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EVALUATION OF CERTAIN CLASS OF EULERIAN INTEGRALS OF MULTIVARIABLE SISTER CELINE'S POLYNOMIALS

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Abstract-In this paper, we evaluate a key Eulerian integral involving Sister Celine's polynomials of several complex variables defined by the author. Our general Eulerian integral formula are shown to provide the key formula from which numerous other potentially useful results involving polynomials such as Jacobi, Laguerre, Hermite, Bessel, Bateman, Rice etc and also discrete polynomials like Hahn, Krawtchouk, Pasternak, Meixner, Poisson-Charlier are derived.

Keywords- Sister Celine's polynomials, Eulerian integrals, Lauricella function, Generalized hypergeometric series.

1. INTRODUCTION

The well-known Eulerian Beta integral

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt=\frac{\Gamma(\alpha)\ \Gamma(\beta)}{\Gamma(\alpha+\beta)}\ ,\quad (where\ \mathrm{Re}(\alpha),\ \mathrm{Re}(\beta)>0) \tag{1.1}$$

Can be rewritten (by suitably manoeuvred) in the form

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}dt=(b-a)^{\alpha+\beta-1}B(\alpha,\beta)\ ,\qquad (where\ \mathrm{Re}(\alpha),\ \mathrm{Re}(\beta)>0, a\neq b) \tag{1.2}$$

Sister M. Celine Fasenmyer [3,4 ; see also 3, p.290] in 1947 defined a polynomials known as Sister Celine's polynomials generated by

$$(1-t)^{-1}\,{}_pF_q\left(\begin{matrix}a_1,...,a_p\\ b_1,...,b_q\end{matrix};\frac{-4xt}{(1-t)^2}\right)=\sum_{n=0}^{\infty} f_n\left(\begin{matrix}a_1,...,a_p\\ b_1,...,b_q\end{matrix};x\right)t^n\ , \tag{1.3}$$

which yields

$$f_n\left(\begin{matrix}a_1,...,a_p\\ b_1,...,b_q\end{matrix};x\right)={}_{p+2}F_{q+2}\left(\begin{matrix}-n,n+1,a_1,...,a_p\\ \frac{1}{2},1,b_1,...,b_q\end{matrix};x\right)\ , \tag{1.4}$$

Author [12] has extended the polynomials and defined the Sister Celine's polynomials of two and more complex variables interms of generating function as follows :

$$\sum_{\substack{n_i=0\\(i=1,...,r)}}^{\infty} f_{n_1,...,n_r}\left(\begin{matrix}(a_j:\alpha'_j,...,\alpha_j^{(r)})_{1,p}:(c',\gamma'_j)_{1,p_1};...;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\\ (b_j:\beta'_j,...,\beta_j^{(r)})_{1,q}:(d',\delta'_j)_{1,q_1};...;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix};z_1^{\lambda_1},...,z_r^{\lambda_r}\right)\ t_1^{n_1}...t_r^{n_r}$$

$$=\prod_{i=1}^{r}(1-t_i)^{-1-c_i}F^{p:p_1;...;p_r}_{q:q_1;...;q_r}\left(\begin{matrix}(a_j:\alpha'_j,...,\alpha_j^{(r)})_{1,p}:(c',\gamma'_j)_{1,p_1};...;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\\ (b_j:\beta'_j,...,\beta_j^{(r)})_{1,q}:(d',\delta'_j)_{1,q_1};...;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix};\frac{(-4x_1t_1)^{\lambda_1}}{(1-t_1)^{2\lambda_1}},...,\frac{(-4x_rt_r)^{\lambda_r}}{(1-t_r)^{2\lambda_r}}\right), \tag{1.5}$$

which gives

$$f_{n_1,...,n_r}(z_1^{\lambda_1},...,z_r^{\lambda_r})\equiv f_{n_1,...,n_r}\left(\begin{matrix}(a_j:\alpha'_j,...,\alpha_j^{(r)})_{1,p}:(c',\gamma'_j)_{1,p_1};...;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\\ (b_j:\beta'_j,...,\beta_j^{(r)})_{1,q}:(d',\delta'_j)_{1,q_1};...;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix};z_1^{\lambda_1},...,z_r^{\lambda_r}\right)$$

$$=\prod_{i=1}^{r}\frac{(1+c_i)_{n_i}}{(n_i)!}F^{p:p_1+2;...;p_r+2}_{q:q_1+2;...;q_r+2}\left(\begin{matrix}(a_j:\alpha'_j,...,\alpha_j^{(r)})_{1,p}:(-n_1,\lambda_1),(1+n_1+c_1,\lambda_1),(c',\gamma'_j)_{1,p_1};...;\\ (b_j:\beta'_j,...,\beta_j^{(r)})_{1,q}:\left(\frac{1+c_1}{2},\lambda_1\right),\left(\frac{2+c_1}{2},\lambda_1\right),(d',\delta'_j)_{1,q_1};...;\end{matrix}\right.$$

$$\left.\begin{matrix}(-n_r,\lambda_r),(1+n_r+c_r,\lambda_r),(c_j^{(r)},\gamma_j^{(r)})\\ (\frac{1+c_r}{2},\lambda_r),(\frac{2+c_r}{2},\lambda_r),(d_j^{(r)},\delta_j^{(r)})\end{matrix};z_1^{\lambda_1},\dots,z_r^{\lambda_r}\right), \tag{1.6}$$

In (1.5) and (1.6), if we take r = 2 and replace $n_1,n_2,\lambda_1,\lambda_2,z_1,z_2,t_1,t_2,c_1,c_2$ respectively by $n,m,\lambda,\mu,x,y,t,h,c,d,$ we get the definition of Sister Celine's polynomials of two variables as follows :

$$\sum_{n,m=0}^{\infty} f_{n,m}\left(\begin{matrix}(a_j:\alpha_j',\alpha_j'')_{1,p}:(c',\gamma_j')_{1,p_1};(c_j'',\gamma_j'')_{1,p_2}\\ (b_j:\beta_j',\beta_j'')_{1,q}:(d',\delta_j')_{1,q_1};(d_j'',\delta_j'')_{1,q_2}\end{matrix};x^{\lambda}y^{\mu}\right) t^n h^m$$
$$=(1-t)^{-1-c}(1-h)^{-1-d}F^{p:p_1;p_2}_{q:q_1;q_2}\left(\begin{matrix}(a_j:\alpha_j',\alpha_j'')_{1,p}:(c',\gamma_j')_{1,p_1};(c_j'',\gamma_j'')_{1,p_2}\\ (b_j:\beta_j',\beta_j'')_{1,q}:(d',\delta_j')_{1,q_1};(d_j'',\delta_j'')_{1,q_2}\end{matrix};\frac{(-4xt)^{\lambda}}{(1-t)^{2\lambda}},\frac{(-4yh)^{\mu}}{(1-h)^{2\mu}}\right), \tag{1.7}$$

and

$$f_{n,m}(x^{\lambda},y^{\mu})\equiv f_{n,m}\left(\begin{matrix}(a_j:\alpha_j',\alpha_j'')_{1,p}:(c',\gamma_j')_{1,p_1};(c_j'',\gamma_j'')_{1,p_2}\\ (b_j:\beta_j',\beta_j'')_{1,q}:(d',\delta_j')_{1,q_1};(d_j'',\delta_j'')_{1,q_2}\end{matrix};x^{\lambda},y^{\mu}\right)$$
$$=\frac{(1+c)_n(1+d)_m}{n!\ m!}F^{p:p_1+2;p_2+2}_{q:q_1+2;q_2+2}\left(\begin{matrix}(a_j:\alpha_j',\alpha_j'')_{1,p}:(-n,\lambda),(1+n+c,\lambda),(c',\gamma_j')_{1,p_1};(-m,\mu),(1+m+d,\mu),(c_j'',\gamma_j'')_{1,p_2}\\ (b_j:\beta_j',\dots,\beta_j'')_{1,q}:\left(\frac{1+c}{2},\lambda\right),\left(\frac{2+c}{2},\lambda\right),(d',\delta_j')_{1,q_1};\left(\frac{1+d}{2},\mu\right),\left(\frac{2+d}{2},\mu\right),(d_j'',\delta_j'')_{1,q_2}\end{matrix};x^{\lambda},y^{\mu}\right), \tag{1.8}$$

where $\lambda,\mu,\lambda_1,\dots,\lambda_r$ are positive real number and $c,d,c_1,\dots,c_r$ are complex numbers. The variables are complex**.** If we put in (1.7) and (1.8) $p=q=m=c=d=0$ and $\lambda,\mu,\gamma's,\delta's$ to be unity, we at once obtain the Sister Celine's polynomials of single variable (1.3) and (1.4).The series involved in (1.5), (1.6); (1.7),(1.8) are well-known generalized Lauricella function of r-variables and 2-variables respectively defined by Srivastava and Daoust [13 see also 15, p.37] .

The multiple series of the Lauricella function and it's special case when r = 2 converge absolutely [14; section 5 (p.157-158), section 3,4 (p.153-157);1, section (3.7); 2, section 1.4]. It is worth to mention here that

(i) For the sake of space, in subsequent sections, we have derived the results of Sister Celine's polynomials of two variables instead of several variables.
(ii) The results are new and even for single variable, the results are also believed to be new results.
(iii)When each of the positive real numbers $\alpha's,\beta's,\gamma's,\delta's$ is equated to unity, the generalized Lauricella series (1.9) reduces to a multiple Kampe-de-Feriet series

2. EULERIAN INTEGRAL

In this section, we shall prove our main general Eulerian integral involving Sister Celine's polynomials of r – complex variables :

$$\prod_{i=1}^{r}\int_{a_i}^{b_i}(t_i-a_i)^{\alpha_i-1}(b_i-t_i)^{\beta_i-1}(u_it_i+v_i)^{e_i}(f_it_i+g_i)^{-s_i}\times f_{n_1,\dots,n_r}\left(\begin{matrix}(z_1x_1)^{k_1}\prod_{i=1}^{r}(u_it_i+v_i)^{\rho_i^{(1)}}(f_it_i+g_i)^{-\sigma_i^{(1)}}\\ (z_rx_r)^{k_r}\prod_{i=1}^{r}(u_it_i+v_i)^{\rho_i^{(r)}}(f_it_i+g_i)^{-\sigma_i^{(r)}}\end{matrix};\dots;\right)dt_1\dots dt_r$$
$$=\prod_{i=1}^{r}(b_i-a_i)^{\alpha_i+\beta_i-1}(a_iu_i+v_i)^{e_i}(b_if_i+g_i)^{-s_i}\sum_{\substack{\lambda_i,\mu_i=0\\(i=1,\dots,r)}}\prod_{i=1}^{r}\frac{(-e_i)_{\lambda_i}(s_i)_{\mu_i}}{\lambda_i!\ \mu_i!}B(\alpha_i+\lambda_i,\beta_i+\mu_i)$$
$$\times\left(\frac{-u_i(b_i-a_i)}{a_iu_i+v_i}\right)^{\lambda_i}\left(\frac{f_i(b_i-a_i)}{b_if_i+g_i}\right)^{\mu_i}f_{n_1,\dots,n_r}\left(\begin{matrix}a,(1+e_i:\rho_i^{(1)},\dots,\rho_i^{(r)})_{1,r}(s_i+\mu_i:\sigma_i^{(1)},\dots,\sigma_i^{(r)})_{1,r}:c';\dots;c^r\\ b,(1+e_i-\lambda_i:\rho_i^{(1)},\dots,\rho_i^{(r)})_{1,r}\quad(s_i:\sigma_i^{(1)},\dots,\sigma_i^{(r)})_{1,r}:d';\dots;d^r\end{matrix}\right);$$

$$(z_1x_1)^{k_1}\prod_{i=1}^{r} (a_iu_i+v_i)^{\rho_i^{(1)}}(b_if_i+g_i)^{-\sigma_i^{(1)}},\ldots,(z_rx_r)^{k_r}\prod_{i=1}^{r} (a_iu_i+v_i)^{\rho_i^{(r)}}(b_if_i+g_i)^{-\sigma_i^{(r)}}\Bigg), \tag{2.1}$$

provided min $\{\rho's,\sigma's\}>0$, min $\{\mathrm{Re}(\alpha_i),\mathrm{Re}(\beta_i\}>0$, $\max\left\{\left|\frac{u_i(b_i-a_i)}{a_iu_i+v_i}\right|,\left|\frac{f_i(b_i-a_i)}{b_if_i+g_i}\right|\right\}<1, a_i\neq b_i\,(i=1,\ldots,r)$.

Where $(1+e_i:\rho_i^{(1)},\ldots,\rho_i^{(r)})_{1,r}$ abbreviates the r- parameters array $(1+e_1:\rho_1^{(1)},\ldots,\rho_1^{(r)}),\ldots,(1+e_r:\rho_r^{(1)},\ldots,\rho_r^{(r)})$ and similarly others.

Proof : Taking LHS, expressing Celine's polynomials interms of series (1.6), changing the order of summation and integration which is justifiable , collecting the powers of $(u_it_i+v_i)$ and $(f_it_i+g_i)$ & using the binomial expansion :

$$(ut+v)^r=(au+v)^r\sum_{m=0}^{\infty}\frac{(-r)_m}{m!}\left(\frac{-u(t-a)}{au+v}\right)^m, (ut+v)^{-r}=(bu+v)^{-r}\sum_{n=0}^{\infty}\frac{(r)_n}{n!}\left(\frac{-u(t-b)}{bu+v}\right)^n,$$

evaluating inner integrals with the help of definition (1.2), using known relations :

$$(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \text{ and } (-n)_k=\frac{(-1)^K n!}{(n-k)!}$$

on respectively positive and negative factors, finally once again using the definition of Sister Celine's polynomials of r – variables (1.6), we obtain the required result.

3. APPLICATIONS

The generalized Eulerian integral (2.1) has manifold generality. By specializing the various parameters and variables involved, the formula (and indeed their several variations obtained by letting any desired number of exponents $\rho_1^{(1)},\ldots,\rho_1^{(r)};\sigma_1^{(1)},\ldots,\sigma_1^{(r)};\rho_r^{(1)},\ldots,\rho_r^{(r)};\sigma_r^{(1)},\ldots,\sigma_r^{(r)}$ decrease to zero in such a manner that both the sides of resulting equation exist) can suitably applied to derive the corresponding results involving a remarkably wide variety of useful polynomials (or product of several such polynomials). We have obtained few of them such as Jacobi, Laguerre, Hermite, Legendre, Gegenbauer, Bessel and Chebyshev polynomials of I & II kind. Some discrete polynomials like – Hahn, Krawtchowk, Pasternak, Bateman, Meixner, Poisson-Charlier are also taken. Sister Celine's polynomials of r – variables are it self a most generalized polynomials, if we put p = q = 0, the multivariable Sister Celine's polynomials would immediately reduces to the product of r different Sister Celine's polynomials of single variables.

(i) In (2.1), taking r = 2 and replacing $\rho_1^{(1)},\sigma_1^{(1)},\rho_2^{(1)},\sigma_2^{(1)},\rho_1^{(2)},\sigma_1^{(2)},\rho_2^{(2)},\sigma_2^{(2)}$, respectively by $\rho_1,\sigma_1,\delta_1,\theta_1,\rho_2,\sigma_2,\delta_2,\theta_2$; we get the general Eulerian integral involving Sister Celine's polynomials of two – variables:

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}(u_1t_1+v_1)^{e_1}(f_1t_1+g_1)^{-s_1}(u_2t_2+v_2)^{e_2}\times(f_2t_2+g_2)^{-s_2} f_{n\,m}\left(X_1^{k_1},Y_1^{k_2}\right)dt_1dt_2,$$

$$=A.\ f_{n,m}\left(\begin{matrix}(1+e_1:\rho_1,\rho_2),(s_1+\mu_1:\sigma_1,\sigma_2),(1+e_2:\delta_1,\delta_2),(s_2+\mu_2:\theta_1,\theta_2),a:c';c''\\(1+e_1-\lambda_1:\rho_1,\rho_2),(s_1:\sigma_1,\sigma_2),(1+e_2-\lambda_2:\delta_1,\delta_2),(s_2:\theta_1,\theta_2),b:d';d''\end{matrix};X_2^{k_1},Y_2^{k_2}\right), \tag{3.1}$$

where

$$X_1^{k_1}\equiv(z_1x_1)^{k_1}(u_1t_1+v_1)^{\rho_1}(f_1t_1+g_1)^{-\sigma_1}(u_2t_2+v_2)^{\delta_1}(f_2t_2+g_2)^{-\theta_1}$$

$$Y_1^{k_2}\equiv(z_2x_2)^{k_2}(u_1t_1+v_1)^{\rho_2}(f_1t_1+g_1)^{-\sigma_2}(u_2t_2+v_2)^{\delta_2}(f_2t_2+g_2)^{-\theta_2}$$

$$X_2^{k_1}\equiv(z_1x_1)^{k_1}(a_1u_1+v_1)^{\rho_1}(b_1t_1+g_1)^{-\sigma_1}(a_2u_2+v_2)^{\delta_1}(b_2f_2+g_2)^{-\theta_1}$$

$$Y_2^{k_2}\equiv(z_2x_2)^{k_2}(a_1u_1+v_1)^{\rho_2}(b_1f_1+g_1)^{-\sigma_2}(a_2u_2+v_2)^{\delta_2}(b_2f_2+g_2)^{-\theta_2}$$

$$A \equiv (b_1-a_1)^{\alpha_1+\beta_1-1}(b_2-a_2)^{\alpha_2+\beta_2-1}(a_1u_1+v_1)^{e_1}(b_1f_1+g_1)^{-s_1}(a_2u_2+v_2)^{e_2}(b_2f_2+g_2)^{-s_2}$$
$$\times\sum_{\lambda_1=0}^{\infty}\sum_{\mu_1=0}^{\infty}\sum_{\lambda_2=0}^{\infty}\sum_{\mu_2=0}^{\infty}\frac{(-e_1)_{\lambda_1}(s_1)_{\mu_1}(-e_2)_{\lambda_2}(s_2)_{\mu_2}}{\lambda_1!\,\lambda_2!\,\mu_1!\,\mu_2!}B(\alpha_1+\lambda_1,\beta_1+\mu_1)B(\alpha_2+\lambda_2,\beta_2+\mu_2)$$
$$\times\left(\frac{-u_1(b_1-a_1)}{a_1u_1+v_1}\right)^{\lambda_1}\left(\frac{f_1(b_1-a_1)}{b_1f_1+g_1}\right)^{\mu_1}\left(\frac{-u_2(b_2-a_2)}{a_2u_2+v_2}\right)^{\lambda_2}\left(\frac{f_2(b_2-a_2)}{b_2f_2+g_2}\right)^{\mu_2}$$

(ii) In (3.1), putting p = q = 0, m = 0, c= d= 0, $k_1=1,\alpha_2=\beta_2=1,e_2=s_2=\delta_1=\theta_1=0$ and replacing $\gamma\,'s,\delta\,'s$ by unity $a_1,b_1,z_1,x_1,\alpha_1,\beta_1,e_1,s_1,\rho_1,\sigma_1,u_1,v_1,f_1,g_1,t_1$ by respectively $a,b,z,x,\alpha,\beta,e,s,\rho,\ \sigma,u,v,f,g,t$; we obtain immediately the Eulerian integral involving Sister Celine's polynomials of single variables :

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^e(ft+g)^{-s}f_n(zx\ (ut+v)^{\rho}(ft+g)^{\sigma})dt$$
$$=(b-a)^{(\alpha+\beta-1)}(au+v)^e(bf+g)^{-s}\sum_{\lambda,\mu=0}^{\infty}\frac{(-e)_\lambda(s)_\mu}{\lambda!\ \mu!}B(\alpha+\lambda,\beta+\mu)\left(\frac{-u(b-a)}{au+v}\right)^{\lambda}\left(\frac{f(b-a)}{bf+g}\right)^{\mu}$$
$$\times f_n\left(\begin{matrix}(1+e:\rho),(s+\mu:\sigma),c'\\(1+e-\lambda:\rho),(s:\sigma),d'\end{matrix};(zx)(au+v)^{\rho}(bf+g)^{-\sigma}\right), \tag{3.2}$$

even this result is believed to be new.

(iii) In (3.1), replacing $\alpha\,'s,\beta'\ s,\gamma'\ s,\delta\,'s$ by unity, $k_1=k_2=1,c=d=0$ with

(a) $p=1,a_1=-n;q=0;p_1=3,c_1'=1+\alpha+\beta+n,c_2^1=\frac{1}{2},c_3^1=1;q_1=3,d_1^1=1+\alpha,d_2^1=-n,d_3^1=n+1;$
$p_2=3,c_1''=1+\alpha\,'+\beta\,'+n,\ c_2''=\frac{1}{2},c_3''=1;q_2=3,d_1''=1+\alpha\,',d_2''=-m,d_3''=m+1,\ X_1\to\frac{1-X_1}{2},Y_1\to\frac{1-Y_1}{2}$

(b) $p=0,q=0;p_1=3,c_1'=1+\alpha+\beta+n,c_2^1=\frac{1}{2},c_3^1=1;q_1=2,d_1^1=1+\alpha,d_2^1=n+1;p_2=3,$
$c_1''=1+\alpha\,'+\beta\,'+m,c_2''=\frac{1}{2},c_3''=1;q_2=2,d_1''=1+\alpha\,',\ d_2''=m+1,X_1\to\frac{1-X_1}{2},Y_1\to\frac{1-Y_1}{2}.$

We get Eulerian integral involving Jacobi polynomials of two variables defined by the author [7,11] and product of Jacobi polynomials of single variables [5] respectively :

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_2-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}(u_1t_1+v_1)^{e_1}(f_1t_1+g_1)^{-s_1}(u_2t_2+v_2)^{e_2}(f_2t_2+g_2)^{-s_2}$$
$$\times P_n^{(\alpha,\beta;\alpha',\beta')}(X_1,Y_1)\,dt_1dt_2$$
$$=\frac{A(1+\alpha)_n(1+\alpha\,')_n}{(n!)^2}f_{n,m}\left(\begin{matrix}(1+e_1:\rho_1,\rho_2)\quad,(s_1+\mu_1:\sigma_1,\sigma_2),(1+e_2:\delta_1,\delta_2),(s_2+\mu_2:\theta_1,\theta_2),\\(1+e_1,-\lambda_1:\rho_1,\rho_2),\quad(s_1:\sigma_1,\sigma_2),(1+e_2,-\lambda_2:\delta_1,\delta_2),(s_2:\theta_1,\theta_2)\end{matrix}\right.$$
$$\left.\begin{matrix}(-n:1,1):(1+\alpha+\beta+n,1),\left(\frac{1}{2},1\right),(1,1);(1+\alpha\,'+\beta\,'+n,1),\left(\frac{1}{2},1\right),(1,1)\\:(1+\alpha,1),(-n,1),(n+1,1)\quad;(1+\alpha\,',1),(-m,1),(m+1,1)\end{matrix};\frac{1-X_2}{2},\frac{1-Y_2}{2}\right), \tag{3.3}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}(u_1t_1+v_1)^{e_1}(f_1t_1+g_1)^{-s_1}(u_2t_2+v_2)^{e_2}(f_2t_2+g_2)^{-s_2}$$
$$\times P_n^{(\alpha,\beta)}(X_1)P_m^{(\alpha',\beta')}(Y_1)\ dt_1dt_2\ ,$$
$$=\frac{A(1+\alpha)_n(1+\alpha\,')_m}{n!\ m!}f_{n,m}\left(\begin{matrix}(1+e_1:\rho_1,\rho_2)\quad,(s_1+\mu_1:\sigma_1,\sigma_2),(1+e_2:\delta_1,\delta_2),(s_2+\mu_2:\theta_1,\theta_2):\\(1+e_1,-\lambda_1:\rho_1,\rho_2),\quad(s_1:\sigma_1,\sigma_2),(1+e_2,-\lambda_2:\delta_1,\delta_2),(s_2:\theta_1,\theta_2):\end{matrix}\right.$$
$$\left.\begin{matrix}(1+\alpha+\beta+n,1),\left(\frac{1}{2},1\right),(1,1);(1+\alpha\,'+\beta\,'+n,1),\left(\frac{1}{2},1\right),(1,1)\\(1+\alpha,1),(n+1,1)\quad;\quad(1+\alpha\,',1),(m+1,1)\end{matrix};\frac{1-X_2}{2},\frac{1-Y_2}{2}\right), \tag{3.4}$$

(iv)In(3.3)&(3.4),taking $\alpha=\alpha\,'=\beta=\beta\,'=0;\alpha=\alpha\,'=\beta=\beta\,'=v-\frac{1}{2};\alpha=\alpha\,'=\beta=\beta\,'=-\frac{1}{2},$

$\alpha=\alpha'=\beta=\beta'=\frac{1}{2};$ we can easily obtain respectively Eulerian integrals involving Legendre, Gegenbauer, Chebyshev polynomials of two variables defined by the author [7,11].

(v) In (3.1), replacing α's, β's, γ's, δ's by unity, $k_1=k_2=1, c=d=0$ with putting

(a) $p=1, a_1=-n; q=0; p_1=2, c_1'=\frac{1}{2}, c_2'=1; q_1=3, d_1'=-n, d_2'=n+1, d_3'=1+\alpha; p_2=2, c_1''=\frac{1}{2}, c_2''=1;$
$q_3=3, d_1''=-m, d_2''=m+1, d_3''=1+\alpha'$ and $p=q=0; p_1=2, c_1'=\frac{1}{2}, c_2'=1; q_1=2, d_1'=n+1, d_2'=1+\alpha;$
$p_2=2, c_1''=\frac{1}{2}, c_2''=1; q_2=2, d_1''=m+1, d_2''=1+\alpha'$

(b) $p=1, a_1=-n; q=0; p_1=3, c_1'=\frac{1}{2}, c_2'=1; c_3'=-n+\frac{1}{2}; q_1=2, d_1'=-n, d_2'=n+1; p_2=3, c_1''=\frac{1}{2},$
$c_2''=1, c_3''=-m+\frac{1}{2}; q_2=2, d_1''=-m, d_2''=m+1; X_1\to-\frac{1}{X_1^2}, Y_1\to-\frac{1}{Y_1^2}$, and $p=q=0; p_1=4, c_1'=\frac{1}{2}, c_2'=1,$
$c_3'=-\frac{n}{2}, c_4'=\frac{-n+1}{2}, q_1=2, d_1'=-n, d_2'=n+1; p_2=4, c_1''=\frac{1}{2}, c_2''=1, c_3''=-\frac{m}{2}, c_4''=\frac{-m+1}{2}; q_2=2, d_1''=-m,$
$d_2''=m+1; X_1\to-\frac{1}{X_1^2}, Y_1\to-\frac{1}{Y_1^2}$

(c) $p=1, a_1=-n; q=0; p_1=3, c_1'=\frac{1}{2}, c_2'=1; c_3'=1+\alpha+n; q_1=2, d_1'=-n, d_2'=n+1; p_2=3, c_1''=\frac{1}{2},$
$c_2''=1, c_3''=1+\beta+m; q_2=2, d_1''=-m, d_2''=m+1; X_1\to-\frac{X_1}{2}, Y_1\to-\frac{Y_1}{2}$, and $p=q=0; p_1=3, c_1'=\frac{1}{2},$
$c_2'=1, c_3'=1+\alpha+n, q_1=1, d_1'=n+1; p_2=3, c_1''=\frac{1}{2}, c_2''=1, c_3''=1+\beta+m; q_2=1, d_1''=m+1; X_1\to-\frac{X_1}{2}, Y_1\to-\frac{-Y_1}{2}$

(d) $p=1, a_1=-n; q=0; p_1=3, c_1'=\frac{1}{2}, c_2'=1; c_3'=2v+n; q_1=4, d_1'=-n, d_2'=n+1, d_3'=v+\frac{1}{2},$
$d_4'=b+1; p_2=3, c_1''=\frac{1}{2}, c_2''=1, c_3''=2v'+m; q_2=4, d_1''=-m, d_2''=m+1, d_3''=v'+\frac{1}{2}, d_4''=b'+1$ and
$p=q=0; p_1=3, c_1'=\frac{1}{2}, c_2'=1; c_3'=2v+n; q_1=3, d_1'=n+1, d_2'=v+\frac{1}{2}, d_3'=b+1; p_2=3,$
$c_1''=\frac{1}{2}, c_2''=1, c_3''=2v'+m; q_2=3, d_1''=m+1, d_2''=v'+\frac{1}{2}, d_3''=b'+1$

(e) $p=1, a_1=-n; q=0; p_1=2, c_1'=\frac{1}{2}, c_2'=\xi; q_1=2, d_1'=-n, d_2'=\lambda; p_2=2, c_1''=\frac{1}{2}, c_2''=\xi'; q_2=2,$
$d_1''=-m, d_2''=\lambda'$ and $p=q=0; p_1=2, c_1'=\frac{1}{2}, c_2'=\xi; q_1=1, d_1'=\lambda; p_2=2, c_1''=\frac{1}{2}, c_2''=\xi'; q_2=1, d_1''=\lambda'$

We get, Eulerian integrals involving respectively Laguerre polynomials of two variables [9] & product of Laguerre polynomials of single variable [5] ; Hermite polynomials of two variables [9] & product of Hermite polynomials of single variable [5] ; Bessel polynomials of two variables [10] & product of Bessel polynomials of single variable [5]; generalized Bateman's polynomials of two variables [12] & product of generalized Bateman's polynomials of single variables [5]; Rice polynomials of two variable [12] & product of Rice polynomials of single variable [5] :

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}F(t_1,t_2)\,L_n^{(\alpha,\alpha')}(X_1,Y_1)\,dt_1dt_2$$
$$=\frac{A(1+\alpha)_n(1+\alpha')_n}{(n!)^2}\,f_{n,m}\left(\begin{matrix}\Delta_1,(-n:1,1):\left(\frac{1}{2},1\right),(1,1);\left(\frac{1}{2},1\right),(1,1)\\ \Delta_2:(-n,1),(n+1,1),(1+\alpha,1)\quad;(-m,1),(m+1,1),(1+\alpha',1)\end{matrix};X_2,Y_2\right),\quad(3.5)$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}F(t_1,t_2)\,L_n^{(\alpha)}(X_1)L_m^{(\alpha')}(Y_1)\,dt_1dt_2$$
$$=\frac{A(1+\alpha)_n(1+\alpha')_m}{n!\;m!}\,f_{n,m}\left(\begin{matrix}\Delta_1:\left(\frac{1}{2},1\right),(1,1);\left(\frac{1}{2},1\right),(1,1)\\ \Delta_2:(n+1,1),(1+\alpha,1)\quad;(m+1,1),(1+\alpha',1)\end{matrix};X_2,Y_2\right),\quad(3.6)$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}F(t_1,t_2)\,H_{2n,2m}(X_1,Y_1)\,dt_1dt_2$$

$$=A\,f_{n,m}\left(\begin{matrix}\Delta_1,(-n:1,1):\left(\frac{1}{2},1\right),(1,1),\left(-n+\frac{1}{2},1\right);\left(\frac{1}{2},1\right),(1,1),\left(-m+\frac{1}{2},1\right);X_2,Y_2\\ \Delta_2:\ (-n,1),(n+1,1)\qquad\qquad;(-m,1),(m+1,1)\end{matrix}\right), \tag{3.7}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}F(t_1,t_2)\,H_n(X_1)H_m(Y_1)\,dt_1dt_2$$

$$=A\,f_{n,m}\left(\begin{matrix}\Delta_1:\left(\frac{1}{2},1\right),(1,1),\left(-\frac{n}{2},1\right),\left(\frac{-n+1}{2},1\right);\left(\frac{1}{2},1\right),(1,1),\left(\frac{-m}{2},1\right),\left(\frac{-m+1}{2},1\right);X_2,Y_2\\ \Delta_2\ :(-n,1),(n+1,1)\qquad\qquad;(-m,1),(m+1,1)\end{matrix}\right), \tag{3.8}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}F(t_1,t_2)\,y_{n,m}^{(\alpha,\beta)}(X_1,Y_1)\,dt_1dt_2$$

$$=A\,f_{n,m}\left(\begin{matrix}\Delta_1,(-n:1,1):\left(\frac{1}{2},1\right),(1,1),(1+\alpha+n,1);\left(\frac{1}{2},1\right),(1,1),(1+\beta+m,1);-\frac{X_2}{2},\frac{-Y_2}{2}\\ \Delta_2\qquad:(-n,1),(n+1,1)\qquad\qquad;(-m,1),(m+1,1)\end{matrix}\right), \tag{3.9}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}F(t_1,t_2)\,y_n^{(\alpha)}y_m^{(\beta)}\,dt_1dt_2$$

$$=A\,f_{n,m}\left(\begin{matrix}\Delta_1:\left(\frac{1}{2},1\right),(1,1),(1+\alpha+n,1);\left(\frac{1}{2},1\right),(1,1),(1+\beta+m,1);-\frac{X_2}{2},\frac{-Y_2}{2}\\ \Delta_2\ :(n+1,1)\qquad\qquad;(m+1,1)\end{matrix}\right), \tag{3.10}$$

The following obvious relationships between Bessel polynomials of two variables $y_{n,m}^{(\alpha,\beta)}(x,y)$ and generalized Bessel polynomials $y_n(a,b,x)$ & simple Bessel polynomials $y_n(x)$ are as follows :

$$y_{n,m}^{(a-2,\beta)}\left(\frac{2x}{b},0\right)=y_n(a,b,x)\ ,\ y_{n,m}^{(0,\beta)}(x,0)=y_n(x)$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}F(t_1,t_2)\,Z_{n,m}(X_1,Y_1)\,dt_1dt_2$$

$$=A\,f_{n,m}\left(\begin{matrix}\Delta_1,(-n:1,1):\ \left(\frac{1}{2},1\right),(1,1),(2v+n,1);\left(\frac{1}{2},1\right),(1,1),(2v'+m,1)\\ \Delta_2\qquad:(-n,1),(n+1,1),\left(v+\frac{1}{2},1\right),(b+1,1);(-m,1),(m+1,1),\left(v'+\frac{1}{2},1\right),(b'+1,1)\end{matrix};X_2,Y_2\right), \tag{3.11}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}F(t_1,t_2)\,Z_n(X_1)\,Z_m(Y_1)\,dt_1dt_2$$

$$=A\,f_{n,m}\left(\begin{matrix}\Delta_1:\left(\frac{1}{2},1\right),(1,1),(2v+n,1)\qquad;\left(\frac{1}{2},1\right),(1,1),(2v'+m,1)\\ \Delta_2:(n+1,1),\left(v+\frac{1}{2},1\right),(b+1,1);(m+1,1),\left(v+\frac{1}{2},1\right),(b'+1,1)\end{matrix};X_2,Y_2\right), \tag{3.12}$$

The relationships between generalized Bateman's polynomials of two variables and generalized Bateman's polynomials of single variable & simple Bateman's polynomials of single variable are as follows :

$$Z_{n,m}(x,0)=Z_n(x)\ ,\ Z_{n,m}(x,0)=Z_n(x);\ \ with\ v=v'=\tfrac{1}{2},\ b=b'=0$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}F(t_1,t_2)\,H_{n,m}(\xi,\lambda,X_1:\xi',\lambda',Y_1)\,dt_1dt_2$$

$$= A\, f_{n,m}\left(\begin{matrix}\Delta_1,(-n:1,1):\left(\frac{1}{2},1\right),(1,1),(\xi,1);\left(\frac{1}{2},1\right),(\xi',1)\\ \Delta_2 \qquad :(-n,1),(\lambda,1) \quad ;(-m,1),(\lambda',1)\end{matrix}; X_2, Y_2\right), \tag{3.13}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}F(t_1,t_2)\,H_n(\xi,\lambda,X_1)H_m(\xi',\lambda',Y_1)\,dt_1dt_2$$

$$= A\, f_{n,m}\left(\begin{matrix}\Delta_1:\left(\frac{1}{2},1\right),(\xi,1);\left(\frac{1}{2},1\right),(\xi',1)\\ \Delta_2:(\lambda,1) \qquad ;(\lambda',1)\end{matrix}; X_2, Y_2\right), \tag{3.14}$$

(vi) In (3.1), replacing α's, β's, γ's, δ's, by unity, $k_1 = k_2 = 1, c = d = 0$ with putting

(a) $p = q = 0;\ p_1 = 2, c_1' = \frac{1}{2}, c_2' = 1;\ q_1 = 3, d_1' = -n, d_2' = n+1, d_3' = n+1;\ p_2 = 2, c_1'' = \frac{1}{2}, c_2'' = 1;\ q_2 = 3,$
$d_1'' = -m, d_2'' = m+1, d_3'' = n+1;\ X_1 \to -X_1, Y_1 \to -Y_1$ and $p = q = 0;\ p_1 = 2, c_1' = \frac{1}{2}, c_2' = 1;\ q_1 = 3,$
$d_1' = -n, d_2' = n+1, d_3' = n+1;\ p_2 = 2, c_1'' = \frac{1}{2}, c_2'' = 1;\ q_2 = 3, d_1'' = -m, d_2'' = m+1, d_3'' = m+1;\ X_1 \to -X_1, Y_1 \to -Y_1.$

We can easily get Eulerian integrals involving Bessel function of two variables defined by the author [8; eq(2.1), p.181] and product of Bessel functions of single variable [5] .
(vii) Now, we discuss some discrete polynomials. First of all we define them for two variables (similarly can be defined for several variables) then establish the relationships between their limiting cases and Sister Celine polynomials of two variables. Finally we give Eulerian integrals of these discrete polynomials.

Pasternak Polynomials :

Pasternak polynomials of single variable is defined as [5;eq(3),p.291]:

$$F_n^{\lambda}(z) = {}_3F_2\left(\begin{matrix}-n, n+1, \frac{z+1+\lambda}{2}\\ 1, \lambda+1\end{matrix};1\right)$$

We define Pasternak polynomials of two variables as :

$$F_{n,m}^{z,z'}(z,z') = \sum_{m_1=0}^{n}\sum_{m_2=0}^{n-m_1}\frac{(-n)_{m_1+m_2}(n+1)_{m_1}(m+1)_{m_2}\left(\frac{z+1+\lambda}{2}\right)_{m_1}\left(\frac{z'+1+\lambda'}{2}\right)_{m_2}}{m_1!\ m_2!\ (1)_{m_1}(1)_{m_2}(\lambda+1)_{m_1}(\lambda'+1)_{m_2}},$$

obviously $F_{n,m}^{\lambda,\lambda'}(z,-1-\lambda') = F_n^{\lambda}(z)$

$$\lim_{\lambda,\lambda'\to\infty} F_{n,m}^{\lambda,\lambda'}(\lambda(z-1),\lambda'(z'-1)) = f_{n,m}\left(\begin{matrix}(-n:1,1):\left(\frac{1}{2},1\right);\left(\frac{1}{2},1\right)\\ \text{—}:(-n,1);(-m,1)\end{matrix};\frac{z}{2},\frac{z'}{2}\right),$$

Hahn Polynomials

Hahn polynomials of single variable is defined as [6;eq(1.31),p.541] ;

$$Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\begin{matrix}-n, 1+\alpha+\beta+n, -x\\ 1+\alpha, -N\end{matrix};1\right)$$

We define Hahn polynomials of two variables as:

$$Q_{n,m}(x;\alpha,\beta,N:y;\alpha',\beta',N') = \sum_{m_1=0}^{n}\sum_{m_2=0}^{n-m_1}\frac{(-n)_{m_1+m_2}(1+\alpha+\beta+n)_{m_1}(1+\alpha'+\beta'+m)_{m_2}(-x)_{m_1}(-y)_{m_2}}{m_1!\ m_2!\ (1+\alpha)_{m_1}(1+\alpha')_{m_2}(-N)_{m_1}(-N')_{m_2}},$$

Clearly $Q_{n,m}(x;\alpha,\beta,N:0;\alpha',\beta',N') = Q_n(x;\alpha,\beta,N)$,

$$\lim_{N,N'\to\infty} Q_{n,m}(xN;\alpha,\beta,N:yN';\alpha',\beta',N')$$

$$= f_{n,m}\left(\begin{matrix}(-n:1,1):\left(\frac{1}{2},1\right),(1,1),(1+\alpha+\beta+n,1);\left(\frac{1}{2},1\right),(1,1),(1+\alpha'+\beta'+m,1)\\ \text{—}: \quad (-n,1),(n+1,1),(1+\alpha,1);(-m,1),(m+1,1),(1+\alpha',1)\end{matrix};x,y\right),$$

Krawtchouk Polynomials

Krawtchouk polynomials of single variable is defined as [6;eq(1.33),p.542] :

$$K_n(x;\lambda,N)={}_2F_1\left(\begin{matrix}-n,-x\\-N\end{matrix};\frac{1}{\lambda}\right),$$

We define Krawtchouk polynomials of two variables as :

$$K_n(x;\lambda,N:y;\lambda',N')=\sum_{m_1=0}^{n}\sum_{m_2=0}^{n-m_1}\frac{(-n)_{m_1+m_2}(-x)_{m_1}(-y)_{m_2}}{m_1!\,m_2!\,(-N)_{m_1}(-N')_{m_2}}\left(\frac{1}{\lambda}\right)^{m_1}\left(\frac{1}{\lambda'}\right)^{m_2},$$

Clearly $K_n(x;\lambda,N:0;\lambda',N')=K_n(x;\lambda,N)$,

$$\lim_{N,N'\to\infty}K_n(xN;\lambda,N:yN';\lambda',N')=f_{n,m}\left(\begin{matrix}(-n:1,1):\left(\frac{1}{2},1\right),(1,1);\left(\frac{1}{2},1\right),(1,1)\\ —:(-n,1),(n+1,1);(-m,1),(m+1,1)\end{matrix};\frac{x}{\lambda},\frac{y}{\lambda'}\right),$$

Meixner Polynomials

Meixner polynomials of single variable is defined as [6;eq(1.34),p.542] :

$$M_n(x;\beta,\lambda)=(\beta)_n\ {}_2F_1\left(\begin{matrix}-n,-x\\ \beta\end{matrix};1-\frac{1}{\lambda}\right),$$

we define Meixner polynomials of two variables as :

$$M_n(x;\beta,\lambda:y;\beta',\lambda')=(\beta)_n(\beta')_n\sum_{m_1=0}^{n}\sum_{m_2=0}^{n-m_1}\frac{(-n)_{m_1+m_2}(-x)_{m_1}(-y)_{m_2}}{m_1!\,m_2!\,(\beta)_{m_1}(\beta')_{m_2}}\left(1-\frac{1}{\lambda}\right)^{m_1}\left(1-\frac{1}{\lambda'}\right)^{m_2},$$

clearly $M_n(x;\beta,\lambda:0;\beta',\lambda')=(\beta')_n M_n(x;\beta,\lambda)$,

$$\lim_{\lambda,\lambda'\to\infty}M_n\left(-\lambda x;\beta,\frac{\lambda}{\lambda-1}:-\lambda'y;\beta',\frac{\lambda'}{\lambda'-1}\right)=(\beta)_n(\beta')_n\,f_{n,m}\left(\begin{matrix}(-n:1,1):\left(\frac{1}{2},1\right),(1,1);\left(\frac{1}{2},1\right),(1,1)\\ —:(-n,1),(n+1,1),(\beta,1);(-m,1),(m+1,1),(\beta',1)\end{matrix};x,y\right),$$

Poisson – Charlier Polynomials

Poisson-Charlier polynomials of single variable is defined as [6;eq(1.35),p.542] :

$$C_n(x;\alpha)={}_2F_0\left(\begin{matrix}-n,-x\\ —\end{matrix};-\frac{1}{\alpha}\right),$$

we define Poisson – Charlier polynomials of two variables as :

$$C_n(x;\alpha:y,\alpha')=\sum_{m_1=0}^{n}\sum_{m_2=0}^{n-m_1}\frac{(-n)_{m_1+m_2}(-x)_{m_1}(-y)_{m_2}}{m_1!\,m_2!}\left(-\frac{1}{\alpha}\right)^{m_1}\left(-\frac{1}{\alpha'}\right)^{m_2},$$

clearly $C_n(x;\alpha:0;\alpha')=C_n(x;\alpha)$,

$$\lim_{\alpha,\alpha'\to\infty}C_n\left(\alpha x;\frac{1}{\alpha}:\alpha'y;\frac{1}{\alpha'}\right)=f_{n,m}\left(\begin{matrix}(-n:1,1):\left(\frac{1}{2},1\right),(1,1);\left(\frac{1}{2},1\right),(1,1)\\ —:(-n,1),(n+1,1);(-m,1),(m+1,1)\end{matrix};x,y\right),$$

The Eulerian integrals of above discrete polynomials can be written as follows :

The Celine's polynomials with replacing α's,β's,γ's,δ's by unity and $k_1=k_2=1,c=d=0$

$$\lim_{\lambda,\lambda'\to\infty}\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}F(\mathrm{t}_1,\mathrm{t}_2)F_{n,m}^{\lambda,\lambda'}\left(\lambda(X_1-1),\lambda'(Y_1-1)\right)dt_1dt_2$$

$$=A\ \ f_{n,m}\left(\begin{matrix}\Delta_1,(-n:1,1):\left(\frac{1}{2},1\right);\left(\frac{1}{2},1\right)\\ \Delta_2\qquad :(-n,1);(-m,1)\end{matrix};\frac{X_2}{2},\frac{Y_2}{2}\right),\tag{3.15}$$

$$\lim_{\lambda,\lambda'\to\infty}\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}\times F(t_1,t_2)F_n^{\lambda}(\lambda(X_1-1))F_m^{\lambda'}(\lambda'(Y_1-1))dt_1dt_2$$

$$=A\ f_{n,m}\left(\begin{matrix}\Delta_1:\left(\frac{1}{2},1\right);\left(\frac{1}{2},1\right);\frac{X_2}{2},\frac{Y_2}{2}\\ \Delta_2:—\quad;—\end{matrix}\right),\tag{3.16}$$

$$\lim_{N,N'\to\infty}\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}F(t_1,t_2)\times Q_{n,m}(X_1N;\alpha,\beta,N:Y_1N';\alpha',\beta',N')\ dt_1dt_2$$

$$=A\ f_{n,m}\left(\begin{matrix}\Delta_1,(-n:1,1):\left(\frac{1}{2},1\right),(1,1),(1+\alpha+\beta+n,1);\left(\frac{1}{2},1\right),(1,1),(1+\alpha'+\beta'+m,1)\\ \Delta_2\qquad:\quad(-n,1),(n+1,1),(1+\alpha,1);(-m,1),(m+1,1),(1+\alpha',1)\end{matrix};X_2,Y_2\right),\tag{3.17}$$

$$\lim_{N,N'\to\infty}\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}F(t_1,t_2)\times Q_n(X\ N;\alpha,\beta,N)\ Q_m(Y_1N';\alpha',\beta')\ dt_1dt_2$$

$$=A\ f_{n,m}\left(\begin{matrix}\Delta_1:\left(\frac{1}{2},1\right),(1,1),(1+\alpha+\beta+n,1);\left(\frac{1}{2},1\right),(1,1),(1+\alpha'+\beta'+m,1)\\ \Delta_2:(n+1,1),(1+\alpha,1)\qquad;(m+1,1),(1+\alpha',1)\end{matrix};X_2,Y_2\right),\tag{3.18}$$

$$\lim_{N,N'\to\infty}\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}F(t_1,t_2)\times K_n(NX_1;\lambda\ ,N:N'Y_1;\lambda',N')\ dt_1dt_2$$

$$=A\ f_{n,m}\left(\begin{matrix}\Delta_1,(-n:1,1):\left(\frac{1}{2},1\right),(1,1)\quad;\left(\frac{1}{2},1\right),(1,1)\\ \Delta_2\qquad:(-n,1),(n+1,1);(-m,1)(m+1,1)\end{matrix};\frac{X_2}{\lambda},\frac{Y_2}{\lambda'}\right),\tag{3.19}$$

$$\lim_{N,N'\to\infty}\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}F(t_1,t_2)\times K_n(NX_1;\lambda\ ,N)K_m(N'Y_1;\lambda',N')\ dt_1dt_2$$

$$=A\ f_{n,m}\left(\begin{matrix}\Delta_1:\left(\frac{1}{2},1\right),(1,1);\left(\frac{1}{2},1\right),(1,1)\\ \Delta_2:(n+1,1)\quad;(m+1,1)\end{matrix};\frac{X_2}{\lambda},\frac{Y_2}{\lambda'}\right),\tag{3.20}$$

$$\lim_{\lambda,\lambda'\to\infty}\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}F(t_1,t_2)\times M_n\left(-\lambda X_1;\beta\ ,\frac{\lambda}{\lambda-1}:-\lambda'Y_1;\beta',\frac{\lambda'}{\lambda'-1}\right)dt_1dt_2$$

$$=A\ (\beta)_n(\beta')_n\ f_{n,m}\left(\begin{matrix}\Delta_1,(-n:1,1):\left(\frac{1}{2},1\right),(1,1)\quad;\left(\frac{1}{2},1\right),(1,1)\\ \Delta_2\qquad:(-n,1),(n+1,1)(\beta,1);(-m,1)(m+1,1),(\beta',1)\end{matrix};X_2,Y_2\right),\tag{3.21}$$

$$\lim_{\lambda,\lambda'\to\infty}\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}F(t_1,t_2)\times M_n\left(-\lambda X_1;\beta\ ,\frac{\lambda}{\lambda-1}\right)M_m\left(-\lambda'Y_1;\beta',\frac{\lambda'}{\lambda'-1}\right)dt_1dt_2$$

$$=A\ (\beta)_n(\beta')_m\ f_{n,m}\left(\begin{matrix}\Delta_1:\left(\frac{1}{2},1\right),(1,1)\quad;\left(\frac{1}{2},1\right),(1,1)\\ \Delta_2:(n+1,1),(\beta,1);(m+1,1),(\beta',1)\end{matrix};X_2,Y_2\right),\tag{3.22}$$

$$\lim_{\alpha,\alpha'\to\infty}\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}F(t_1,t_2)\times C_n\left(\alpha X_1;\frac{1}{\alpha}:\alpha'Y_1:\frac{1}{\alpha'}\right)dt_1dt_2$$

$$=A\,f_{n,m}\left(\begin{matrix}\Delta_1,(-n:1,1):\left(\frac{1}{2},1\right),(1,1)\quad;\left(\frac{1}{2},1\right),(1,1)\\ \Delta_2\qquad:(-n,1),(n+1,1);(-m,1)(m+1,1)\end{matrix};X_2,Y_2\right),\tag{3.23}$$

$$\lim_{\alpha,\alpha'\to\infty}\int_{a_1}^{b_1}\int_{a_2}^{b_2}(t_1-a_1)^{\alpha_1-1}(b_1-t_1)^{\beta_1-1}(t_2-a_2)^{\alpha_2-1}(b_2-t_2)^{\beta_2-1}F(t_1,t_2)\times C_n\left(\alpha X_1;\frac{1}{\alpha}\right)C_m\left(\alpha'Y_1:\frac{1}{\alpha'}\right)dt_1dt_2$$

$$=A\ f_{n,m}\left(\begin{matrix}\Delta_1:\left(\frac{1}{2},1\right),(1,1)\quad;\left(\frac{1}{2},1\right),(1,1)\\ \Delta_2:\ (n+1,1)\quad;(m+1,1)\end{matrix};X_2,Y_2\right),\tag{3.24}$$

It is worth to note here that there exists a relationship between Krawtchouk polynomials and Meixner polynomials as follows :

$$M_n\left(x;N,\frac{p}{p-1}\right)=(N)_n K_n(x;p,N)\ , \tag{3.25}$$

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