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DIFFERENTIAL TRANSFORMATION METHOD FOR SOLVING DIFFERENTIAL EQUATIONS OF LANE-EMDEN TYPE

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Abstract- Using differential transformation method to solve the Lane-Emden equations as singular initial value problems is introduced in this study. Some numerical examples are presented to illustrate the efficiency and reliability of the method.

Keywords- Differential transformation method, Lane-Emden equations

1. INTRODUCTION

Singular initial value problems in the second order ordinary differential equations occur in several models of mathematical physics and astrophysics [1-3] such as the theory of stellar structure, the thermal behaviour of a spherical cloud of gas, isothermal gas spheres and theory of thermoionic currents which are modelled by means of the following Lane–Emden equation:

$$u''(x)+\frac{\alpha}{x}u'(x)+f(x,u)=g(x),\quad 0<x\le 1,\quad \alpha\ge 0, \tag{1}$$

under the following initial conditions

$$u(0)=A,\quad u'(0)=B, \tag{2}$$

where A and B are constants, $f(x,u)$ is a continuous real valued function and $g(x)\in C[0,1]$.

Eq. (2) has attracted many mathematicians. Wazwaz [4,5] has given a general study to construct exact and series solutions to Lane-Emden equations by employing the Adomian decomposition method. Russel and Shampine [6] have investigated three-point difference methods of second-order. Moreover, three-point difference methods of second-order have been also used by Chawla and Katti [7], Chawla et al. [8] and Iyengar and Jain [9]. However, Jain et al [10] derived three-point difference methods of fourth and sixth orders to solve this problem. On the other hand, El-Sayed [11] used a multi-integral method to investigate the nonlinear problem (1), and Legendre wavelets method [12] has been implemented independently to handle the initial value problem (1)-(2).

In this paper, we extend the application of the differential transformation method [13], which is based on Taylor series expansion, to construct analytical approximate solutions of the initial value problem (1)-(2). The concept of differential transformation was introduced first by Zhou [13] , and it was applied to solve linear and nonlinear initial value problems in electric circuit analysis. With this technique, it is possible to obtain highly accurate results or exact solutions for differential equations.

This paper is organized as follows: In Section 2, the differential transformation method is described. In Section 3, the method is implemented to three examples, and conclusion is given in Section 4.

2. DIFFERENTIAL TRANSFORMATION METHOD

The differential transformation of the kth derivative of function $u(x)$ is defined as follows:

$$U(k) = \frac{1}{k!}\left[\frac{d^k u(x)}{dx^k}\right]_{x=x_0}. \tag{3}$$

and the differential inverse transformation of $U(k)$ is defined as follows:

$$u(x) = \sum_{k=0}^{\infty} U(k)(x-x_0)^k. \tag{4}$$

In real applications, function $u(x)$ is expressed by a finite series and Eq.(4) can be written as

$$u(x) = \sum_{k=0}^{n} U(k)(x-x_0)^k. \tag{5}$$

Eq. (5) implies $\sum_{k=n+1}^{\infty} U(k)(x-x_0)^k$ is negligibly small. In fact, n is decided by the convergence of natural frequency in this study.

The following theorems that can be deduced from Eqs. (3) and (4) are given below [14,15]:

Theorem 1. If $u(x) = y(x) \pm z(x)$, then $U(k) = Y(k) \pm Z(k)$.

Theorem 2. If $u(x) = ay(x)$, then $U(k) = aY(k)$, where a is a constant.

Theorem 3. If $u(x) = (d^m y(x)/dx^m)$, then $U(k) = \frac{(m+k)!}{k!}Y(k+m)$.

Theorem 4. If $u(x) = y(x)z(x)$, then $U(k) = \sum_{k_1=0}^{k} Y(k_1)Z(k-k_1)$.

Theorem 5. If $u(x) = x^n$, then $U(k) = \delta(k-n), \delta(k-n) = \begin{cases} 1 & k = n, \\ 0 & k \neq n. \end{cases}$

3. NUMERICAL EXAMPLES

To demonstrate the method introduced in this study, three examples are solved here.

Example 1 We first start by considering the following Lane-Emden equation given in [12]

$$u''(x) + \frac{2}{x}u'(x) + u(x) = 6 + 12x + x^2 + x^3, \quad 0 < x \le 1, \tag{6}$$

with initial conditions

$$u(0) = 0, \quad u'(0) = 0. \tag{7}$$

By multiplying both sides of Eq. (6) by x and then taking differential transformation of both sides of the resulting equation using Theorems 1-5 , the following recurrence relation is obtained:

$$U(k+1)=\frac{1}{(k+1)(k+2)}\times \left(6\delta(k-1)+12\delta(k-2)+\delta(k-3)+\delta(k-4)-\sum_{l=0}^{k}\delta(l-1)U(k-l)\right). \tag{8}$$

By using Eqs. (3) and (7), the following transformed initial conditions at $x_0=0$ can be obtained:

$$U(0)=0, \tag{9}$$

$$U(1)=0. \tag{10}$$

Substituting Eqs. (9) and (10) at $k=1$ into Eq. (8), we have

$$U(2)=1. \tag{11}$$

Following the same recursive procedure, we find $U(k+1)=0$, $k=3,4,5,\ldots$ and listing the computation and result corresponding to $n=3$, we have

$$U(3)=1. \tag{12}$$

Using Eqs.(9)-12) and the inverse transformation rule in Eq. (5), we get the following solution:

$$u(x)=x^2+x^3. \tag{13}$$

Note that for $n>3$ one evaluates the same solution, which is the exact solution of Eq. (6) with the initial conditions in Eq. (7).

Example 2 We next consider the the following Lane-Emden equation given in [12]

$$u''(x)+\frac{8}{x}u'(x)+xu(x)=x^5-x^4+44x^2-30x,\ \ 0<x\le 1, \tag{14}$$

with initial conditions

$$u(0)=0,\ \ u'(0)=0. \tag{15}$$

By multiplying both sides of Eq. (14) by *x* and then taking differential transformation of both sides of the resulting equation using Theorems 1-5, we obtain the following recurrence relation

$$U(k+1)=\frac{1}{(k+1)(k+8)}\times \left(\delta(k-6)-\delta(k-5)+44\delta(k-3)-30\delta(k-2)-\sum_{l=0}^{k}\delta(l-2)U(k-l)\right). \tag{16}$$

We apply the differential transformation at $x_0=0$, therefore, the initial conditions given in Eq. (15) are transformed as follows:

$$U(0)=0, \tag{17}$$

$$U(1)=0. \tag{18}$$

Substituting Eqs. (17) and (18) at $k=1$ into Eq. (16), we have

$$U(2)=0. \tag{19}$$

Following the same recursive procedure, we find $U(k+1)=0$, $k=4,5,\ldots$ and listing the computation and result corresponding to $n=4$, we have

$$U(3)=-1, \tag{20}$$

$$U(4)=1. \tag{21}$$

Using Eqs. (17)-(21) and the inverse transformation rule in Eq. (5), we get the following solution:

$$u(x) = -x^3 + x^4. \tag{22}$$

For $n > 4$, one evaluates that the solution (22), which is the exact solution of Eq. (14) under the initial conditions in Eq. (15).

Example 3 We finally close our analysis by studying the following Lane-Emden equation

$$u''(x) + \frac{2}{x}u'(x) + u(x) = x^5 + 30x^3, \quad 0 < x \le 1, \tag{23}$$

subject to initial conditions

$$u(0) = 0, \quad u'(0) = 0. \tag{24}$$

By multiplying both sides of Eq. (23) by x and then taking differential transformation of both sides of the resulting equation using Theorems 1-5, we obtain the following recurrence relation

$$U(k+1) = \frac{1}{(k+1)(k+2)}\left(\delta(k-6) + 30\delta(k-4) - \sum_{l=0}^{k}\delta(l-1)U(k-l)\right). \tag{25}$$

The initial conditions in Eq. (24) can be transformed at $x_0 = 0$ as

$$U(0) = 0, \tag{26}$$

$$U(1) = 0. \tag{27}$$

Substituting Eqs. (26) and (27) at $k = 1$ into (25), we have

$$U(2) = 0. \tag{28}$$

Following the same procedure, $U(3) - U(5)$ can be solved as follows:

$$U(3) = 0, \tag{29}$$

$$U(4) = 0, \tag{30}$$

$$U(5) = 1. \tag{31}$$

For $n > 5$, by the same way, we have $U(k+1) = 0$, $k = 5,6,7,\ldots$. By using the inverse transformation rule in Eq. (5), we obtain the solution in a closed form by

$$u(x) = x^5, \tag{32}$$

which is the exact solution of Eq. (23) subject to the initial conditions in Eq. (24).

4. CONCLUSION

In this study, the differential transformation method is implemented to the Lane-Emden differential equations as singular initial value problems. Three equations are solved and exact solutions are obtained. It is shown that differential transformation method is a very fast convergent, precise and cost efficient tool for solving the Lane-Emden equations.

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