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COUPLING OF FINITE AND BOUNDARY ELEMENT METHODS WITH INCOMPATIBLE INTERFACES

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Abstract- In this study, an algorithm is presented for the coupling of finite and boundary element method with incompatible interfaces for two dimensional elasticity problems. In this approach, the number of nodes at the interface line of finite element domain can be different from the number of nodes at the interface line of boundary element region. The subregion technique has been used to obtain the coupling model and the compatibility requirement is satisfied by distributing the forces for incompatible interface. The continuity requirement is satisfied by interpolation equations. After satisfying the continuity and compatibility requirements at the interface line, the distribution matrix has been used for transformation of nodal forces into nodal tractions. Two different case studies have been solved and the results are compared with each other, FEM, BEM, ANSYS.
Keywords- Coupling, FEM, BEM, incompatible interfaces.

1. INTRODUCTION

The main purpose of coupling of finite element (FE) method (FEM) and boundary element (BE) method (BEM) is to use the advantages of both methods for the solution of various engineering problems. Both methods have some advantages for certain applications. While the BEM gives better results for the surface type problems like contact problems, FEM is more effective technique for the domain type problems.

Zienkiewicz et al.[1] are one of the frontiers of coupling process. They discussed the coupling in a general context. Then, Kelly et al.[2] have proposed a method obtaining the symmetric stiffness matrices of the BE region which satisfy the equilibrium equation and applied the method to a number of field problems in fluid mechanics. Furthermore Felippa [3] used the coupling methods for a three dimensional structure submerged in an acoustic fluid.

Later, a number of researchers have been studied on coupling of BE and FE [4-25]. Finally, authors have been proposed a different approach for coupling process [26, 27]. The common point of previous studies is to use equal interface nodes for FE and BE regions. However, a coupling procedure can be developed for incompatible interfaces.

2. COUPLING FOR COMPATIBLE INTERFACES

Although the subregion is a technique in BEMs [25], coupling process may be thought as a subregional technique due to the division of problem domain into the two subdomains as BE and FE. The general form of FE equation can be written as follows:

$$\underline{F} = \underline{K}\,\underline{u} \tag{1}$$

It can be rewritten in a form including FE domain and interface sub-matrices as:

$$\begin{vmatrix} \underline{F}_{FE} \\ \underline{F}^{*}_{FE_i} \end{vmatrix} = \begin{vmatrix} \underline{K}_{FE} & \underline{K}_{FE_i} \end{vmatrix} \begin{vmatrix} \underline{u}_{FE} \\ \underline{u}^{*}_{FE_i} \end{vmatrix} \tag{2}$$

Where $\underline{F}_{FE}$ is the force vector at the FE domain, $\underline{F}^*_{FE\,i}$ is the internal reaction force vector at the interface line, $\underline{u}_{FE}$ is the displacement vector at the FE domain and $\underline{u}^*_{FE\,i}$ is the internal displacement vector at the interface line.

The above equation may be solved for internal reaction force vector, $\underline{F}^*_{FE_i}$

After finding the internal reaction force vector at the interface line, the ordinary distribution matrix, $\underline{M} = \int_{s_e} \underline{N}^t \, \underline{N} \, ds$, may be used to transform it into the internal traction vector as follows:

$$\underline{F}^*_{FE_i} = \underline{M} \; \underline{t}^*_{FE_i} \tag{3}$$

The Equation (2) can be rewritten in a new form which is similar to general BE equation form using the Equation (3);

$$\begin{vmatrix} \underline{I} & \underline{M}_{FE_i} \end{vmatrix} \begin{vmatrix} \underline{F}_{FE} \\ \underline{t}^*_{FE_i} \end{vmatrix} = \begin{vmatrix} \underline{K}_{FE} & \underline{K}_{FE_i} \end{vmatrix} \begin{vmatrix} \underline{u}_{FE} \\ \underline{u}^*_{FE_i} \end{vmatrix} \tag{4}$$

Where $\underline{I}$ is the unit matrix, $\underline{M}_{FE_i}$ is the distribution matrix at the interface line, $\underline{F}_{FE}$ is the force vector at the FE domain and $\underline{t}^*_{FE}$ is the internal traction vector at the interface line.

The general BE equation is as follows;

$$\underline{H}\,\underline{u} = \underline{G}\,\underline{t} \tag{5}$$

It can also be rewritten including BE and interface BE sub-matrix as follows:

$$\begin{vmatrix} \underline{H}_{BE\,i} & \underline{H}_{BE} \end{vmatrix} \begin{vmatrix} \underline{u}_{BE_i} \\ \underline{u}_{BE} \end{vmatrix} = \begin{vmatrix} \underline{G}_{BE_i} & \underline{G}_{BE} \end{vmatrix} \begin{vmatrix} \underline{t}^*_{BE_i} \\ \underline{t}_{BE} \end{vmatrix} \tag{6}$$

Where $\underline{t}^*_{BE_i}$ is the internal traction vector and $\underline{u}_{BE_i}$ is the real displacement vector.

The traction equilibrium must be satisfied for coupling at the interface line as follows:

$$\underline{t}^*_{FE_i} = -\underline{t}^*_{BE_i} \tag{7}$$

Then the Equation (4) may be solved for real displacement vector, $\underline{u}_{BE_i}$. After finding the real displacement vector at the interface line, the displacement continuity requirement can be satisfied for coupling purpose.

$$\underline{u}_{FE_i} = \underline{u}_{BE_i} = \underline{u}_i \tag{8}$$

As a result, the general coupling equation can be written using Equation (4) and (6);

$$\begin{vmatrix} \underline{K}_{FE} & \underline{K}_{FE_i} & \underline{0} \\ \underline{0} & \underline{H}_{BE_i} & \underline{H}_{BE} \end{vmatrix} \begin{vmatrix} \underline{u}_{FE} \\ \underline{u}_i \\ \underline{u}_{BE} \end{vmatrix} = \begin{vmatrix} \underline{I} & \underline{M}_{FE_i} & \underline{0} \\ \underline{0} & -\underline{G}_{BE_i} & \underline{G}_{BE} \end{vmatrix} \begin{vmatrix} \underline{F}_{FE} \\ \underline{t}_i \\ \underline{t}_{BE} \end{vmatrix} \tag{9}$$

Due to the formulation of Equation (9), the only interface part of FE force vector has been converted to tractions so the remaining part has been kept as original force vector.

3. COUPLING FOR INCOMPATIBLE INTERFACES

Due to continuity and compatibility requirements at the interface line, the number of nodes must be equal to each other for FE and BE interface in the ordinary coupling procedure. However, it can be different and continuity and compatibility can also be satisfied for incompatible interfaces using the method developed in this study. Basically, in a coupling model, there are three different cases. In the first case, the number of nodes of FE and BE sides are equal to each other. In the second case, the number of nodes of FE interface can be greater than the number of nodes of boundary interface. The last case is the reverse of second case as shown in Figure 1.

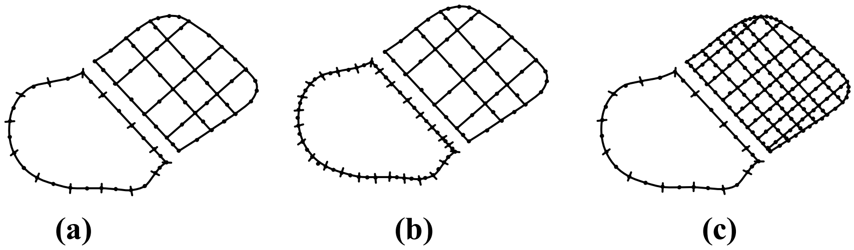

Figure 1 (a) Ordinary coupling model ,

(b) Coupling model-1 with uniform meshes,

(c) Coupling model-2 with uniform meshes.

In the first case, the ordinary coupling procedures can be used as discussed in Section 2. In the second and third cases, however, the force equilibrium can be satisfied as discussed in the Section 3.1. The displacement continuity can also be satisfied by following the procedure discussed in the Section 3.2.

3.1. Force Distribution

Subregional coupling technique gives independent solutions for boundary element and finite element region. Because of this, each internal reaction force found by the FE region can be treated as concentrated forces given on this point. A concentrated force can also be considered as a resultant force of a pressure on one side of a two-dimensional finite element.

The pressure acting at an infinitesimal length (ds), as shown in Figure 2, can be expressed as follows;

$$dF = -P\, ds\, h\, \hat{n} \tag{10}$$

Where h is the thickness of the element in the z-direction, $\hat{n}$ is the unit vector in the direction of outward normal to ds.

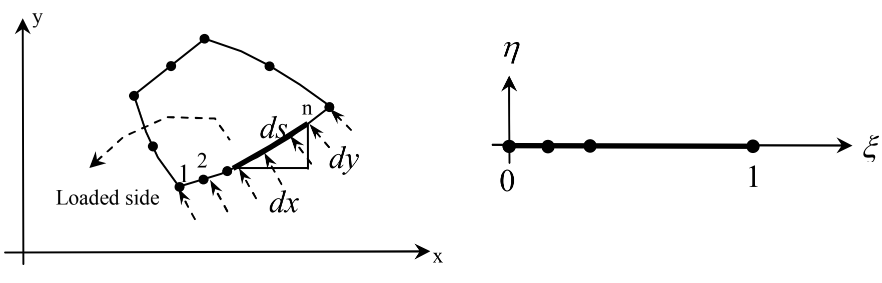

Figure 2 A pressure loaded 2D finite element from one side.

Hence, the virtual work done by the infinitesimal increment of the pressure load, dF, on the virtual displacement δq can be expressed as follows;

$$\delta(dW) = d\vec{F}\,\delta\vec{q} = P\,h\,(-\delta u\,dy + \delta v\,dx) \quad \text{or} \quad \delta W = \int\limits_{\substack{pressure\\element}} P\,h\,(-\delta u\,dy + \delta v\,dx) \tag{11}$$

The virtual displacements can be interpolated over pressurised line as follows;

$$\delta u = \sum_{i=1}^{n} \delta u_i N_i(\xi) \quad \text{and} \quad \delta v = \sum_{i=1}^{n} \delta v_i N_i(\xi) \tag{12}$$

Equation (11) can be written explicitly as follows;

$$\delta W = \sum_{i=1}^{n} \left(\int P\,h\,N_i(-\delta u_i\,dy + \delta v_i\,dx)\right) \tag{13}$$

The work done by the actual forces during any virtual displacement can be written as follows;

$$\delta W = \delta \underline{q}^t\, \underline{F} \tag{14}$$

and it can be written for a 2-D finite element as follows;

$$\delta W = \sum_{i=1}^{n} (Fx_i\,\delta u_i + Fy_i\,\delta v_i) \tag{15}$$

By comparing Equations (13) and (15), it can be proved that

$$Fx_i = -\int P\,h\,N_i(\xi)\,dy \quad \text{and} \quad Fy_i = \int P\,h\,N_i(\xi)\,dx \tag{16}$$

They can be written in an explicit form as follows;

$$Fx_i = -\int_0^1 P\,h\,N_i(\xi)\left(\frac{dy}{d\xi}\right)d\xi \quad \text{and} \quad Fy_i = \int_0^1 P\,hN_i(\xi)\left(\frac{dx}{d\xi}\right)d\xi \tag{17}$$

For a 2-noded element, they are equal to

$$Fx_1 = -\frac{1}{2}P\,h\,l_y = -\frac{1}{2}R\,h \quad \text{and} \quad Fx_2 = -\frac{1}{2}P\,h\,l_y = -\frac{1}{2}R\,h$$

Where l_y is the length of the element and R is the resultant force acting on a point on the pressurized element. So that each internal reaction force found by FE region can be distributed to nodes for incompatible interfaces using Equations (21) and (22). For a 3-noded element, they are equal to

$$Fx_1 = -\frac{1}{6}R\,h, \quad Fx_2 = -\frac{4}{6}R\,h \quad \text{and} \quad Fx_3 = -\frac{1}{6}R\,h$$

and similar equations can be derived for y-components. Internal reaction forces can be distributed in the same manner when the FE interface nodes are greater or smaller than BE interface nodes.

3.2. Displacement Continuity

When the number of node of FE interface is smaller then the BE interface nodes (Figure 3-a), the displacement continuity is satisfied in matching nodes. In the Figure 3-a, the displacements, $u_1^b,...,u_5^b$ are known internal displacements and found by BE region.

$$u_1^f = u_1^b \qquad u_2^f = u_3^b \qquad u_3^f = u_5^b$$

In the reverse case (Figure 3-b), however, the displacements, u_1^f, u_2^f, u_3^f are known internal displacements and found by FE region. The number of known displacements are not enough to satisfy the displacement continuity. So that well-known interpolation functions can be used to satisfy the displacement continuity as follows:

$$u_1^f = u_1^b \qquad u_3^f = u_2^b \qquad u_5^f = u_3^b$$

$$u_2^f(\xi_2) = \sum_{i=1}^{3} u_i^b\, N_i(\xi) \quad \text{and} \quad u_4^f(\xi_4) = \sum_{i=1}^{3} u_i^b\, N_i(\xi)$$

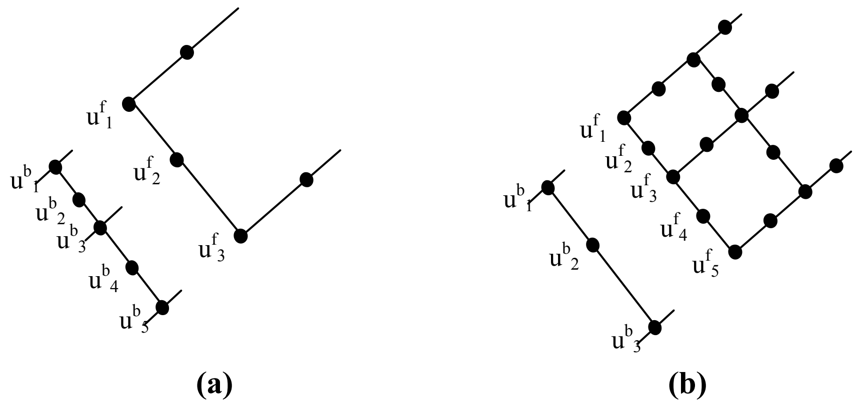

Figure 3 FE and BE displacements for incompatible interface line.

4. CASE STUDIES

Two different cases have been used for the validations of the developed approaches and three different coupling models are used for each case. They refers equal number of nodes (CM1), BE interface nodes are greater than FE interface nodes (CM2) and FE interface nodes are greater than BE interface nodes (CM3).

4.1. Axially Loaded Square Plate

This is a simple axially loaded case as plane stress problem. The dimensions and models are shown in the Figure 4. Linear elements are used in models. In this case an extra coupling model (CM4) is used to show the effect of nonuniformed boundary elements in a coupling model. It contains serious erros as seen in Figure 5 and 6. All other models have exactly same results. So nonuniformed BE meshes should not be preferred in coupling models.

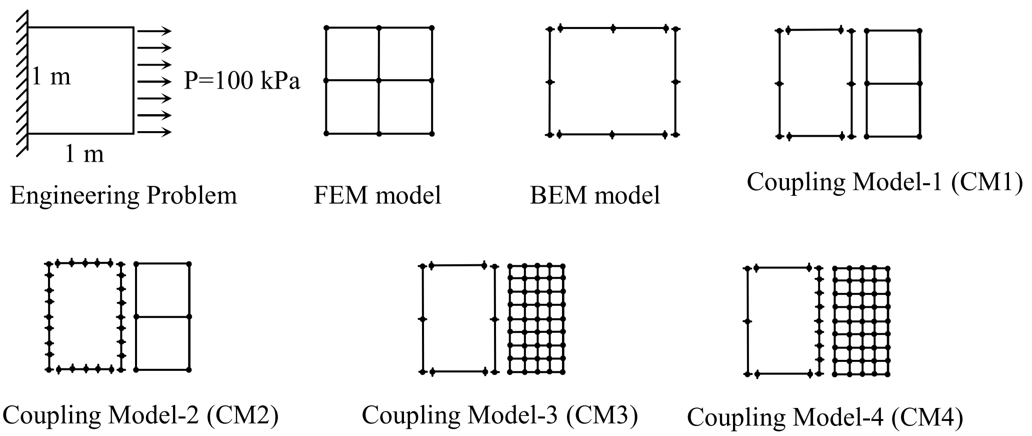

Figure 4 Axially loaded plate and its FEM, BEM and coupling models.

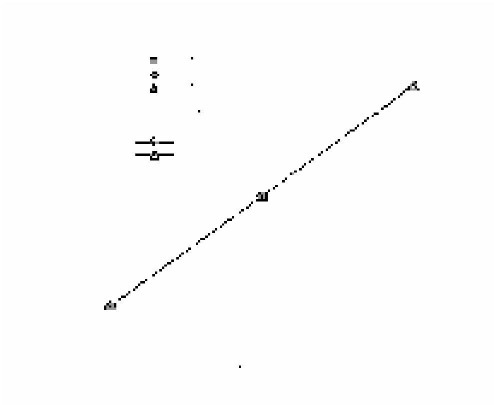

Figure 5 Ux distribution along upper surface of bar.

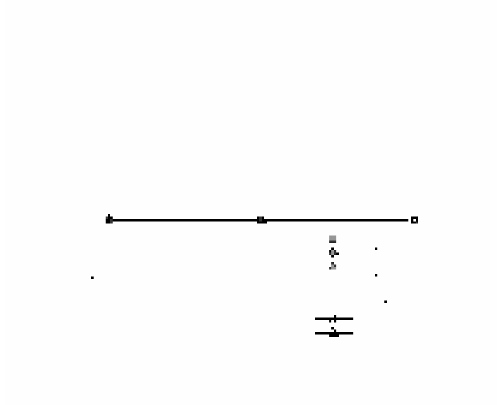

Figure 6 Sx distribution along upper surface of bar.

4.2. Cantilever Beam with a Distributed Load

This case represents a steel cantilever beam under the action of a linearly distributed load (Figure 7). The vertical displacements and axial stress distributions along the upper surface of the beam can be seen in Figure 8 and 9. All methods are in good agreement for the vertical displacement distribution. In the axial stress distribution, CM1 results are improved using CM2 and CM3 around the interface line.

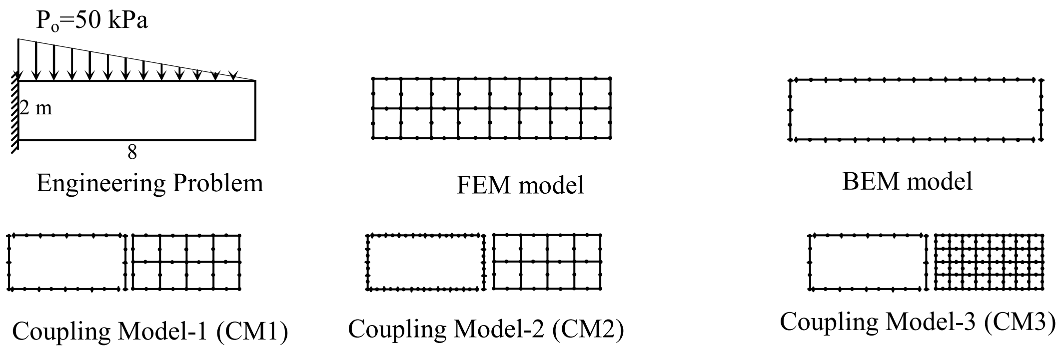

Figure 7 Rectangular plate with a distributed load and its FEM, BEM and coupling models.

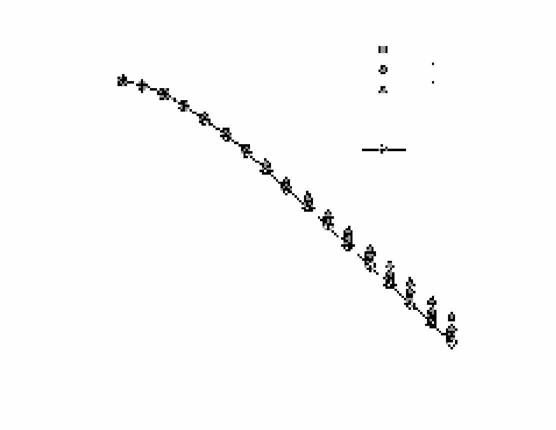

Figure 8 Uy distribution along upper surface of beam.

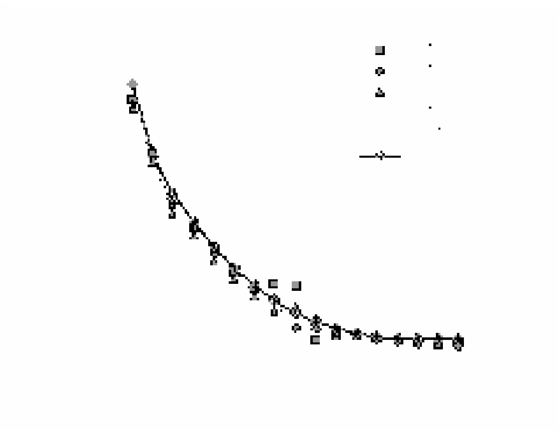

Figure 9 Sx distribution along upper surface of beam.

4.3. Slideway Base

In this case, a slideway base under the action of weight of the inner part is considered. The material of the base is the grey cast iron. Because of the symmetry, half base part is modelled as shown in Figure 10 [24]. All methods are in good agreement in the axial stress and vertical displacement distributions along line AB (Figure 11 and 12). However, the CM1 shows some errors in the vertical stress distribution along line CE. It’s errors are reduced in CM2 and CM3 (Figure 13). In the axial stress distribution, CM3 results more accurate than other coupling models along the same line (Figure 14). In the vertical stress distribution along line BDF, CM2 and CM3 includes less error than CM1 (Figure 15). CM2 and CM3 have similar improvements for CM1 results in the horizontal displacement distributions along line BDF as shown in Figure 16.

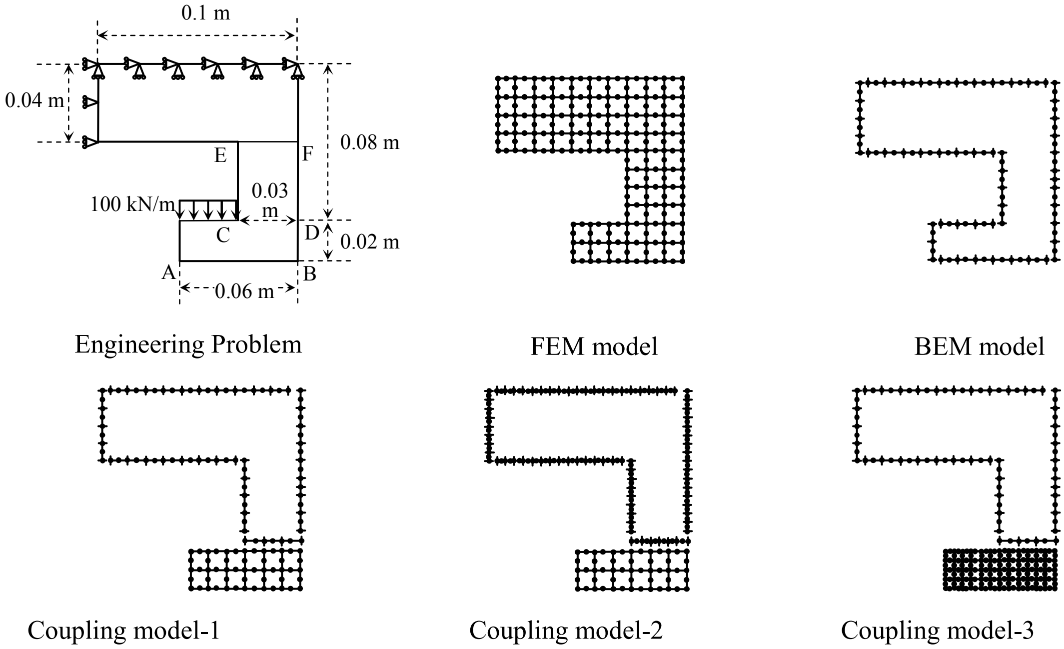

Figure 10 Slideway base and its FEM, BEM and coupling models.

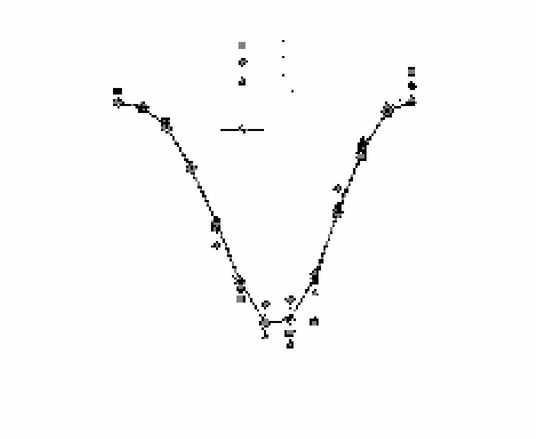

Figure 11 Sx distribution along line AB.

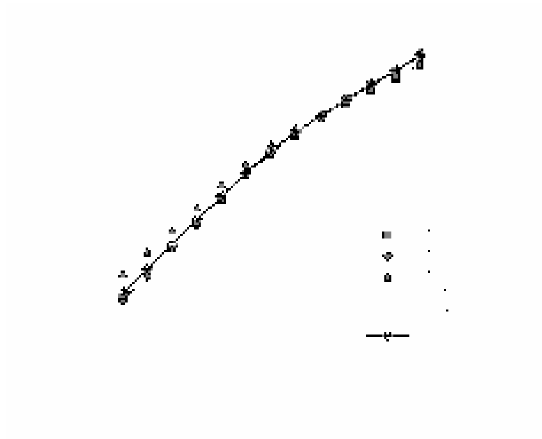

Figure 12 Uy distribution along line AB.

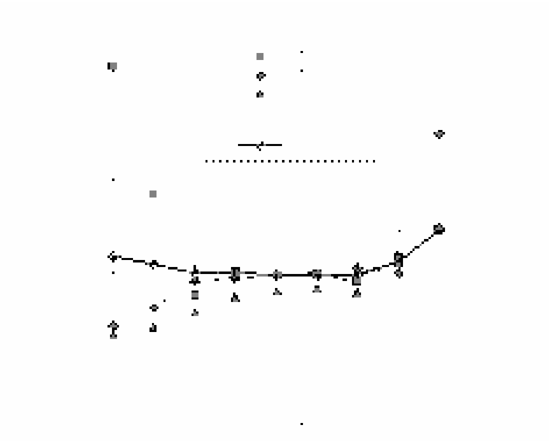

Figure 13 Sy distribution along line CE.

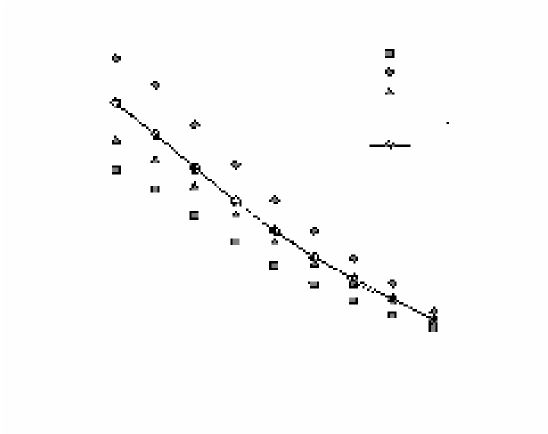

Figure 14 Ux distribution along line CE.

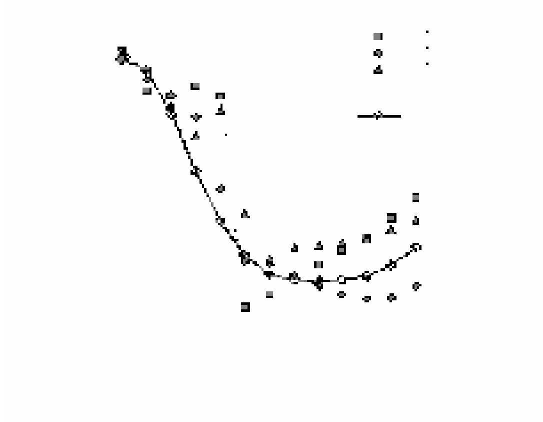

Figure 15 Sy distribution along line BDF.

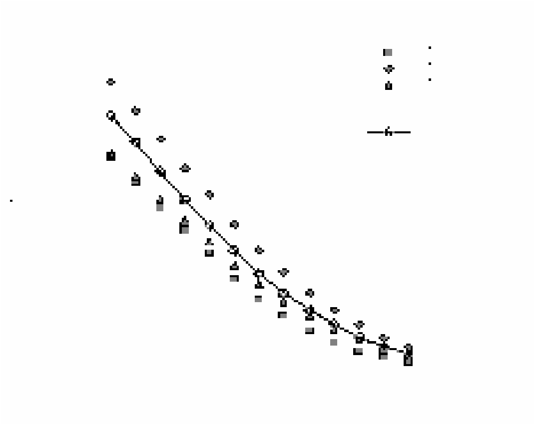

Figure 16 Ux distribution along line BDF.

5. CONCLUSION

It has been proved that, the coupling procedure of finite and boundary element methods may be also carried with incompatible interfaces. The result of ordinary coupling method may be improved by independent mesh refinements in both regions. The results of coupling models depend on the correctness of the internal forces found by FE solutions. So the local mesh refinements may be achieved for the FE meshes without changing BE meshes with developed method. All of the problems used in this work are 2-D elasticity problems. So the idea may be extended for plasticity, coupled and other types of problems.

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