Mathematical and Computational Applications, Vol. 11, No. 2, pp. 103-110, 2006.

THERMAL SHOCK PROBLEM FOR ONE DIMENSIONAL GENERALIZED THERMOELASTIC LAYERED COMPOSITE MATERIAL

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Abstract– The dynamic treatment of one-dimensional generalized thermoelastic problem of heat conduction is made for a layered thin plate, which is exposed, to a uniform thermal shock. The basic equations are transformed by Laplace transform and solved by a direct method. The solution was applied for a plate of sandwich structure. The inverses of Laplace transforms are obtained numerically. The temperature, the stress and the displacement distributions are represented in graphs, which show the coupled and the generalized cases.
Keywords– Thermoelasticity, Laplace Transforms, Layered Composite

1. INTRODUCTION

Lord and Shulman [1] obtained the governing equations of generalized thermoelasticity involving one relaxation time for isotropic homogeneous media. These equations predict finite speeds of propagation of heat and displacement distribution, the corresponding equations for an isotropic case were obtained by Dhaliwal and Sherief [2]. Due to the complexity of the governing equations and the mathematical difficulties associated with their solution several simplifications have been used. For example some authors [3,4] use the framework of coupled thermoelastic\city where the relaxation time is taken as zero resulting in a parabolic system of partial differential equations. The solution of this system exhibits infinite speed of propagation of heat signals contradictory to physical observation. Some other authors use still further simplifications by ignoring the inertia effects in a coupled theory [5] or by neglecting the coupling effect.

This work deals with a plate consisting of layers of unidentical substances, each of which is homogeneous and isotropic. When this plate, which is initially at rest and having a uniform temperature, is suddenly heated at the free surfaces, a heat flow occurs in the plate and change in thermal and the mechanical field is brought about.

2. THE BASIC EQUATIONS

The coordinate system is so chosen that the x-axis is taken perpendicularly to the layer, and the y-and z-axes in parallel. We are dealing with one-dimensional generalized thermoelasticity with one relaxation time.
The equation of motion

$$\frac{\partial^2 \sigma}{\partial x^2} = \rho \ddot{e}, \tag{1}$$

The constitutive equation

$$\sigma = (\lambda + 2\mu)e - \gamma(T - T_0), \tag{2}$$

The heat equation

$$K\frac{\partial^2 T}{\partial x^2}=\left(\frac{\partial}{\partial t}+\tau_o\frac{\partial^2}{\partial t^2}\right)\left(\rho C_E T+T_o\gamma e\right), \tag{3}$$

where $\gamma=(3\lambda+2\mu)\alpha_T$, $e=\frac{\partial u}{\partial x}$ and.

The above equations can be put into a more convenient form by using the following non-dimensional variables

$$x'=v\eta x\,,\; t'=v^2\eta t\,,\; \tau_o=v^2\eta\tau_o\,,\; \theta=\frac{(T-T_o)(3\lambda+2\mu)\alpha_T}{(\lambda+2\mu)}\,,\; \sigma'=\frac{\sigma}{\lambda+2\mu}\,,\; u'=v\eta u\,,$$

where $v=\sqrt{\frac{\lambda+2\mu}{\rho}}$ and $\eta=\frac{\rho C_E}{K}$.

After dropping the primes for convenience, we obtain

$$D^2\sigma=\ddot{e}\,, \tag{4}$$

$$\sigma=(e-\theta), \tag{5}$$

$$D^2T=\left(\frac{\partial}{\partial t}+\tau_o\frac{\partial^2}{\partial t^2}\right)\left[\theta+\varepsilon e\right], \tag{6}$$

where $\varepsilon=\frac{(3\lambda+2\mu)^2\alpha_T^2 T_o}{(\lambda+2\mu)\rho C_E}$.

Taking Laplace transform as define

$$\bar{f}(s)=\int_0^\infty f(t)e^{-st}dt\,.$$

Then, equations (4), (5) and (6) will take the forms

$$D^2\bar{\sigma}=s^2\bar{e}\,, \tag{7}$$

$$\left[D^2-\left(s+\tau_o s^2\right)\right]\bar{\theta}=\varepsilon h^2\bar{e}\,, \tag{8}$$

$$\bar{\sigma}=\left(\bar{e}-\bar{\theta}\right), \tag{9}$$

where $D=\frac{\partial}{\partial x}$.

By eliminating $\bar{e}$, we get

$$\left[D^2-s^2\right]\bar{\sigma}=s^2\bar{\theta}\,, \tag{10}$$

$$\left[D^2-(\varepsilon+1)\left(s+\tau_o s^2\right)\right]\bar{\theta}=\varepsilon h^2\bar{\sigma}\,, \tag{11}$$

Using the above two equations, we obtain

$$\left(D^4-LD^2+M\right)\bar{\theta}=0\,, \tag{12}$$

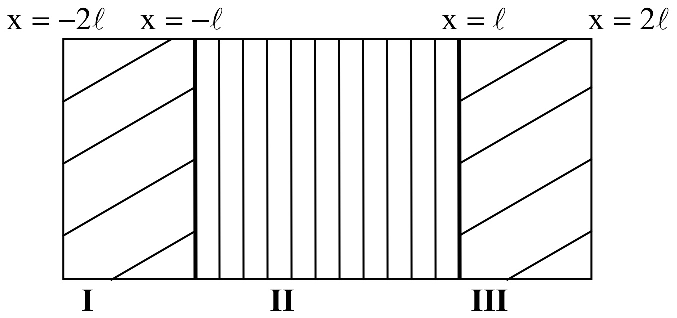
$$\left(D^4-LD^2+M\right)\bar{\sigma}=0\,, \tag{13}$$

where

$$L=s^2+\left(s+\tau_o s^2\right)(\varepsilon+1)+\varepsilon\left(s+\tau_o s^2\right)s^2 \text{ and } M=s^2\left(s+\tau_o s^2\right)(1+\varepsilon)\,.$$

3. APPLICATION

Considering a layered plat of sand-witch structure such as shown in Figure 1, where layers **I** , **III** made from the same metal, and the layer **II** is a deferent metal. Layer **II** is put in the middle of the plate, and its thickness is a half of that of the plate.



(Figure.1)

1-In region **I** where $-2\ell \le x \le -\ell$

The solution of the equations (12) and (13) take the form

$$\overline{\theta}^{I} = A_1\left(k_1^2 - s^2\right)\cosh(k_1 x) + A_2\left(k_2^2 - s^2\right)\cosh(k_2 x), \tag{14}$$

$$\overline{\sigma}^{I} = A_1 s^2\cosh(k_1 x) + A_2 s^2\cosh(k_2 x), \tag{15}$$

where the parameters k_1 and k_2 satisfy the equation

$$k^4 - L^I k^2 + M^I = 0$$

where

$$L^I = s^2 + \left(s + \tau_o^I s^2\right)\left(\varepsilon^I + 1\right) + \varepsilon\left(s + \tau_o^I s^2\right)s^2$$

$$M^I = s^2\left(s + \tau_o^I s^2\right)\left(1 + \varepsilon^I\right).$$

2- In region **II** where $-\ell \le x \le \ell$

The solution of the equations (12) and (13) take the form

$$\overline{\theta}^{II} = B_1\left(p_1^2 - s^2\right)\cosh(p_1 x) + B_2\left(p_2^2 - s^2\right)\cosh(p_2 x), \tag{16}$$

$$\overline{\sigma}^{II} = B_1 s^2\cosh(p_1 x) + B_2 s^2\cosh(p_2 x), \tag{17}$$

where the parameters p_1 and p_2 satisfy the equation

$$p^4 - L^{II} p^2 + M^{II} = 0$$

where

$$L^{II} = s^2 + \left(s + \tau_o^{II} s^2\right)\left(\varepsilon^{II} + 1\right) + \varepsilon\left(s + \tau_o^{II} s^2\right)s^2$$

$$M^{II} = s^2\left(s + \tau_o^{II} s^2\right)\left(1 + \varepsilon^{II}\right).$$

1- In region **III** where $\ell \le x \le 2\ell$

The solution of the equations (12) and (13) take the form

$$\overline{\theta}^{III} = c_1\left(k_1^2 - s^2\right)\cosh(k_1 x) + c_2\left(k_2^2 - s^2\right)\cosh(k_2 x), \tag{18}$$

$$\bar{\sigma}^{III} = c_1 s^2 \cosh(k_1 x) + c_2 s^2 \cosh(k_2 x), \tag{19}$$

The Boundary Conditions:

(1)- The thermal boundary conditions

$\theta = \theta_o H(t)$ for $x = \pm 2\ell$,

which takes the form

$$\bar{\theta} = \frac{\theta_o}{s} \text{ for } x = \pm 2\ell. \tag{20}$$

(2)- The mechanical boundary conditions

$\sigma = 0$ for $x = \pm 2\ell$,

which takes the form

$$\bar{\sigma} = 0 \text{ for } x = \pm 2\ell. \tag{21}$$

(3)- The continuity conditions

$$\text{(i)-}\ \bar{\theta}^{I} = \bar{\theta}^{II} \text{ at } x = -\ell, \text{ and } \bar{\theta}^{II} = \bar{\theta}^{III} \text{ at } x = \ell, \tag{22}$$

$$\text{(ii)-}\ \bar{\sigma}^{I} = \bar{\sigma}^{II} \text{ at } x = -\ell, \text{ and } \bar{\sigma}^{II} = \bar{\sigma}^{III} \text{ at } x = \ell. \tag{23}$$

Applying the pervious conditions into equations (14)-(19), we obtain

$$\bar{\theta}^{I} = \bar{\theta}^{III} = \frac{1}{s\left(k_1^2 - k_2^2\right)}\left[\frac{\left(k_1^2 - s^2\right)}{\cosh(2\ell k_1)}\cosh(k_1 x) - \frac{\left(k_2^2 - s^2\right)}{\cosh(2\ell k_2)}\cosh(k_2 x)\right], \tag{24}$$

$$\bar{\sigma}^{I} = \bar{\sigma}^{III} = \frac{s}{\left(k_1^2 - k_2^2\right)}\left[\frac{\cosh(k_1 x)}{\cosh(2\ell k_1)} - \frac{\cosh(k_2 x)}{\cosh(2\ell k_2)}\right], \tag{25}$$

$$\begin{aligned}\bar{\theta}^{II} = {} & \frac{\left(p_1^2 - s^2\right)}{2s\left(k_1^2 - k_2^2\right)\left(p_1^2 - p_2^2\right)\cosh(\ell p_1)}\left[\frac{k_1^2 - p_2^2}{\sinh(\ell k_1)} - \frac{k_2^2 - p_2^2}{\sinh(\ell k_2)}\right]\cosh(p_1 x) \\ & - \frac{\left(p_2^2 - s^2\right)}{2s\left(k_1^2 - k_2^2\right)\left(p_1^2 - p_2^2\right)\cosh(\ell p_2)}\left[\frac{k_1^2 - p_1^2}{\sinh(\ell k_1)} - \frac{k_2^2 - p_1^2}{\sinh(\ell k_2)}\right]\cosh(p_2 x),\end{aligned} \tag{26}$$

$$\begin{aligned}\bar{\sigma}^{II} = {} & \frac{s}{2\left(k_1^2 - k_2^2\right)\left(p_1^2 - p_2^2\right)\cosh(\ell p_1)}\left[\frac{k_1^2 - p_2^2}{\sinh(\ell k_1)} - \frac{k_2^2 - p_2^2}{\sinh(\ell k_2)}\right]\cosh(p_1 x) \\ & - \frac{s}{2\left(k_1^2 - k_2^2\right)\left(p_1^2 - p_2^2\right)\cosh(\ell p_2)}\left[\frac{k_1^2 - p_1^2}{\sinh(\ell k_1)} - \frac{k_2^2 - p_1^2}{\sinh(\ell k_2)}\right]\cosh(p_2 x).\end{aligned} \tag{27}$$

We can get the displacement by using equation (4), such that

$$\bar{u} = \frac{1}{s^2} D\bar{\sigma},$$

so, we obtain

$$\bar{u}^{I} = \bar{u}^{III} = \frac{1}{s\left(k_1^2 - k_2^2\right)}\left[\frac{k_1 \sinh(k_1 x)}{\cosh(2\ell k_1)} - \frac{k_2 \sinh(k_2 x)}{\cosh(2\ell k_2)}\right], \tag{28}$$

and

$$\bar{u}^{II} = \frac{p_1}{2s\left(k_1^2-k_2^2\right)\left(p_1^2-p_2^2\right)\cosh(\ell p_1)}\left[\frac{k_1^2-p_2^2}{\sinh(\ell k_1)}-\frac{k_2^2-p_2^2}{\sinh(\ell k_2)}\right]\sinh(p_1 x)$$
$$-\frac{p_2}{2s\left(k_1^2-k_2^2\right)\left(p_1^2-p_2^2\right)\cosh(\ell p_2)}\left[\frac{k_1^2-p_1^2}{\sinh(\ell k_1)}-\frac{k_2^2-p_1^2}{\sinh(\ell k_2)}\right]\sinh(p_2 x). \quad (29)$$

4. THE SOLUTION IN THE PHYSICAL DOMAIN

In order to invert the Laplace transform in equations (24)-(29), we adopt a numerical inversion method based on a Fourier series expansion [6].

By this method the inverse $f(t)$ of the Laplace transform $\bar{f}(s)$ is approximated by

$$f(t)=\frac{e^{ct}}{t_1}\left[\frac{1}{2}\bar{f}(c)+\mathrm{Re}\sum_{k=1}^{N}\bar{f}\left(c+\frac{ik\pi}{t_1}\right)\exp\left(\frac{ik\pi t}{t_1}\right)\right] \quad , \quad 0<t_1<2t,$$

where N is a sufficiently large integer representing the number of terms in the truncated Fourier series, chosen such that

$$\exp(ct)\,\mathrm{Re}\left[\bar{f}\left(c+\frac{iN\pi}{t_1}\right)\exp\left(\frac{iN\pi t}{t_1}\right)\right]\le\varepsilon_1,$$

where ε_1 is a prescribed small positive number that corresponds to the degree of accuracy required. The parameter c is a positive free parameter that must be greater than the real part of all the singularities of $\bar{f}(s)$.

The optimal choice of c was obtained according to the criteria described in [7].

The copper material and the type 316 stainless steel are chosen for purposes of numerical evaluations [6], [8].

Table1. Material constant

The constant	Copper	Stainless steel
Thermal expansion $\times 10^{-6}$	17.8	17.7
Mass density $\times 10^{3}$	8.95	7.97
Heat capacity $\times 10^{3}$	0.3831	0.560
Thermal conductivity	386	19.5
Poisson's ratio	0.343	0.294
Relaxation time	0.02	0.015
ε	0.0150	0.0141

The computations were carried out for value of time, namely t = 0.2 and for length $\ell = 1$ (unit length)

The numerical values of the temperature, displacement component and stress component for the two cases, coupled and generalized are obtained and represented graphically.

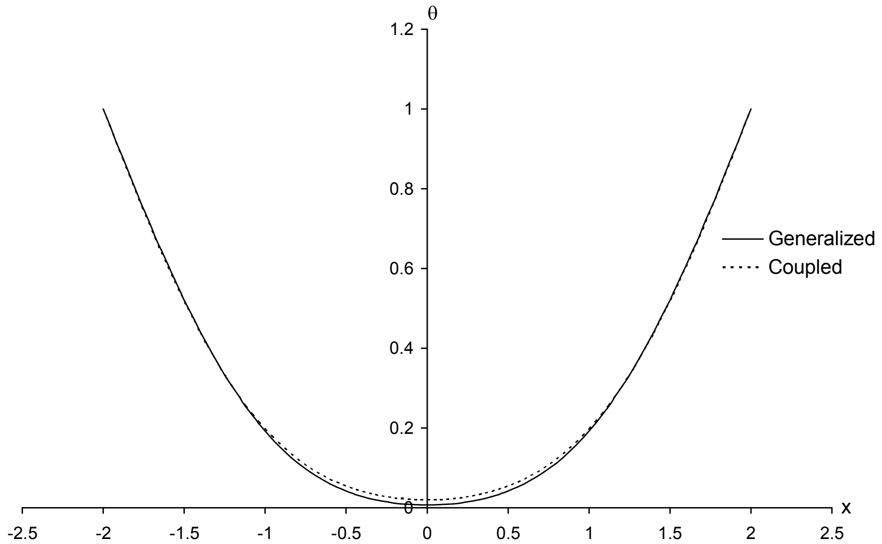


Figure 2: Temperature Distribution

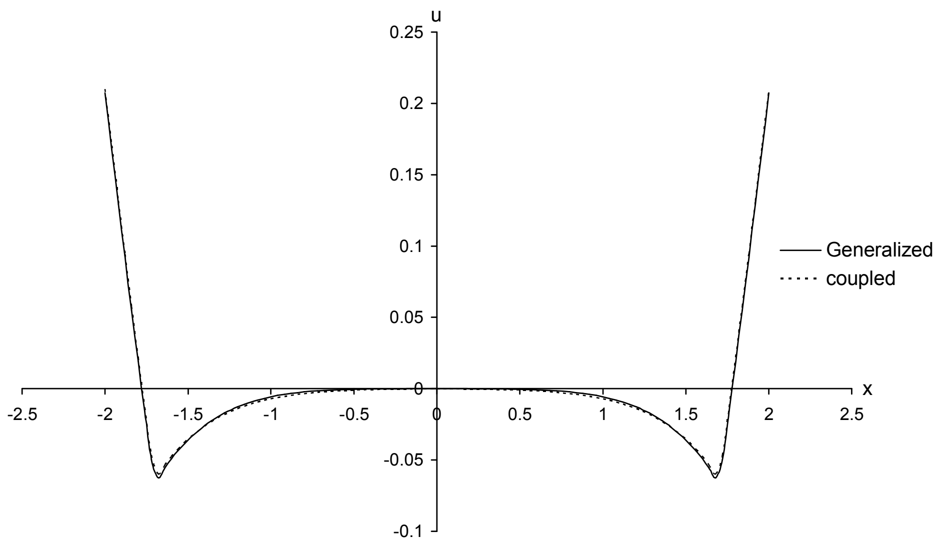


Figure 3: Displacement Distribution

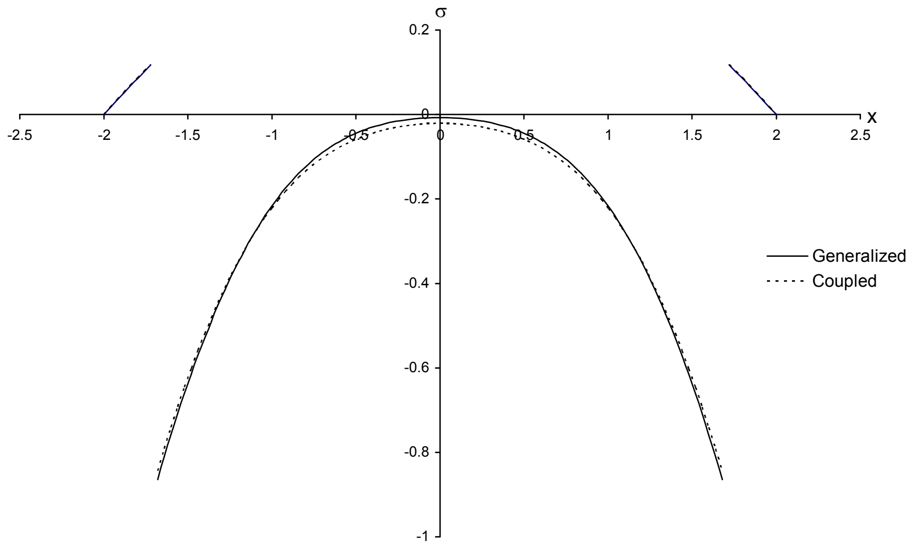


Figure 4: The Displacement Distribution

Nomenclature

λ , μ Lame's constants
ρ Density
C_E Specific heat at constant strain
t Time
τ_0 One relaxation time
T Absolute temperature
T_0 Reference temperature
σ Components of stress tensor
e Components of strain tensor
u Components of displacement vector
k Thermal conductivity
α_t Thermal expansion

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