Mathematical and Computational Applications, Vol. 13, No. 3, pp. 175-182, 2008.

NETWORK OF TANDEM AND BI-TANDEM QUEUEING PROCESS WITH RENEGING AND JOCKEYING

A.B. Chandramouli
Department of P.G. Studies and Research in Mathematics,
M.S. College, Saharanpur 247001, India

Abstract - The steady state behaviour of a queueing model where two bi-tandem channels are linked in tandem with a common channel has been studied using the concept of reneging and jockeying.
Keywords - Reneging, Jockeying, Bi-tandem channels.

1. INTRODUCTION

O, Brien [1], Jackson [2], Suzuki [3], Maggu [4] etc. studied the model of tandem (series) queues. In 1957 the concept of reneging in queueing system was introduced by Barrer [5]. Further this concept has been discussed in different ways by many researchers as Haight [6] Blackburn [7] etc. in different models.

The concept of jockeying was first discussed by Glazer. Then using this concept in different models many researchers discussed as Keonigsberg [8], Disney and Mitchell [9] etc. The network of queues was studied by Finch [10], Kelly [11], Melamed [12] and recently Chandramouli [13] discussed a model in which two bi-tandem channels are linked with a common channel taking the concept of non-linear service growth rate. In the present paper the concept of reneging and jockeying has been introduced in this model when the service rates do not depend upon the queue length. The steady behaviour of this model has been discussed. The practical situation corresponding to this model can be realized in banks or in a publishing company etc. For example, consider a publishing company which has three types of machines say S_1, S_2 and S_3. Let S_1 print the matter in red ink and S_2 in blue ink. and S_3 denote the binding machine. We suppose that the arrivals (matters for printing) are printed in two colours (red or blue) and finally go to the binding process at S_3. It has also been assumed that the binding machine S_3 undertakes outside printed matter for binding. Now in this situation, the reneging and jockeying at the arrivals may also be observed.

2. FORMULATION AND SOLUTION

Let S_1, S_2 and S_3 denote the three service channels in which it is supposed that S_1 □ S_2, that is S_1 and S_2 are in bi-tandem and $S_1 \to S_3$ or $S_2 \to S_3$, that is, each is further linked in tandem with S_3. An arriving unit for service at either S_1 or S_2 may follow one of the following routes for terminal services :

$$S_1 \to S_2 \to S_3 \text{ or } S_2 \to S_1 \to S_3.$$

This unit which arrives directly at S_3 departs from the system after servicing at S_3. Let Q_1, Q_2 and Q_3 be waiting line formed before S_1, S_2 and S_3. If they are busy. It has been supposed that an arriving unit after intolerable waiting time in the queue Q_1 or Q_2 may renege (leave) at S_1 or S_2 without service. Also it has been assumed that units may jockey (move) from $Q_1 \to Q_2$ or from $Q_2 \to Q_1$ for personal economic gains.

Let λ_i denote the Possion mean rate of arrivals at Q_i before S_i (i = 1, 2, 3), we assume that the input source is infinite. Let μ_i denote the Possion mean departure rates at S_i. Also let b_r denote the constant rate of reneging from queues Q_r (r = 1, 2). Further, let J_{ir} (i $\neq$ r, i, r = 1, 2) denote the constant rates of jockeying from $Q_i \to Q_r$. Let p_{12} and p_{13} denote the probabilities that a unit after service at S_1 departs to join the respective queues Q_2 and Q_3. Again let p_{21} and p_{23} denote the probabilities that a unit after service at S_2 join the respective queues Q_1 and Q_3, where $p_{ij} \geq 0$ (i $\neq$ j, i = 1, 2, j = 1, 2, 3) and $p_{12} + p_{13} = 1$, $p_{21} + p_{23} = 1$. Let P (k, m, n) denote the steady-state probability that there are waiting k units in Q_1, m units in Q_2 and n units in Q_3. Each queue includes service also and k, m, n $\leq$ 0.

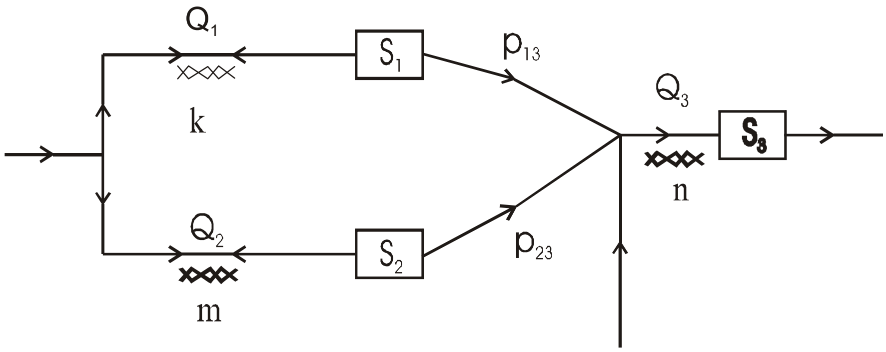

Figure :- Queue model with Reneging and Jockeying

For steady state situation the following difference equation exists for k, m, n > 0

$$\begin{aligned}
&(\lambda_1+\lambda_2+\lambda_3+\mu_1+\mu_2+\mu_3+b_1+b_2+J_{12}+J_{21})P(k,m,n)=\lambda_1\, P(k-1,m,n) \\
&+\lambda_2\, P(k,m-1,n)+\lambda_3\, P(k,m,n-1)+b_1\,(k+1,m,n)+b_2\, P(k,m+1,n) \\
&+J_{12}\, P(k+1,m-1,n)+J_{21}\, P(k-1,m+1,n)+\mu_1\, p_{12}\, P(k+1,m-1,n)+\mu_1 \\
&p_{13}\, P(k+1,m,n-1)+\mu_2\, p_{21}\,(k-1,m+1,n)+\mu_2 p_{23}\, P(k,m+1,n-1) \\
&+\mu_3\, P(k,m,n+1) \qquad for\, k,m,n>0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.1)
\end{aligned}$$

If one of k, m, n is zero and other two are non-zero e.g. k = 0, m, n > 0 then in this case for P (o, m, n) $b_1 = 0 = \mu_1 = J_{12}$ and negative of P (k, m, n) is zero. Substituting these value is (1) we get the equation. Similarly for m = 0, n > 0 and also for n = 0, k m > 0. We get three equations in this manner. Again, if two of k, m, n are zero and other one is non-zero e.g. k = 0 = m, n > 0 then in this case for P (o, o, n), $b_1 = 0 = b_2 = J_{12} = J_{21} = \mu_1 = \mu_2$ and negative of P (k, m, n) is zero. Substituting these values in (1) we get the equation. Similarly, for k = 0 = n, m > 0 and also m = 0 = n, k > 0. We get three equations in this manner also.

Again, if k, m, n are also zero. Then in this case for P (0, 0, 0), $b_1 = 0 = b_2 = b_3 = \mu_1 = \mu_2 = J_{12} = J_{21}$ and negative of P (k, m, n) in zero and substitute these values in (1) we get one equation.

Hence, the above set of eight difference equations govern the model in steady state situation.

To solve the above set of difference equations we use the generating function technique and similar steps as Chandramouli [14] has taken in his paper. Now, define the generating function as

$$F(x,y,z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} P(k,m,n)\, x^k\, y^m\, z^n \;\; Where|\,x\,| \le 1 \;\; and |\,z\,| \le 1 \text{ ....(2.2)}$$

Using the following partial generating functions and simplification

$$F_{m,n}(x) = \sum_{k=0}^{\infty} P\big(k,m,n\big)\, x^k$$

$$G_{k,n}(y) = \sum_{m=0}^{\infty} P\big(k,m,n\big)\, y^m \qquad \text{............(2.3)}$$

$$I_n(x,y) = \sum_{m=0}^{\infty} F_{m,n}(x)\, y^m$$

$$A_k(y,z) = \sum_{n=0}^{\infty} G_{k,n}(y)\, z^n \qquad \text{................(2.4)}$$

$$B_m(x,z) = \sum_{n=0}^{\infty} F_{m,n}(x)\, z^n$$

We get the following equation : F (x,y,z) =

$$\left[\left\{\mu_1\left(1-\frac{z}{x}p_{13}-\frac{y}{x}p_{12}\right) + J_{12}\left(1-\frac{y}{x}\right)\right\} A_0(y,z) + \left\{\mu_2\left(1-\frac{z}{y}\,p_{23}-\frac{x}{y}p_{21}\right) + b_2\left(1-\frac{1}{y}\right) + J_{21}\left(1-\frac{x}{y}\right)\right\} B_0(x,z) + \mu_3\left(1-\frac{1}{z}\right) I_0(x,y)\right]$$

$$\left[\lambda_1(1-x) + \lambda_2(1-y) + \lambda_3(1-z) + b_1\left(1-\frac{1}{x}\right) + b_2\left(1-\frac{1}{y}\right) + J_{12}\left(1-\frac{y}{x}\right) + J_{21}\left(1-\frac{x}{y}\right)\right] \Big/$$

$$+\,\mu_1\left(1-\frac{z}{x}\,p_{13}-\frac{y}{z}p_{12}\right) + \mu_2\left(1-\frac{z}{y}\,p_{23}-\frac{x}{y}p_{21}\right) + \mu_3\left(1-\frac{1}{z}\right) \qquad \text{.....................(2.5)}$$

Using L' Hospital's rule for indeterminate form %, and using F (x, 1, 1) = 1 as $x \to 1$ and similarly other also we have the following set of equations :

$$1 = \frac{\mu_1\, p_{13}\, A_0(1,1) + \mu_2\, p_{23}\, B_0(1,1) + \mu_3\, I_0(1,1)}{\lambda_3 + \mu_1\, p_{13} + \mu_2\, p_{23} - \mu_3} \qquad \text{...................(2.6)}$$

$$1=\frac{(\mu_1+b_1+J_{12})\ A_0(1,1)-(\mu_2\ p_{21}+J_{21})\,B_0(1,1)}{-\lambda_1+b_1+J_{12}-J_{21}\ +\ \mu_1-\mu_2 p_{21}} \qquad \text{..................(2.7)}$$

$$1=\frac{-(\mu_1\ p_{12}+J_{12})\ A_0(1,1)-(\mu_2+b_2+J_{21})\,B_0(1,1)}{-\lambda_2+b_2-J_{12}+J_{21}\ -\ \mu_1\ p_{12}+\mu_2} \qquad \text{..................(2.8)}$$

In matrix notations, the equations (2.6), (2.7) and (2.8) can be written as

$$AX = B; \qquad \text{…………….(2.9)}$$

$$\text{where } A = \begin{bmatrix} \mu_3 & -\mu_1\ p_{13} & -\mu_2\ p_{23} \\ 0 & \mu_1+b_1+J_{12} & -\mu_2\ p_{21}-J_{12} \\ 0 & -\mu_1\ p_{12}-J_{12} & \mu_2+b_2+J_{21} \end{bmatrix}$$

$$X = \begin{bmatrix} I_0(1,1) \\ A_0(1,1) \\ B_0\ (1,1) \end{bmatrix} \ and\ B = \begin{bmatrix} -\lambda_3-\mu_1\ p_{13}-\mu_2\ p_{23}+\mu_3 \\ -\lambda_1+b_1+J_{12}-J_{21}-\mu_2 p_{21}+\mu_1 \\ -\lambda_2+b_2+J_{21}-J_{12}-\mu_1 p_{12}+\mu_2 \end{bmatrix}$$

The augmented matrix [A : B] after the elementary row transformation,

$$R_3 \to R_3 + \left[\frac{\mu_1 p_{12}+J_{12}}{\mu_1+b_1+J_{12}}\right] R_2 \ \text{ becomes}$$

$$\left[A:B\right] \approx \begin{bmatrix} \mu_3 & -\mu_1\ p_{13} & -\mu_2\ p_{23} & : & -\lambda_3-\mu_1\ p_{13}-\mu_2\ p_{23}+\mu_3 \\ 0 & \mu_1+b_1+J_{12} & -\mu_2\ p_{21}-J_{21} & : & -\lambda_1+\mu_1-\mu_2 p_{21}+J_{12}-J_{21}+b_1 \\ 0 & 0 & M & : & M-N \end{bmatrix} \qquad \text{......(2.10)}$$

Where M

$$=\mu_1\mu_2\left(1-p_{12}\ p_{21}\right)+\mu_2\ J_{12}p_{23}+\mu_1\ J_{21}\ p_{13}+\mu_1 b_2+\mu_2 b_1+b_2 J_{12}+b_1 J_{21}+b_1 b_2$$

And $N=\lambda_1\left(\mu_1 p_{12}+J_{12}\right)+\lambda_2\left(\mu_1+J_{12}+b_1\right)$

By matrix algebra, the system of equation (2.9) are consistent. Thus, the value of three unknowns $B_0(1,1)$, $A_0(1,1)$ and $I_0(1,1)$, after simplification are as follows :

$$B_0(1,1) = 1-\frac{\lambda_1\left(\mu_1 p_{12}+J_{21}\right)+\lambda_2\ \left(\mu_1+b_1+J_{21}\right)}{M},$$

$$A_0(1,1) = 1-\frac{\lambda_1\left(\mu_2+b_2+J_{21}\right)+\lambda_2\left(\mu_2\ p_{21}+J_{21}\right)}{M},$$

$$and\ I_0(1,1)=1-\big[\lambda_1\{\mu_1\mu_2\left(p_{21}+p_{23}p_{12}\right)+\mu_1\left(J_{21}+b_2\right)p_{13}+\mu_2 J_{12}p_{23}\}+\lambda_2\{\mu_1\mu_2\left(p_{23}+p_{13}p_{21}\right)+\mu_1 J_{21}p_{13}+\mu_2\left(J_{12}+b_1\right)p_{23}\}+\lambda_3 \\ \{\mu_1\mu_2\left(1-p_{12}p_{21}\right)+\mu_1 J_{21}p_{13}+\mu_2 J_{12}p_{23}+b_1\mu_2+\mu_1 b_2+b_2 J_{12}+b_1 J_{21}+b_1 b_2\}/\mu_3 M\big]$$

$$\text{……………….(2.11)}$$

Now, the steady state solution of M/M/1, when there are h persons (including service) in the queue is given by :

$$p_h = p_0 (1 - p_0)^h$$
$$p_0 = (1 - \rho) \ and \ \rho = \frac{\lambda}{\mu} < 1 \text{with h} \geq 0 \qquad .....(2.12)$$

Now, if p_k, q_m, r_n denote the probabilities that there are k units in Q_1, m units in Q_2 and n units in Q_3 and since in our model all the probability distribution are mutually independent, therefore, the joint probability that there are k units in Q_1, m units in Q_2 and n units in Q_3, including service, if any, in the system is given by :

$$P(k,m,n) = p_k q_m r_n \qquad ..... (2.13)$$

Hence, by virtue of (12) we have :

$$P_k = p_0(1-p_0)^k, p_k \text{ converge if } 1 - p_0 < 1 \qquad ......(2.14)$$

and similarly q_m and r_n also.

Now, using (2.13) and (2.14), we obtain

$$P(k,m,n) = p_0 q_0 r_0 (1-p_0)^k (1-q_0)^m (1-r_0)^n \qquad .....(2.15)$$

Where $p_0, q_0, r_0 > 0$

Now, $A_0(1,1)$ denotes the marginal probability generating function (m.p.g.f.) of 0 units in Q_1 when Q_2 and Q_3 have been eliminated from the consideration in the system. Similarly $B_0(1,1)$ and $I_0(1,1)$ for consideration in the system. Therefore, we can easily see that :

$$A_0(1,1) = p_0, B_0(1,1) = q_0 \text{ and } I_0(1,1) = r_0 \qquad ......(2.16)$$

Therefore, using (2.11) and (2.16) the steady state solution in (15) can be written as

$$P(k,m,n) = P(0,0,0)\rho_1^k \rho_2^m \rho_3^n \qquad .....(2.17)$$

$$Where \ \ \rho_1 = \frac{\lambda_1(\mu_2 + b_2 + J_{21}) + \lambda_2(\mu_2 p_{21} + J_{21})}{M}$$

$$\rho_2 = \frac{\mu_1 p_{12}(\mu_{12} + J_{12}) + \lambda_2(\mu_1 + b_1 + J_{12})}{M}$$

$$\rho_3 = [\lambda_1\{\mu_1\mu_2(p_{13} + p_{12}p_{23}) + \mu_1(J_{21} + b_2)p_{13} + \mu_2 J_{12} p_{23}\} +$$
$$\lambda_2\{\mu_1\mu_2(p_{23} + p_{21}p_{13}) + \mu_2 J_{21} p_{13} + \mu_2 p_{23}(J_{12} + b_1)\} + \lambda_3 M] / \mu_3 M$$

where $M = \mu_1\mu_2(1 - p_{12}p_{21}) + \mu_2 J_{12} p_{23} + \mu_1 J_{21} p_{13} + b_2(\mu_1 + J_{12} + b_1) + b_1(\mu_2 + J_{21}),$

with $P(0,0,0) = (1-\rho_1)(1-\rho_2)(1-\rho_3)$ ....(2.18)

Under the assumption that $p_0, q_0, r_0 > 0$ otherwise (2.17) may diverge to ∞.

The marginal probability P (k . .) of k units are in waiting and in service in Q_1 can be obtained by using the value of P (k, m, n) from (17) in the formula :

$$P(k..) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P(k,m,n)$$

$$= P(0,0,0) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \rho_1^k \rho_2^m \rho_3^n$$

$$= \rho_1^k (1-\rho_1), \ for\, k = 0,1,2 \qquad \text{.........(2.19)}$$

Similarly, the marginal probabilities P (. m .) of m units in Q_2 and P (. . n) of n units in Q_3 are

$$P(.m.) = \rho_2^m (1-\rho_2), \ for \ m = 0,1,2... \text{ and}$$

$$P(..n) = \rho_3^n (1-\rho_3), \ for \ m = 0,1,2... \ Q_1, Q_2 \ Q_3$$

3. SOME CHARACTERISTICS OF THE SYSTEM

3.1. Mean queue length : It is denoted by L and is equal to the sum of marginal queue lengths of the queues Q_1, Q_2 and Q_3 which are denoted by L_1, L_2 and L_3 respectively.

Hence, $L = L_1 + L_2 + L_3$ ....(3.1)

Now, the marginal queue length L_1 in the queue Q_1 is obtained by the formula

$$L_1 = \sum_{k=0}^{\infty} kP\,(k..)$$

Using (2.19) for P (k . .) in the above relation and simplifying, we have

$$L_1 = \frac{\rho_1}{1-\rho_1} \qquad \text{.......(3.2)}$$

Similarly, the marginal mean queue lengths L_2 and L_3 for the queues, Q_2 and Q_3 respectively are :

$$L_2 = \frac{\rho_2}{1-\rho_2} \qquad \text{.......(3.3)}$$

$$\text{And } L_3 = \frac{\rho_3}{1-\rho_3} \qquad \text{.......(3.4)}$$

Now, using (21), (22) and (23) in (20), we have

$$L = \frac{\rho_1}{1-\rho_1} + \frac{\rho_2}{1-\rho_2} + \frac{\rho_3}{1-\rho_3} \qquad \text{......(3.5)}$$

Where P_1, P_2 and P_3 are defined in (17),

3.2. Fluctuation in the queue length : Fluctuation is denoted by Var θ for $\theta = k + m + n$ and is evaluated as

$$Var\,\theta = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\theta - L)^2 \ \rho(k,m,n)$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (k + m + n)^2 \ \rho(k,m,n) - L^2 \qquad \text{........(3.6)}$$

using ρ (k, m, n), from (2.17) in (3.6) and the value of L from (3.5), we have

$$Var\,\theta = P(0,0,0)\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(k+m+n)^2\,\rho_1^k\rho_2^m\rho_3^n - L^2$$

$$= \sum_{i=1}^{3}\frac{\rho_i(1-\rho_i)}{(1-\rho_i)^2} - \left[\sum_{i=1}^{3}\frac{\rho_i}{1-\rho_i}\right]^2 \qquad \text{…….(3.7)}$$

4. PARTICULAR CASES

Case-I. If we take $\lambda_3 = 0 = b_1 = b_2 = J_{12} = J_{21} = p_{12} = p_{21}$

Then, equation (3.5) becomes $L = \dfrac{\lambda_1}{\mu_1-\lambda_1} + \dfrac{\lambda_2}{\mu_2-\lambda_2} + \dfrac{\lambda_1+\lambda_2}{\mu_3-(\lambda_1+\lambda_2)}$ …(4.1)

Which coincides with the result given by Maggu [4].

Case-II. If we consider, $\lambda_2 = 0 = \lambda_3 = b_1 = b_2 = J_{12} = J_{21} = p_{21}$ and $p_{12} = 1$ with $\mu_3 \to \infty$ in the equation (3.5),

we have :

$$L = \frac{\lambda_1}{\mu_1-\lambda_1} + \frac{\lambda_2}{\mu_2-\lambda_1} \qquad \text{….(4.2)}$$

This result coincides with the result given by Jackson [2].

Case-III. If we assume $\lambda_2 = 0 = \lambda_3 = b_1 = b_2 = J_{12} = J_{21}$ and $p_{12} = 1$ with $\mu_3 \to \infty$ in equation (2.17), we have

$$P(k,m) = \left[1-\frac{\lambda_1}{\mu_1(1-p_{21})}\right]\left[1-\frac{\lambda_1}{\mu_2(1-p_{21})}\right]\left[\frac{\lambda_1}{\mu_1(1-p_{21})}\right]^k\left[\frac{\lambda_1}{\mu_2(1-p_{21})}\right]^m \qquad \text{.(4.3)}$$

This results gives the solution of the cyclic queues with terminal feedback which was given by Finch [10].

REFERENCES

1. G.C.O' Brien, The solutions of some queueing problems, *J.Soc. Ind. App. Math* 2, 132-142, 1954.
2. R.R.P. Jackson, Queueing system with phase type service, *Operations Research Quarterly* 5, 109-120, 1954.
3. T. Suzuki, Two queues in series, *J.O.R. Soc. of Japan* 5, 149-155, 1963.
4. P.L. Maggu, On certain type of queues in series, *Statistica Netherlandica* 24, 89-97, 1970.
5. D.Y. Barrer, Queueing with impatient customer and ordered service, *Operations Research* 5, 650-656, 1957.
6. F.A. Haight, Queueing with reneging, *Metrika,* 2, 186-197, 1959.
7. J.D. Blackburn, Optimal control of a single server queue with balking and reneging, *Management Science* 11, 297-313, 1972.
8. E. Koenigsberg, On Jockeying in Queues, *Unpublished paper presented to CORS-ORSA Joint National Conference Montreala,* 1964.
9. R.L. Disney and W.E. Mitchell, A solution for queues with instantaneous Jockeying and other customer selection rule, ***Navel*** *Research Logistics Quarterly*. 17, 315-325, 1970.
10. P.D. Finch, Cyclic queues with feedback, *J. Roy. Stat. Soc. Ser-B* 21, 153-157, 1959.
11. F.P. Kelly, Networks of Queues, *Adv. App. Prob*. 8, 416-432, 1976.
12. B. Melaned, Characterizations of Poisson traffic streams in Jackson queueing networks, *Adv. App. Prob*. 11, 422-438, 1979.
13. A.B. Chandramouli, Queues in bi-tandem channels linked in tandem with a common channel with non-linear service growth rate, *Acta Ciencia Indica*, XXII (M), 197-206, 1996.
14. A.B. Chandramouli, Queues in bi-tandem channels, *Acta Ciencia Indica* XXII (M), 171-179, 1996.