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FUNDAMENTAL THEOREMS FOR THE HYPERBOLIC GEODESIC TRIANGLES

H.H. Uğurlu, M. Kazaz and A.Özdemir
Department of Mathematics, Faculty of Science, Universty of Celal Bayar,
Muradiye Campus, 45047, Manisa, Turkey. m_kazaz@hotmail.com

Abstract- In this work, we state and prove the sine, cosine I, cosine II, sine-cosine and cotangent rules for spherical triangles on the hyperbolic unit sphere H_0^2 in the Lorentzian space R_1^3.
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1. INTRODUCTION

In plane Lorentzian geometry it is studied points, timelike, spacelike, and lightlike lines, triangles, etc [5]. On the hyperbolic sphere, there are points, but there are no straight lines, at least not in the usual sense. However the straight timelike lines in the Lorentzian plane are characterized by the fact that they are the shortest paths between points. The curves on the hyperbolic sphere with the same property are the great hyperbolic circles. Thus it is natural to use the great hyperbolic circles as (geodesic) replacements for timelike lines.

The formulas for the sine, cosine-I, cosine-II, sine-cosine and cotangent rules for Euclidean sphere S^2 are given in [2, 6]. In this study, we obtain the sine, cosine-I, cosine-II, sine-cosine and cotangent rules for spherical triangles on the hyperbolic unit sphere H_0^2.

2. BASIC CONCEPTS

In this section, we give a brief summary of the theory of Lorentzian concepts.

Let L^3 be vector space R^3 provide with Lorentzian inner product $<,>$ given by

$$< a,b >= a_1b_1 + a_2b_2 - a_3b_3,$$

where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3) \in R^3$.

A vector $a = (a_1, a_2, a_3)$ of R^3 is said to be *timelike* if $< a,a >< 0$, *spacelike* if $< a,a >> 0$, and *lightlike* (or *null*) if $< a,a >= 0$. The norm of a vector a is defined by $|a| = \sqrt{|< a,a >|}$.

Let $e = (0,0,1)$. A timelike vector $a = (a_1, a_2, a_3)$ is *future pointing* (resp. *past pointing*) if $< a,e >< 0$ (resp. $< a,e >> 0$. Thus a timelike vector $a = (a_1, a_2, a_3)$ is future pointing if and only if $a_1^2 + a_2^2 < a_3^2$ and $a_3 > 0$.

The set of all timelike unit vectors is called hyperbolic unit sphere and denoted by $H_0^2 = \{a = (a_1, a_2, a_3) \in L^3 \big| < a,a >= -1\}$. There are two components of the hyperbolic unit sphere H_0^2. The components of H_0^2 through (0,0,1) and (0,0,-1) are called the future pointing hyperbolic unit sphere and the past pointing hyperbolic unit sphere and denoted by H_0^{+2} and H_0^{-2}, respectively. Thus we have

$$H_0^{+2} = \{a = (a_1, a_2, a_3) \in L^3 \mid a \text{ is a future pointing vector}\}$$

and

$$H_0^{-2} = \{a = (a_1, a_2, a_3) \in L^3 \mid a \text{ is a past pointing vector}\}.$$

From now on, we will use the notation H_0^2 instead of H_0^{+2}.

Now let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ be two vectors in L^3, then the Lorentzian cross product of a and b is given by

$$a \times b = (a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_3 b_2 - a_2 b_3,\ a_1 b_3 - a_3 b_1,\ a_1 b_2 - a_2 b_1).$$

Lemma 1.1 *Let* $a, b, c, d \in L^3$. *Then we have*

$$< a \times b, c >= -\det(a, b, c)$$

$$a \times b = -b \times a,$$

$$(a \times b) \times c = -< a, c > b + < b, c > a,$$

$$< a \times b, c \times d >= -< a, c >< b, d > + < a, d >< b, c >,$$

$$< a \times b, a >= 0; \textit{ and } < a \times b, b >= 0.$$

2. FUNDAMENTAL THEOREMS FOR HYPERBOLIC GEODESIC TRIANGLES

In this section we prove the sine, cosine-I, cosine-II, sine-cosine, cotangent rules for hyperbolic geodesic triangles.

Fundamental relations of hyperbolic spherical trigonometry can be given on a trihedron. With the aid of this trihedron, both angles and sides of the spherical triangle can be represented as the spacelike angles between the hyperbolic angles corresponding to the sides of hyperbolic geodesic triangles. The radius of the sphere is not important while getting the fundamental relations related with the hyperbolic spherical triangles. That is, these relations are independent from the radius. Therefore, we consider the unit sphere in our work.

Lemma 2.1 ***(Hyperbolic Sine Rule)*** *Let* ABC *be a hyperbolic geodesic triangle on the hyperbolic unit sphere* H_0^2. *Then the hyperbolic sine rule is given by*

$$\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \frac{\sinh c}{\sin C}.$$

Proof: Let *ABC* be a hyperbolic geodesic triangle on the hyperbolic unit sphere H_0^2 with center *O*. A trihedron can be obtained by joining the vertices of the triangle to the center *O*. Let E_1 and E_2 be the spacelike planes passing through the point *C* and perpendicular to the lines *OA* and *OB*, respectively. Then, it follows that the line *CN* of the intersection of the planes E_1 and E_2 are perpendicular to the lines *OA* and *OB* lie on the same plane *OAB*, respectively. Therefore the triangles *OCP* and *OCQ* are the right triangles as the triangles *CNP* and *CNQ,* see Figure 2.1.

Firstly, from the right triangles *OCQ* and *CNP*, we can write

$$\sinh a = \frac{CQ}{OC}, \text{ and } \sin A = \frac{CN}{CP},$$

respectively. Then it follows that

$$\frac{\sinh a}{\sin A} = \frac{CQ}{OC}\frac{CP}{CN}. \tag{1}$$

Similarly, from the right triangles *OCP* and *CQN*, we get

$$\sinh b = \frac{CP}{OC}, \text{ and } \sin B = \frac{CN}{CQ},$$

respectively. Then we have

$$\frac{\sinh b}{\sin B} = \frac{CP}{OC}\frac{CQ}{CN}. \tag{2}$$

Comparing the equations (1) and (2) gives

$$\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \frac{CQ}{OC}\frac{CP}{CN}. \tag{3}$$

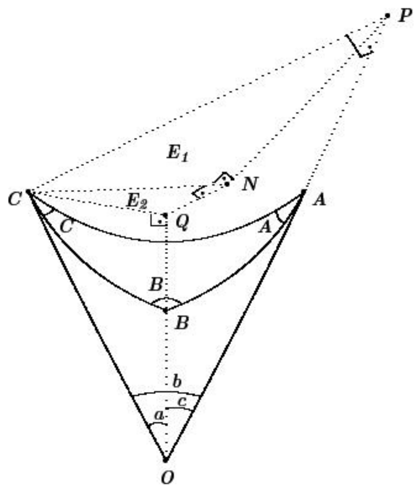

Figure 2.1: Hyperbolic Spherical Triangle.

Note that the expression on the right hand side of the equation (3) is invariant for this hyperbolic spherical triangle. Thus

$$\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \text{constant}. \tag{4}$$

Similarly, if we take the spacelike planes passing through the point *B* and perpendicular to the lines *OC* and *OA*, respectively, then we get

$$\frac{\sinh a}{\sin A} = \frac{\sinh c}{\sin C} = \text{constant}. \tag{5}$$

From the equations (4) and (5), we obtain

$$\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \frac{\sinh c}{\sin C} = m = \text{constant}.$$ ■

The common ratio *m* is called modulo of the hyperbolic spherical triangle. This value changes for different hyperbolic spherical triangles.

Lemma 2.2 ***(The Hyperbolic Cosine Rule I)*** *Let ABC be a spherical triangle on the hyperbolic unit sphere H_0^2. Then the hyperbolic cosine rule I is given by*

$$\cos A = \frac{\cosh b \cosh c - \cosh a}{\sinh b \sinh c}. \tag{6}$$

Proof: From the right triangle *OCQ* of Figure 2.1, we write

$$\cosh a = \frac{OQ}{OC} . \tag{7}$$

Let us draw a perpendicular line *PF* from the point *P* to the line segment *OB*. From Figure 2.2 (which shows the plane *OAB* of the trihedron of Figure 2.1), we can write

$$OQ = OF + FQ . \tag{8}$$

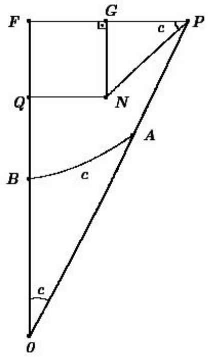

Figure 2.2.

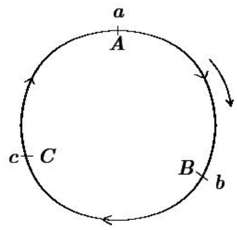

Figure 2.3

From the right triangles *OPF* in Figure 2.2 and *OPC* in Figure 2.1, we can write

$$OF = OP \cosh c \tag{9}$$

and

$$OP = OC \cosh b , \tag{10}$$

respectively. If we put the formula (10) in the formula (9), then we have

$$OF = OC \cosh b \cosh c . \tag{11}$$

If we draw the perpendicular line *NG* from the point *N* to the line segment *PF,* it is easily seen that $FQ = GN$. The angle *NPG* of the right triangle *PNG* is equal to the angle *c* of the right triangle *POF.* Then it follows from the right triangle *PNG* in Figure 2.2 that

$$GN = FQ = PN \sinh c , \tag{12}$$

and from the right triangle *PNC* in Figure 2.1 that

$$PN = CP \cos A , \tag{13}$$

and finally from the right triangle *OPC* in Figure 2.1, we write

$$CP = OC \sinh b . \tag{14}$$

Firstly, putting the value of *CP* in (13), and then the value of *PN* in (12) gives

$$FQ = OC \sinh b \sinh c \cos A . \tag{15}$$

Secondly, substitution of the equations (11) and (15) into (8) gives

$$OQ = OC \cosh b \cosh c - OC \sinh b \sinh c \cos A . \tag{16}$$

Finally, from the equations (7) and (16) we obtain

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos A . \qquad \blacksquare$$

If we change the elements of the hyperbolic triangle in a certain direction (see Figure 2.3), we obtain the similar formulas for $\cosh b$ and $\cosh c$ as follows:

$$\cosh b = \cosh c \cosh a - \sinh c \sinh a \cos B , \tag{17}$$

$$\cosh c = \cosh a \, \cosh b - \sinh a \, \sinh b \, \cos C \,. \tag{18}$$

Lemma 2.3 ***(The Hyperbolic Cosine Rule II)*** *Let ABC be a geodesic triangle on the hyperbolic unit sphere H_0^2. Then the hyperbolic cosine rule II is given by*

$$\cosh c = \frac{\cos A \, \cos B + \cos C}{\sin A \, \sin B}. \tag{19}$$

Proof: For brevity, let *X, Y* and *Z* be $\cosh a$, $\cosh b$ and $\cosh c$, respectively. Then the cosine rule I yields

$$\cos C = \frac{XY - Z}{(X^2-1)^{\frac{1}{2}}(Y^2-1)^{\frac{1}{2}}} \tag{20}$$

$$\cos B = \frac{XZ - Y}{(X^2-1)^{\frac{1}{2}}(Z^2-1)^{\frac{1}{2}}} \tag{21}$$

$$\cos A = \frac{YZ - X}{(Y^2-1)^{\frac{1}{2}}(Z^2-1)^{\frac{1}{2}}}. \tag{22}$$

On the other hand, since $\cos^2 A + \sin^2 B = 1$, it follows that

$$\sin^2 A = \frac{D}{(Y^2-1)(Z^2-1)},$$

where $D = 2XYZ - \left(X^2 + Y^2 + Z^2\right)$. We note that *D* is positive and symmetric in *X, Y* and *Z*. Then we obtain

$$\sin A = \frac{\sqrt{D}}{(Y^2-1)^{\frac{1}{2}}(Z^2-1)^{\frac{1}{2}}}, \tag{23}$$

$$\sin B = \frac{\sqrt{D}}{(X^2-1)^{\frac{1}{2}}(Z^2-1)^{\frac{1}{2}}}, \tag{24}$$

$$\sin C = \frac{\sqrt{D}}{(X^2-1)^{\frac{1}{2}}(Y^2-1)^{\frac{1}{2}}}.$$

If we write the formulas (20)-(24) in the right side of the formula (19), then the equality is satisfied:

$$\frac{\cos A \cos B + \cos C}{\sin A \sin B} = \frac{(YZ-X)(XZ-Y) + (XY-Z)(Z^2-1)}{(Y^2-1)^{\frac{1}{2}}(X^2-1)^{\frac{1}{2}}(Z^2-1)}.$$

$$\frac{(X^2-1)^{\frac{1}{2}}(Y^2-1)^{\frac{1}{2}}(Z^2-1)}{D}$$

$$= \frac{XYZ^2 - Y^2Z - X^2Z + XY + XYZ^2 - XY - Z^3 + Z}{D}$$

$$= \frac{Z(1 + 2XYZ - X^2 - Y^2 - Z^2)}{D}$$

$$= Z$$
$$= \cosh c\,. \qquad \blacksquare$$

By the same way, we can give the similar formulas for $\cosh b$ and $\cosh a$ as follows:

$$\cosh b = \frac{\cos A \cos C + \cos B}{\sin A \sin C}\,, \tag{25}$$

$$\cosh a = \frac{\cos B \cos C + \cos A}{\sin B \sin C}. \tag{26}$$

Lemma 2.4 ***(The Hyperbolic Sine-Cosine Rule)*** *Let ABC be a spherical triangle on the hyperbolic unit sphere H_0^2. Then the hyperbolic sine- cosine rule is given by*

$$\sinh a \ \cos B = \cosh b \ \sinh c - \sinh b \cosh c \ \cos A \tag{27}$$

Proof: From Figure 2.2, we can write

$$PF = PG + GF = PG + NQ\,. \tag{28}$$

It follows from the right triangle OFP that

$$PF = OP \sinh c\,. \tag{29}$$

If we replace in the equation (29) OP by its value (10), we obtain

$$PF = OC \cosh b \ \sinh \ c\,. \tag{30}$$

On the other hand, from the right triangles CNQ and OCQ in Figure 2.1, we get

$$NQ = QC \cos B \quad \text{and} \quad QC = OC \sinh a \tag{31}$$

respectively. Substitution of the equation (31) into (30) gives

$$NQ = OC \ \sinh a \ \cos B\,.$$

From Figure 2.2, it is clear that

$$PG = PN \cosh c\,.$$

Using the equations (13) and (14), we get

$$PG = OC \sinh b \ \cos A \ \cosh c\,.$$

If we substitute the corresponding values of PF, NQ and PG into (28), we deduce

$$OC \ \cosh b \ \sinh c = \ OC \ \sinh a \ \cos B + OC \ \sinh b \ \cos A \ \cosh c\,.$$

We can therefore obtain from the last equation that

$$\sinh a \cos B = \cosh b \sinh c - \sinh b \cosh c \cos A\,. \qquad \blacksquare$$

In a similar way, we can find two more hyperbolic sine-cosine formulas as follows:

$$\sinh b \ \cos C = \cosh c \sinh a - \sinh c \cosh a \cos B\,,$$
$$\sinh c \cos A = \cosh a \sinh b - \sinh a \cosh b \cos C\,.$$

By using the hyperbolic cosine rule I, we can deduce different formulas of hyperbolic sine-cosine rule. From (18) it follows that

$$\sinh a \cos C = -\frac{1}{\sinh b}(\cosh c - \cosh a \cosh b)\,. \tag{32}$$

Substitutions of the equation (6) into (32) gives

$$\sinh a \cos C = -\frac{1}{\sinh b}[\cosh c - \cosh b(\cosh b \cosh c - \sinh b \sinh c \cos A)]\,,$$

or equivalently,

$$\sinh a \cos C = -\frac{1}{\sinh b}[\cosh c(1-\cosh^2 b)+\sinh b\ \cosh b \sinh c \cos A)].$$

Then we obtain

$$\sinh a\ \cos C\ =\ \cosh c \sinh b\ - \sinh c\ \cosh b\ \cos A.$$

In similar way, we get two more formulas as follows:

$$\sinh b \cos A = \cosh a \sinh c - \sinh a \cosh c \cos B,$$
$$\sinh c \cos B = \cosh b \sinh a - \sinh b \cosh a \cos C.$$

We note that the hyperbolic sine-cosine rule has five elements whereas the others have four elements.

Lemma 2.5 ***(The Hyperbolic Cotangent Rule).*** *Let ABC be a spherical triangle on the hyperbolic unit sphere* H_0^2. *Then the hyperbolic cotangent rule is given by*

$$\cosh c \cos A = \coth b \sinh c - \sin A \cot B\ .$$

Proof: If we replace in (27) $\sinh a$ by

$$\sinh a = \frac{\sinh b \sin A}{\sin B}$$

in (3), we find that

$$\sinh b \sin A \cot B = \cosh b \sinh c - \sinh b \cosh \cos A.$$

Dividing both sides of this equation by $\sinh b$ gives

$$\sin A \cot B = \coth b \sinh c - \cosh c \cos A, \qquad (33)$$

or equivalently,

$$\cosh c \cos A = \coth b \sinh c - \sin A \cot B.$$

We note that, if the elements of the triangle are changed by cyclically, we get

$$\begin{aligned} \cosh c \cos A &= \coth b \sinh c - \sin A \cot B, \\ \cosh a \cos B &= \coth c \sinh a - \sin B \cot C, \\ \cosh b \cos C &= \coth a \sinh b - \sin C \cot A. \end{aligned} \qquad (34)$$

By the same way, if we replace in (27) $\sinh a$ by

$$\sinh a = \frac{\sinh c \sin A}{\sin C}$$

in (5), we find that

$$\sinh c \sin A \cot C = \cosh c \sinh b - \sinh c \cosh b \cos A.$$

Dividing both sides of this equation by $\sinh c$, we deduce that

$$\sin A\ \ \cot C\ =\ \coth c\ \sinh b - \cosh b\ \cos A,$$

or equivalently

$$\cosh b \cos A = \coth c \sinh b - \sin A \cot C.$$

We note that, by changing the elements of the triangle in cyclical order, we get

$$\begin{aligned} \cosh b \cos A &= \coth c \sinh b - \sin A \cot C, \\ \cosh c \cos B &= \coth a \sinh c - \sin B \cot A, \\ \cosh a \cos C &= \coth b \sinh a - \sin C \cot B. \end{aligned} \qquad (35)$$

■

Formulas in (34) and (35) are known as hyperbolic cotangent rules. In each of these formulas, there are four elements of spherical triangle. Furthermore, these four elements are not by chance, they follow each other in order. This property allows us to write the formulas in (34) and (35) in general. That is, by starting any sides of the hyperbolic spherical triangle, these four elements, which followed by each other, can be numbered in any direction. Therefore, hyperbolic cotangent rule can be generalized as follows:

$$\cosh III \cos II = \coth I \sinh III - \sin II \cot IV \ . \tag{36}$$

For example, in Figure 2.4, starting with the side a, the elements of hyperbolic spherical triangle are numbered in clockwise direction:

$$a \to I,\ C \to II,\ b \to III,\ A \to IV .$$

If we replace the numeration by the letters corresponding to the elements of the hyperbolic triangle, we get

$$\cosh b \cos C = \coth a \sinh b - \sin C \cot A .$$

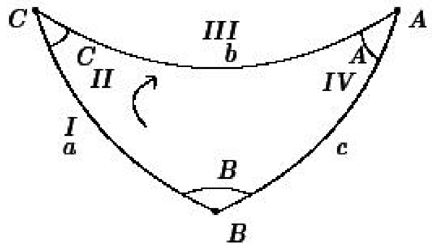

Figure 2.4:

This formula is equal to the last formula of (34). In a similar way, starting with the same side, but anticlockwise direction gives a new formula. For example, as in figure 2.4, starting with the side *a* and replacing the angles *A, B* and *C* by the numbers *I, II* and *III* gives the hyperbolic cotangent rule as follows:

$$\cosh c \cos B = \coth a \sinh c - \sin B \cot A .$$

This formula is equal to the second formula of (35). Since the triangle has three sides, and the numeration can be made in two different directions for each side, then the six formulas of the hyperbolic cotangent rule are generalized with the formula (36).

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