combining the Multiple Scales and Lindstedt Poincare method is proposed for the first time. Instead of the transformation frequency, the natural frequency is expanded in a perturbation series. This choice leads to solutions with good convergence properties in Lindstedt Poincare method [1]. The new method is applied to three models: 1) Damped linear oscillator, 2) Duffing Equation, 3) Damped cubic nonlinear equation. In the first case, even the exact solution is retrieved by the method. In the second and third cases, the new method produced solutions with good agreement with the numerical solutions for strongly nonlinear problems. The classical Multiple Scales method however failed to produce close solutions with the numerical ones for strongly nonlinear problems.

A further study would be to apply this new technique to partial differential equations. The nonlinearities arising in partial differential equations are classified using a suitable operator notation and general solution algorithms were developed for the models previously [15-17].

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