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NUMERICAL VERIFICATION OF THE ORDER OF THE ASYMPTOTIC SOLUTIONS OF A NONLINEAR DIFFERENTIAL EQUATION

Jianping Cai
Department of Mathematics, Zhangzhou Teachers College, Zhangzhou 363000, China
mathcai@hotmail.com

Abstract-A perturbation method, **t**he Lindstedt-Poincare method, is used to obtain the asymptotic expansions of the solutions of a nonlinear differential equation arising in general relativity. The asymptotic solutions contain no secular term, which overcomes a defect in Khuri's paper. A technique of numerical order verification is applied to demonstrate that the asymptotic solutions are uniformly valid for small parameter.
Keywords- perturbation method, asymptotic solution, numerical verification, Lindstedt-Poincare method

1. INTRODUCTION

Asymptotic solutions for nonlinear differential equations by perturbation methods are formally in the form of a power series of small parameter ε:

$$u(t)=\sum_{n=0}^{\infty}\varepsilon^{n}u_{n}=u_{0}(t)+\varepsilon u_{1}(t)+\varepsilon^{2}u_{2}(t)+\cdots \tag{1}$$

Usually, the N-th order approximation is to choose a truncation of Eq.(1):

$$u_{asym}(t)=\sum_{n=0}^{N}\varepsilon^{n}u_{n}=u_{0}(t)+\varepsilon u_{1}(t)+\varepsilon^{2}u_{2}(t)+\cdots+\varepsilon^{N}u_{N}(t) \tag{2}$$

A major objective of perturbation methods is to show that the obtained expansion (1) is uniformly valid, that is, the error committed by truncating the series after N+1 terms is of order $O(\varepsilon^{N+1})$. Typically, one chooses a few test cases and compares the asymptotic solution to either the exact solution (if available) or the numerical solution to show that the error is relatively small. However, such few comparisons are insufficient to demonstrate the asymptotic expansion is accurate to a specified order. Although the quantitative error may be small, it does not become small at the rate expected [1, 2]. Therefore, one needs to further verify that the solution is indeed asymptotic accurate to the order to which it is constructed.

This paper will focus on a nonlinear differential equation arising in general relativity [3]:

$$\ddot{u}+u-\frac{\varepsilon}{c}u^{2}=c \tag{3}$$

where dots denote differentiation with respect to t, and c is a constant. The order of the accuracy of the asymptotic expansions of Eq.(3) has been verified by Khuri [4], but we note that the reversion method is adopt there and the consequent expansion contains a secular term εt sint, which is effective only for small t and small parameter ε. In fact, it is easy to verify that the first order approximate solution in Ref.[4] is no longer effective when $\varepsilon t>0.5$. In this paper, firstly, the Lindstedt-Poincare method [1] is used to obtain the asymptotic expansions of the solutions of Eq.(3), which is effective for small ε and arbitrary t. Secondly, a technique of numerical order verification, first proposed by Bosley [2], is applied to verify the quantitative accuracy as well as the order of the accuracy of the asymptotic solutions. Furthermore, a modification of the technique of Bosley is also made. Instead of only one fixed point $t=t_0$ in Refs.[2, 4-7], an average error is defined at finite fixed points $t=t_i(i=1,2,\ldots,m)$ to give a proper evaluation of the error between the asymptotic and numerical solutions. Finally, numerical verification shows that the asymptotic solutions are uniformly valid for small parameter ε.

2. ASYMPTOIC EXPANSIONS OF SOLUTIONS

Consider Eq.(3) with the initial conditions

$$u(0) = a,\ \dot{u}(0) = 0 \tag{4}$$

According to the standard Lindstedt-Poincare method [1], a new variable

$$\tau=\omega t \tag{5}$$

is introduced, in which ω is frequency of the system. Eq.(3) then becomes

$$\omega^2 u'' + u - \frac{\varepsilon}{c} u^2 = c \tag{6}$$

where primes denote differentiation with respect to τ. Both unknowns u and ω are usually expanded in powers of ε, that is

$$u_{asym} = \sum_{n=0}^{\infty} \varepsilon^n u_n(\tau) \tag{7}$$

$$\omega = \sum_{n=0}^{\infty} \varepsilon^n \omega_n = 1 + \varepsilon\omega_1 + \varepsilon^2 \omega_2 + \cdots \tag{8}$$

Substituting Eqs.(7) and (8) into Eq.(6) and equating coefficients of like powers of ε yields the first three equations:

$$u_0'' + u_0 = c \tag{9}$$

$$u_1'' + u_1 = \frac{1}{c} u_0^2 - 2\omega_1 u_0'' \tag{10}$$

$$u_2'' + u_2 = \frac{2}{c} u_0 u_1 - (2\omega_2 + \omega_1^2) u_0'' - 2\omega_1 u_1'' \tag{11}$$

The initial conditions (4) are replaced by

$$u_0(0) = a, \quad u_0'(0) = 0 \tag{12}$$

$$u_n(0)=0,\ \ u'_n(0)=0,\ \ n=1,2,\cdots \tag{13}$$

The solution of Eq.(9) with initial conditions (12) is

$$u_0=a\cos\tau+c \tag{14}$$

Then Eq.(10) becomes

$$u_1''+u_1=\frac{a^2}{2c}(1+\cos 2\tau)+c+2a(1+\omega_1)\cos\tau \tag{15}$$

To eliminate the secular term from the particular solution for u_1, the coefficient of $\cos\tau$in the right-hand side of Eq.(15) must vanish, that is, $2a(1+\omega_1)=0$, which is used to determine $\omega_1=-1$. Then the solution of Eq.(10) with initial conditions (13) (n=1) is

$$u_1=\frac{1}{6c}(3a^2+6c^2-(2a^2+6c^2)\cos\tau-a^2\cos 2\tau) \tag{16}$$

Substituting Eqs.(14) , (16) and $\omega_1=-1$ into Eq.(11), one obtains

$$u_2''+u_2=\frac{1}{6c^2}(-2a^3+6a^2c-6ac^2+12c^3+(5a^3+18ac^2+12ac^2\omega_2)\cos\tau$$
$$+(-2a^3+6a^2c-6ac^2)\cos 2\tau-a^3\cos 3\tau) \tag{17}$$

Similarly, to eliminate the secular term from the particular solution for u_2, the coefficient of $\cos\tau$ in the right-hand side of Eq.(17) must vanish, that is, $5a^3+18ac^2+12ac^2\omega_2=0$, which is used to determine $\omega_2=-\frac{3}{2}-\frac{5a^2}{12c^2}$. Then the solution of Eq.(11) with initial conditions (13) (n=2) is

$$u_2=\frac{1}{144c^2}(-48a^3+144a^2c-144ac^2+288c^3$$
$$+(29a^3-96a^2c+96ac^2-288c^3)\cos\tau$$
$$+(16a^3-48a^2c+48ac^2)\cos 2\tau+3a^3\cos 3\tau) \tag{18}$$

Therefore, the second order approximate solution of Eqs.(3) and (4) is:

$$u_{asym}(t)=u_0+\varepsilon u_1+\varepsilon^2 u_2+O(\varepsilon^3) \tag{19}$$

where u_0, u_1, and u_2 are determined by Eqs.(14), (16) and (18) respectively, and

$$\omega=1-\varepsilon+(-\frac{3}{2}-\frac{5a^2}{12c^2})\varepsilon^2. \tag{20}$$

3. NUMERICAL ORDER VERIFICATION OF ASYMPTOTIC EXPANSIONS

We first give a brief introduction of the technique of Bosley [2]. Assume that the asymptotic expansion of the solution of a nonlinear equation is

$$u_{asym}(t,\varepsilon)=u_0(t)+\varepsilon u_1(t)+\varepsilon^2 u_2(t)+\cdots+\varepsilon^N u_N(t)+O(\varepsilon^{N+1})\,. \tag{21}$$

The error of the asymptotic expansion is

$$\begin{aligned}\text{Error} &= \mathrm{E_N}(t,\varepsilon)=\left|u_{exact}(t,\varepsilon)-u_{asym}(t,\varepsilon)\right| \\ &= \left|u_{exact}(t,\varepsilon)-\sum_{n=0}^{N}\varepsilon^n u_n(t)\right| = O(\varepsilon^{N+1}) = K\varepsilon^{N+1}\end{aligned} \tag{22}$$

where K is a constant. Taking the logarithm of both sides of Eq.(22) yields

$$\log(\text{Error}) = \log(E_N) = \log K + (N+1)\log\varepsilon\,.$$

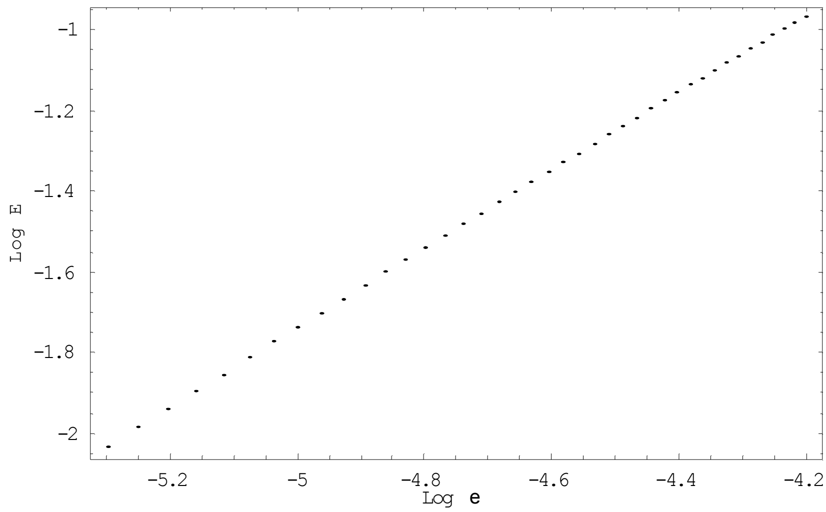
If E_N is of order $O(\varepsilon^{N+1})$ for a fixed $t=t_0$ and for small values of ε, the value of $\log(E_N)$ as a function of log ε should be linear with slope N+1. Therefore, when we graph $\log(E_N)$ versus log ε for different values of ε, these points should be nearly on a line and the linear equation that interpolates these points using a linear least-squares fit should have slope N+1.

In this paper, instead of Eq.(22) with a fixed $t=t_0$, an average error is introduced:

$$\text{Average Error} = \overline{E}_N(t,\varepsilon)=\frac{1}{m}\sum_{i=1}^{m}E_N(t_i,\varepsilon) \tag{23}$$

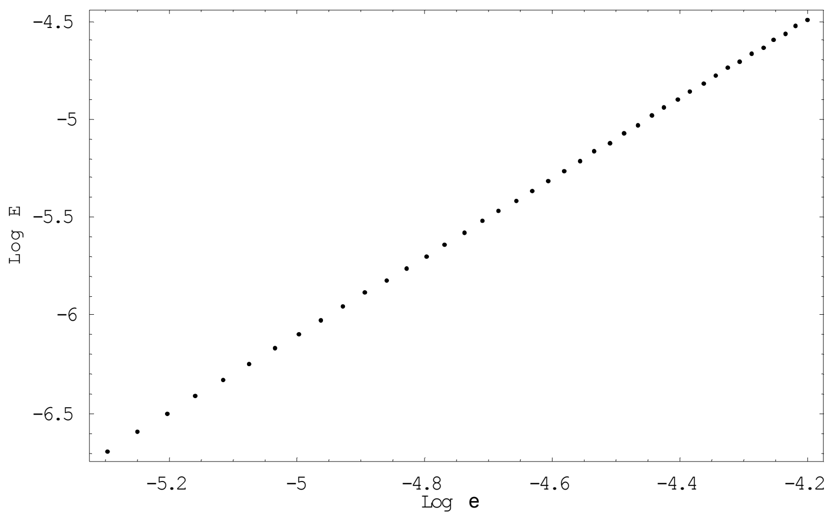
where t_i are fixed points in the concerned domain of t. Such modification can give a better overall estimation of difference between the exact (or numerical) and asymptotic solutions on the domain of interest.

To verify the order of asymptotic expansion (19), we first find the numerical solutions of Eqs.(3) and (4) with a=1, c=1, t_i =i, i=1,2,…,80, and ε starting from 0.005 and ending at 0.015 by a step size 0.00025. Next, we evaluate the asymptotic expansion (19) at the same values of ε and t_i as the numerical solutions for N=0,1 and 2 respectively. Figs.1-3 plot respectively the values of the errors at these 41 points, namely, $\log\overline{E}_0$, $\log\overline{E}_1$ and $\log\overline{E}_2$ as functions of logε. The exact solution $u_{exact}(t,\varepsilon)$ in Eq.(22) is replaced by the numerical solution. For N=0,1 and 2, the least-square fit of the data is used to determine the slopes 0.972435, 2.00396 and 3.01346 respectively, which are in excellent agreement with the exact slopes of N+1=1,2 and 3, respectively. In this paper, software MATHEMATICA is applied to implement relative calculations and plots.



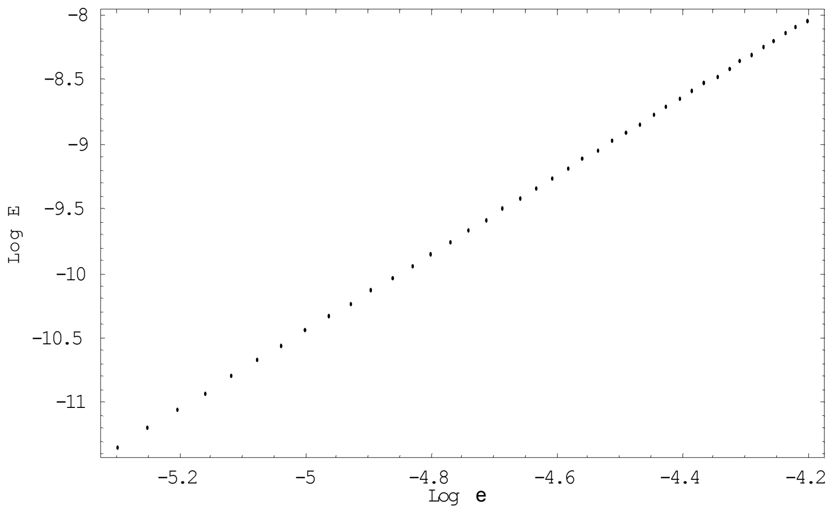
$$\log E = 3.12072 + 0.972435 \log \varepsilon$$

Fig.1 Order verification of the asymptotic expansion (19) using one term $u_{asym} = u_0$



$$\log E = 3.91999 + 2.00396 \log \varepsilon$$

Fig.2 Order verification of the asymptotic expansion (19) using two terms $u_{asym} = u_0 + \varepsilon u_1$



$$\log E = 4.61533 + 3.01346 \log \varepsilon$$

Fig.3 Order verification of the asymptotic expansion (19) using three terms

$$u_{asym} = u_0 + \varepsilon u_1 + \varepsilon^2 u_2$$

4. CONCLUTIONS

The asymptotic expansions of the solutions of the nonlinear differential equation, obtained by the Lindstedt-Poincare method, are uniformly valid for small parameter ε and arbitrary t, which overcomes the defect in Ref.[4].

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