Mathematical and Computational Applications, Vol. 11, No. 2, pp. 121-125, 2006.

SENSITIVITY ANALYSIS OF GENERAL MIXED MULTIVALUED MILDLY NONLINEAR VARIATIONAL INEQUALITIES

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Abstract- Dafermos [1] studied the sensitivity properties of solutions of a variational inequality with respect to a parameter λ In this paper, we extend this analysis for general mixed multivalued mildly nonlinear variational inequalities in the setting of Hilbert spaces.

Keywords- Sensitivity analysis, general mixed multivalued mildly nonlinear variational inequality, monotone and Lipschitz mappings, Hausdorff metric.

1. INTRODUCTION

Dafermos [1] studied the sensitivity property of solutions of a particular kind of variational inequality on a parameter which takes values on an open subset of Euclidean space R^K. Siddiqi et al [4], Tobin [5], Verma et al [6] studied the sensitivity analysis of various types of variational inequalities.

Let H be a Hilbert space and D be a nonempty closed and convex subset of H. Let $A,T:D \to 2^H$ be the multivalued mappings. We consider the problem of finding $u\in D$, $p\in A(u)$, $q\in T(u)$ such that $g(u)\in D$

$$\langle g(v) - g(u), p + q \rangle + b(u, g(v)) - b(u, g(u)) \geq 0, \text{ for all } g(v) \in D, \quad (1)$$

where $b(.,.) : H \times H \to R$ satisfying the following properties:

(i) $b(.,.)$ is linear in the first argument;

(ii) $b(.,.)$ is bounded, that is there exists constant $\gamma > 0$ such that

$$|b(u,v)| \leq \gamma \,||u||\, ||v||, \text{ for all } u,v \in H; \quad (2)$$

(iii) $|b(.,.)|$ is either convex or linear in the second argument;

(iv) for every $u,v,w \in H$

$$|b(u,v) - b(u,w)| \leq b(u,v - w) \quad (3)$$

$$b(u, v \pm w) \leq b(u,v) + b(u,w).$$

In this paper, we study the sensitivity analysis of general mixed multivalued mildly nonlinear variational inequality problem (GMMMNVIP) of the type (1).

2. SENSITIVITY ANALYSIS

To formulate the problem, let M be an open subset of H in which the parameter λ takes values and assume that $\{K_\lambda: \lambda \in M\}$ is a family of closed convex subset of H. The parametric general mixed multivalued mildly nonlinear variational inequality is to find $u\in D$, $(p, \lambda)\in A(u, \lambda)$, $(q, \lambda)\in T(u, \lambda)$ such that $g(u)\in K_\lambda$

$$\langle g(v) - g(u), (p, \lambda) + (q, \lambda)\rangle + b(u, g(v)) - b(u, g(u)) \geq 0, \text{ for all } g(v)\in K_\lambda \quad (4)$$

where $A(u, \lambda)$ and $T(u, \lambda)$ are multivalued mappings, which are define on the set (u, λ) with $\lambda\in M$. We also assume that for some $\bar{\lambda}\in M$, the problem (4) admits a solution $\bar{u}$.
We want to investigate those conditions under which, for each λ is a neighbourhood of $\bar{\lambda}$. The problem (4) has a unique solution $u(\lambda)$ near $\bar{u}$ and the function $u(\lambda)$ is continuous and differentiable. We assume that B is the closure of a ball in H centered at $\bar{u}$.
We need the following concepts.
2.1 Definition- An operator g defined on B to H is said to be locally, $u,v\in B$,
(i) ξ-strongly monotone, if there exists a constant $\xi > 0$ such that

$$\langle g(u) - g(v), u - v \rangle \geq \xi \| u - v \|^2 \quad (5)$$

(ii) σ-h-Lipschitz continuous, if there exists a constant $\sigma > 0$ such that

$$\| g(u) - g(v) \| \leq \sigma \| u - v \|. \quad (6)$$

In particular, it follows that $\xi \leq \sigma$.
2.2 Definition- A multivalued mapping $T(u, \lambda)$ defined on $B\times M$ to $C(H)$ is said to be locally υ-h-Lipschitz continuous, if there exists a constant $\upsilon > 0$ such that

$$h(T(u,\lambda), T(v,\lambda)) \leq \upsilon \| u - v \|, \text{ for all } u,v\in B, \quad (7)$$

where $h(.,.)$ is a Hausdorff metric on $C(H)$, and $C(H)$ denotes the family of all nonempty compact subsets of H.
2.3 Definition- A multivalued mapping $A(u,\lambda)$ defined on $B\times M$ to $C(H)$ is said to be locally α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle (p_1, \lambda) - (p_2, \lambda), u - v\rangle \geq \alpha \| u - v \|^2 \text{ for all } u,v\in B, (p_1,\lambda)\in A(u,\lambda), (p_2,\lambda)\in A(v,\lambda). \quad (8)$$

We have the following lemmas which can be proved by the techniques of Noor [3].
2.1 Lemma- A point $u\in K_\lambda$ is a solution of the parametric general mixed multivalued mildly nonlinear variational inequalities (4) iff it is fixed point of the map

$$\varphi(u, \lambda) = u - g(u) + P_{K_\lambda} [g(u) + F(u) - \rho((p, \lambda) + (q, \lambda))] \quad (9)$$

for all $\lambda\in M$, $(p, \lambda)\in A(u, \lambda)$, $(q, \lambda)\in T(u, \lambda)$ for some $\rho > 0$, where P_{K_λ} is the projection of H on the family of closed convex sets K_λ and $F:H\rightarrow H$ is a single valued function defined by, for every $(p, \lambda)\in A(u, \lambda)$ and $(q, \lambda)\in T(u, \lambda)$

$$\langle F(u), g(v)\rangle = \langle u, g(v)\rangle - \rho \ \langle (p, \lambda) + (q, \lambda), g(v)\rangle - \rho b(u, g(v)), \text{ for all } g(v) \in K_\lambda. \quad (10)$$

Since we are interested in the case, when the solution of the problem (4) lies in the interior of B. So we consider the map $\varphi^*(u, \lambda)$ define by

$$\varphi^*(u, \lambda) = u - g(u) + P_{K_\lambda\cap B} [g(u) + F(u) - \rho((p, \lambda) + (q, \lambda))], \text{ for all } (u, \lambda)\in B\times M. \quad (11)$$

We have to show that the map $\varphi^*(u, \lambda)$ has a fixed point, which by (9) is also a solution of (4). First of all, we prove that the map $\varphi^*(u, \lambda)$ is a contraction map with respect to u, uniformly in $\lambda\in M$, by using locally α-strongly monotonicity and locally β-h-Lipschitz continuity of the operator $A(u,\lambda)$, locally ξ-strongly monotonicity and σ-h-Lipschitz continuity of $g(u)$ and locally υ-h-Lipschitz continuity of $T(u,\lambda)$.
2.2 Lemma- For all $u_1, u_2\in B$ and $\lambda\in M$, we have

$$\| \varphi^*(u_1,\lambda) - \varphi^*(u_2,\lambda) \| \leq \theta \|u_1 - u_2 \|$$

where

$$\theta = k + 2\rho\upsilon + \rho\gamma + 2(1-2\rho\alpha + \rho^2\beta^2)^{1/2}$$

for

$$((1-k)(2\upsilon+\gamma)) 4^{-1} < \alpha < \beta, \qquad k<1,$$

$$\alpha > (1-k)(\gamma + 2\upsilon)\, 4^{-1} + ((\beta^2 - ((2\upsilon + \gamma)\, 2^{-1})^2)(3-k)(1+k)\, 4^{-1})^{1/2}$$

and

$$|\rho - (\alpha + (k-1)(\gamma + 2\upsilon)\, 4^{-1})(\beta^2 - ((2\upsilon + \gamma)\, 2^{-1})^2)^{-1}| <$$

$$\{(\alpha + (k-1)(\gamma + 2\upsilon)\, 4^{-1})^2 - (\beta^2 - ((2\upsilon + \gamma)2^{-1})^2)(1+k)(3-k)4^{-1})\}^{1/2}\{\beta^2 - ((2\upsilon + \gamma)2^{-1})^2\}^{-1}$$

with

$$k = 2\,(1 - 2\xi + \sigma^2)^{1/2}.$$

Proof- Using (11), we have

$$\| \varphi^*(u_1, \lambda) - \varphi^*(u_2, \lambda) \| = \| \{u_1 - g(u_1) + P_{K_\lambda} [g(u_1) + F(u_1) - \rho\, ((p_1,\lambda) + (q_1,\lambda))]\} \\ -\{ u_2 - g(u_2) + P_{K_\lambda} [g(u_2) + F(u_2) - \rho\, ((p_2,\lambda) + (q_2,\lambda))]\}\|$$

and using the fact that the projection operator is nonexpansive, we have

$$\| \varphi^*(u_1, \lambda) - \varphi^*(u_2, \lambda) \| \le 2\, \| u_1 - u_2 - (g(u_1) - g(u_2))\| + \|F(u_1) - F(u_2)\| \\ + \| u_1 - u_2 - \rho\, ((p_1, \lambda) - (p_2, \lambda))\| + \rho\, \|(q_1, \lambda) - (q_2, \lambda)\|. \quad (12)$$

Now, the operator g(u) is both locally ξ-strongly monotone and σ-h-Lipschitz continuous and the operator $A(u, \lambda)$ is locally α-strongly monotone and locally β-h-Lipschitz continuous, so by the method of Noor [2],

$$\| u_1 - u_2 - (g(u_1) - g(u_2))\|^2 = \| u_1 - u_2 \|^2 + \| g(u_1) - g(u_2) \|^2 - 2\, \langle g(u_1) - g(u_2), u_1 - u_2 \rangle \\ \le (1 - 2\xi + \sigma^2)\, \| u_1 - u_2 \|^2, \quad (13)$$

where σ and ξ are Lipschitz and strongly monotone constants of g(u), respectively and

$$\langle F(u_1) - F(u_2), g(v)\rangle = \langle u_1 - u_2 , g(v)\rangle - \rho\, \langle (p_1, \lambda) - (p_2, \lambda), g(v)\rangle \\ - \rho\, \langle (q_1, \lambda) - (q_2, \lambda), g(v)\rangle - \rho\, b(u_1 - u_2, g(v))$$

$$\le \{\| u_1 - u_2 - \rho\, ((p_1, \lambda) - (p_2, \lambda))\| + \rho\, \| (q_1, \lambda) - (q_2, \lambda)\| + \rho\gamma\, \| u_1 - u_2 \| \}\| g(v) \|$$

$$\le \{(1 - 2\alpha\rho + \rho^2\beta^2)^{1/2} + \rho\upsilon + \rho\gamma\}\, \| u_1 - u_2 \|\, \| g(v) \|.$$

Now

$$\| F(u_1) - F(u_2) \| = \sup_{g(v)\in K} \{\langle F(u_1) - F(u_2), g(v)\rangle / \| g(v) \|\}$$

$$\le \{(1 - 2\alpha\rho + \rho^2\beta^2)^{1/2} + \rho\upsilon + \rho\gamma\}\, \| u_1 - u_2 \| \quad (14)$$

and

$$\| u_1 - u_2 - \rho((p_1, \lambda) - (p_2, \lambda))\|^2 = \| u_1 - u_2 \|^2 + \rho^2\, \| (p_1, \lambda) - (p_2, \lambda)\|^2 \\ - 2\rho\, \langle u_1 - u_2, (p_1, \lambda) - (p_2, \lambda)\rangle. \quad (15)$$

Using (6) and (8), we have

$$\| (p_1, \lambda) - (p_2, \lambda)\| \le h(A(u_1, \lambda), A(u_2, \lambda)) \le \beta\, \| u_1 - u_2 \|$$

$$\langle u_1 - u_2 , (p_1, \lambda) - (p_2, \lambda)\rangle \ge \alpha\, \| u_1 - u_2\|^2 .$$

Therefore (15) becomes

$$\| u_1 - u_2 - \rho\, ((p_1, \lambda) - (p_2, \lambda)) \|^2 \le \| u_1 - u_2 \|^2 + \rho^2\beta^2\, \| u_1 - u_2 \|^2 - 2\rho\alpha\, \| u_1 - u_2 \|^2 \\ \le (1 - 2\alpha\rho + \rho^2\beta^2)\, \| u_1 - u_2 \|^2 \quad (16)$$

where β and α are the constants appearing in (5), (6), again

$$\| (q_1, \lambda) - (q_2, \lambda) \| \leq h(T(u_1, \lambda), T(u_2, \lambda)) \leq \upsilon \| u_1 - u_2 \|. \quad (17)$$

Since $T(u, \lambda)$ is locally υ-h-Lipschitz continuous, using (13)-(17), we have

$$\| \varphi^*(u_1,\lambda) - \varphi^*(u_2,\lambda) \| \leq \theta \| u_1 - u_2 \|$$

where $$\theta = k + 2\rho\upsilon + \rho\gamma + 2(1 - 2\rho\alpha + \rho^2\beta^2)^{1/2}$$
with
$$k = 2(1 - 2\xi + \sigma^2)^{1/2}.$$

Now, using the technique of Noor [3], we can show that $\theta < 1$ from which it follows that the map $\varphi^*(u, \lambda)$ defined by (11) is a contraction map.

2.1 Remark- From Lemma 2.2, It is clear that the map $\varphi^*(u, \lambda)$ defined by (11) has a unique fixed point $u(\lambda)$ that is $u(\lambda) = \varphi^*(u,\lambda)$, we also know by assumption, the function $\bar{u}$ for $\lambda = \bar{\lambda}$ is a solution of the parametric general mixed multivalued mildly nonlinear variational inequality problem (4), we see that $\bar{u}$ is a fixed point of $\varphi(\bar{u}, \lambda)$ and it is also a fixed point of $\varphi^*(u, \lambda)$, consequently we have $u(\lambda) = u = \varphi^*(u(\lambda), \lambda)$, now we show that the solution $u(\lambda)$ of problem (4) is β-h-Lipschitz continuous.

2.3 Lemma- If $T(\bar{u}, \lambda)$,$A(\bar{u}, \lambda)$ is the multivalued operators and the map

$$\lambda \rightarrow P_{K_{\lambda \cap B}} [g(\bar{u}) + F(\bar{u}) - \rho((\overline{p, \lambda}) + (\bar{q}, \bar{\lambda}))]$$

are β-h- Lipschitz continuous in λ at $\bar{\lambda}$, then $u(\lambda)$ is β-h- Lipschitz continuous at $\lambda=\bar{\lambda}$.

Proof- Fix $\lambda \in M$, then using the triangle inequality and Lemma 2.2, we have

$$\|u(\lambda) - u(\bar{\lambda})\| \leq \| \varphi^*(u(\lambda), \lambda) - \varphi^*(u(\bar{\lambda}), \lambda) \| + \| \varphi^*(u(\bar{\lambda}), \lambda) - \varphi^*(u(\bar{\lambda}), \bar{\lambda}) \|$$

$$\leq \theta \| u(\lambda) - u(\bar{\lambda}) \| + \| \varphi^*(u(\bar{\lambda}), \lambda) - \varphi^*(u(\bar{\lambda}), \bar{\lambda}) \|. \quad (18)$$

From (11) and the fact that the projection operator is nonexpansive, we have

$$\| \varphi^*(u(\bar{\lambda}), \lambda) - \varphi^*(u(\bar{\lambda}), \bar{\lambda}) \| = \| P_{K_{\lambda \cap B}} [g(u(\bar{\lambda})) + F(u(\bar{\lambda})) - \rho((p(\bar{\lambda}), \lambda) + (q(\bar{\lambda}), \lambda))]$$

$$- P_{K_{\lambda \cap B}} [g(u(\bar{\lambda})) + F(u(\bar{\lambda})) - \rho ((p(\bar{\lambda}), \bar{\lambda}) + (q(\bar{\lambda}), \bar{\lambda}))]\|$$

$$\leq \rho \| (p(\bar{\lambda}), \lambda) - (p(\bar{\lambda}), \bar{\lambda}) \| + \rho \| (q(\bar{\lambda}), \lambda) - (q(\bar{\lambda}), \bar{\lambda})\|$$

$$+ \| P_{K_{\lambda \cap B}} [g(u(\bar{\lambda})) + F(u(\bar{\lambda})) - \rho ((p(\bar{\lambda}), \bar{\lambda}) + (q(\bar{\lambda}), \bar{\lambda}))]$$

$$- P_{K_{\bar{\lambda} \cap B}} [g(u(\bar{\lambda})) + F(u(\bar{\lambda})) - \rho((p(\bar{\lambda}), \bar{\lambda}) + (q(\bar{\lambda}), \bar{\lambda}))]\| . \quad (19)$$

Now from Remark 2.1 and combining (18) and (19), we have

$$\| u(\lambda) - \bar{u} \| \leq \rho(1- \theta)^{-1} \|(p(\bar{\lambda}), \lambda) - (p(\bar{\lambda}), \bar{\lambda}) \| + \rho(1- \theta)^{-1} \| (q(\bar{\lambda}), \lambda) - (q(\bar{\lambda}), \bar{\lambda}) \|$$

$$+ \rho (1- \theta)^{-1} \| P_{K_{\lambda \cap B}} [g(\bar{u}) + F(\bar{u}) - \rho((p(\bar{\lambda}), \bar{\lambda}) +(q(\bar{\lambda}), \bar{\lambda}))]$$

$$- P_{K_{\overline{\lambda} \cap B}} [g(\overline{u}) + F(\overline{u}) - \rho((p(\overline{\lambda}), \overline{\lambda}) + (q(\overline{\lambda}), \overline{\lambda}))]\| .$$

From which the required result follows:

Using the technique of Dafermos [1], we can show that there exists a neighbourhood N containing in M of $\overline{\lambda}$ such that for $\lambda \in N$, $u(\lambda)$ is the unique solution of problem (4) in the interior of B.

Combining the above results we arrive at the following:

2.1 Theorem- Let $\overline{u}$ be the solution of parametric general mixed multivalued mildly nonlinear variational inequality problem (4) at $\lambda = \overline{\lambda}$, the multivalued mapping $T(\overline{u}, \lambda)$ be locally υ-h-Lipschitz continuous and the multivalued mapping $A(\overline{u}, \lambda)$ be locally α-strongly monotone and locally β-h-Lipschitz continuous, the map $g(\overline{u})$ be locally ξ-strongly monotone and σ-h-Lipschitz continuous.
Suppose that $T(\overline{u}, \lambda)$, $A(\overline{u}, \lambda)$, $g(\overline{u})$ and the map

$$\lambda \rightarrow P_{K_{\lambda \cap B}} [g(\overline{u}) + \overline{F}(\overline{u}) - \rho((\overline{p}, \overline{\lambda}) + (\overline{q}, \overline{\lambda}))]$$

are β-h-Lipschitz continuous at $\lambda = \overline{\lambda}$, then there exists a neighbourhood N containing in M of λ such that for $\lambda \in N$, the problem (4) has a unique solution $u(\lambda)$ in the interior of B, $u(\overline{\lambda}) = \overline{u}$ and $u(\lambda)$ are continuous (β-h-Lipschitz continuous) at $\lambda = \overline{\lambda}$.

2.2 Remark- The function $u(\lambda)$ as defined in Theorem 2.1 is continuously differentiable on some neighbourhood N of λ . For this see Dafermos [1].

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