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FORMULAS FOR THE EXPONENTIAL OF A SEMI SKEW-SYMMETRIC MATRIX OF ORDER 4

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Abstract- In this paper the formula of the exponential matrix e^A when A is a semi skew-symmetric real matrix of order 4 is derived. The formula is a generalization of the Rodrigues formula for skew-symmetric matrices of order 3 in Minkowski 3-space.

Keywords- Semi skew-symmetric matrix, Rodrigues formula , Minkowski 3-space

1. INTRODUCTION

The computation of matrix functions has been one of the most challenging problems in numerical linear algebra. Among the matrix functions one of the most interesting is the exponential. In the last years the problem has been studied after the introduction of Lie group methods to solve numerically systems of ordinary differential equations. According to these methods the differential system is solved in a Lie algebra (and not in a Lie group) using a coordinate map defined from the algebra to its related group. In 2001 the paper by Tiziano Politi gave the exponential matrix e^A when A is a skew-symmetric real matrix of order 4. Among the explicit formulas only the Rodrigues formula allows the computation of e^A when A is a skew-symmetric real matrix $\left(i.e.\ \ A^T = -\varepsilon\, A\, \varepsilon,\ \ \varepsilon = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$ of order 3 in Minkowski 3-space.If A is the matrix

$$A = \begin{bmatrix} 0 & a_3 & -a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \leftrightarrow \vec{a} = (a_1, a_2, a_3)$$

the Rodrigues formula is

$$e^A = I + \frac{sh\alpha}{\alpha} A + \frac{(ch\alpha - 1)}{\alpha^2} A^2 \qquad (1.1)$$

where $\alpha = -a_1^2 + a_2^2 + a_3^2 \quad \left(\langle \vec{a}, \vec{a} \rangle = -a_1^2 + a_2^2 + a_3^2 > 0,\ \vec{a}\ spacelike \right)$

and

$$e^A = I + \frac{\sin\alpha}{\alpha} A + \frac{(1 - \cos\alpha)}{\alpha^2} A^2$$

where $\alpha = -a_1^2 + a_2^2 + a_3^2 \quad \left(\langle \vec{a}, \vec{a} \rangle = -a_1^2 + a_2^2 + a_3^2 < 0,\ \vec{a}\ timelike \right)$ [2].

In the following section we derive a generalization of (1.1) or semi skew-symmetric real matrices of order 4.

2. THE MAIN RESULT

Let us consider the following semi skew-symmetric matrix of order 4.

$$A = \begin{bmatrix} 0 & -a_6 & a_5 & a_3 \\ a_6 & 0 & a_4 & -a_2 \\ a_5 & a_4 & 0 & -a_1 \\ a_3 & -a_2 & a_1 & 0 \end{bmatrix} \tag{2.1}$$

where

$$A^T = -\varepsilon A \varepsilon, \ \varepsilon = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now we give some preliminary results.

Lemma 2.1. Let $p(\lambda)$ be the characteristic polynomial of matrix (2.1). Then $p(-\lambda) = p(\lambda)$.

Proof.

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det(A^T - \lambda I) \\ &= \det(-\varepsilon A \varepsilon - \lambda I) \\ &= \det(\varepsilon A \varepsilon + \lambda I) \\ &= \det\left(\varepsilon (A + \lambda I) \varepsilon\right) \\ &= \det(A + \lambda I) \\ &= p(-\lambda) \end{aligned}$$

Lemma 2.2. If A it the matrix (2 .1) then

$$p(\lambda) = \lambda^4 + b_2 \lambda^2 + b_0$$

where

$$\begin{aligned} b_2 &= a_1^2 - a_2^2 - a_3^2 - a_4^2 - a_5^2 + a_6^2 \\ b_0 &= \left(a_1 a_6 - a_3 a_4 - a_2 a_5\right)^2 \end{aligned} \tag{2.2}$$

Proof. From lemma 2.1. it follows that if

$$p(\lambda) = \lambda^4 + b_3 \lambda^3 + b_2 \lambda^2 + b_1 \lambda + b_0$$

is the characteristic polynomial of A then $b_3 = b_1 = 0$. To obtain the expressions for b_0 and b_2 it is sufficient to apply the rule of Laplace to $\det(A - \lambda I)$ computing only the coefficient of λ_2 and the determinant of A (i.e. b_0).

Corollary 2.3. The eigenvalues of (2 .1) are

i) If $b_2^2 - 4b_0 > 0$ and $b_2 > 0$,

$$\lambda_1 = i\sqrt{\frac{-b_2 + \sqrt{b_2^2 + 4b_0}}{2}} = i\alpha_1 \quad , \quad \lambda_2 = -i\sqrt{\frac{-b_2 + \sqrt{b_2^2 + 4b_0}}{2}} = -i\alpha_1$$

$$\lambda_3 = i\sqrt{\frac{b_2 + \sqrt{b_2^2 - 4b_0}}{2}} = i\mu_1 \quad , \quad \lambda_4 = -i\sqrt{\frac{-b_2 - \sqrt{b_2^2 - 4b_0}}{2}} = -i\mu_1$$

ii) If $b_2^2 - 4b_0 > 0$ and $b_2 < 0$,

$$\lambda_1 = \sqrt{\frac{-b_2 + \sqrt{b_2^2 + 4b_0}}{2}} = \alpha_2 \quad , \quad \lambda_2 = -\sqrt{\frac{-b_2 + \sqrt{b_2^2 + 4b_0}}{2}} = -\alpha_2$$

$$\lambda_3 = \sqrt{\frac{-b_2 - \sqrt{b_2^2 - 4b_0}}{2}} = \mu_2 \quad , \quad \lambda_4 = -\sqrt{\frac{-b_2 - \sqrt{b_2^2 - 4b_0}}{2}} = -\mu_2$$

iii) If $b_2^2 - 4b_0 < 0$ and $b_2 \neq 0$,

$$\lambda_1 = \frac{1}{2}\sqrt{2\sqrt{b_0} - b_2} + \frac{i}{2}\sqrt{2\sqrt{b_0} + b_2} = u + iv = z,$$

$$\lambda_2 = -\frac{1}{2}\sqrt{2\sqrt{b_0} - b_2} - \frac{i}{2}\sqrt{2\sqrt{b_0} + b_2} = -u - iv = -z,$$

$$\lambda_3 = -\frac{1}{2}\sqrt{2\sqrt{b_0} - b_2} + \frac{i}{2}\sqrt{2\sqrt{b_0} + b_2} = -u + iv = -\bar{z},$$

$$\lambda_1 = \frac{1}{2}\sqrt{2\sqrt{b_0} - b_2} - \frac{i}{2}\sqrt{2\sqrt{b_0} + b_2} = u - iv = \bar{z},$$

iv) If $b_2^2 - 4b_0 = 0$ and $b_2 > 0$,

$$\lambda_1 = i\sqrt{\frac{b_2}{2}} = i\mu \quad , \quad \lambda_2 = -i\sqrt{\frac{b_2}{2}} = -i\mu \ .$$

If $b_2^2 - 4b_0 = 0$ and $b_2 < 0$,

$$\lambda_1 = \sqrt{\frac{-b_2}{2}} = \alpha \quad , \quad \lambda_2 = -\sqrt{\frac{-b_2}{2}} = -\alpha$$

Remark 2.1. If $b_2^2 = 4b_0$ then A has two eigenvalues with algebraic multiplicity equal to 2. Exploiting the relations (2.2) it is possible to state the conditions on the entries such that A has multiple eigenvalues. In fact, first we note that

$$b_2^2 - 4b_0 > 0 \quad (< 0)$$

and equality occurs only when we have simultaneously

$$a_1 = a_6, \qquad a_2 = a_5, \qquad a_3 = a_4,$$

or

$$a_1 = -a_6, \quad a_2 = -a_5, \quad a_3 = -a_4 \ .$$

Then the following theorem holds:

Theorem 2.4. For the eigenvalues of the matrix (2 .1)

i) If $b_2^2 - 4b_0 > 0$ and $b_2 > 0$,

$$e^A = I + \frac{\sin\mu_1}{\mu_1} A + \frac{1-\cos\mu_1}{\mu_1} A^2 \tag{2.3}$$

where

$$\begin{aligned} a_1 &= \frac{\alpha_1^2 \cos\mu_1 - \mu_1^2 \cos\alpha_1}{\alpha_1^2 - \mu_1^2}, & b_1 &= \frac{\alpha_1^3 \sin\mu_1 - \mu_1^3 \sin\alpha_1}{\alpha_1\mu_1(\alpha_1^2 - \mu_1^2)} \\ c_1 &= \frac{\cos\mu_1 - \cos\alpha_1}{\alpha_1^2 - \mu_1^2}, & d_1 &= \frac{\alpha_1 \sin\mu_1 - \mu_1 \sin\alpha_1}{\alpha_1\mu_1(\alpha_1^2 - \mu_1^2)} \end{aligned} \tag{2.4}$$

In addition if $b_0 = 0$,

$$a_1 = 1, b_1 = 1, \; c_1 = \frac{1-\cos\mu_1}{\mu_1^2}, \; d_1 = \frac{\mu_1 - \sin\mu_1}{\mu_1^3}$$

or

$$e^A = I + \frac{\sin\mu_1}{\mu_1} A + \frac{1-\cos\mu_1}{\mu_1} A^2 .$$

ii) If $b_2^2 - 4b_0 > 0$ and $b_2 < 0$,

$$e^A = a_2 I + b_2 A + c_2 A^2 + d_2 A^3$$

where

$$\begin{aligned} a_2 &= \frac{\mu_2^2 ch\alpha_2 - \alpha_2^2 ch\mu_2}{\mu_2^2 - \alpha_2^2}, & b_2 &= \frac{\alpha_2^3 sh\mu_2 - \mu_2^3 sh\alpha_2}{\alpha_2\mu_2(\alpha_2^2 - \mu_2^2)} \\ c_2 &= \frac{ch\alpha_2 - ch\mu_2}{\alpha_2^2 - \mu_2^2}, & d_2 &= \frac{\mu_2 sh\alpha_2 - \alpha_2 sh\mu_2}{\alpha_2\mu_2(\alpha_2^2 - \mu_2^2)}. \end{aligned} \tag{2.5}$$

In addition if $b_0 = 0$,

$$a_2 = 1, \; b_2 = 1, \; c_2 = \frac{-1 + ch\alpha_2}{\alpha_2^2}, \; d_2 = \frac{-\alpha_2 + sh\alpha_2}{\alpha_2^3}$$

or

$$e^A = I + \frac{sh\alpha_2}{\alpha_2} A + \frac{-1 + ch\alpha_2}{\alpha_2} A^2 .$$

iii) If $b_2^2 - 4b_0 < 0$ and $b_2 \neq 0$,

$$e^A = a_3 I + b_3 A + c_3 A^2 + d_3 A^3,$$

$$\begin{aligned}
a_3 &= \frac{1}{2}\left(2uv\, chu \cos v - (u^2 - v^2) shu \sin v\right) \\
b_3 &= \frac{1}{2uv(u^2+v^2)}\left(v(-v^2+3u^2) shu \cos v + u(-u^2+3v^2) chu \sin v\right) \\
c_3 &= \frac{1}{2}\left(shu \sin v\right) \\
d_3 &= \frac{1}{2uv(u^2+v^2)}\left(u\, chu \sin v - v\, shu \cos v\right)
\end{aligned} \tag{2.6}$$

In addition if $b_2 = 0$,

$$\begin{aligned}
a_3 &= chu \cos u \\
b_3 &= \frac{1}{2u}\left(shu \cos u + chu \sin u\right) \\
c_3 &= \frac{1}{2u^2}\left(shu \sin u\right) \\
d_3 &= \frac{1}{4u^3}\left(chu \sin u - shu \cos u\right).
\end{aligned}$$

where I is the identity matrix of order 4 .

Proof. From the Sylvester formula [1, p.437]

$$e^A = \sum_{i=1}^{4} e^{\lambda_i} A_i \tag{2.7}$$

where A_i are the Frobenius covariants of A that are polynomials of degree 3 evaluated in A. Then e^A has a form given by (2.3), (2.4) and (2.5). Instead of computing the four coefficient from (2.7) we can follow a different way. If λ_i and x_i are the $i-th$ eigenvalue of A and its eigenvector then

$$e^A x_i = ax_i + bAx_i + cA^2 x_i + dA^3 x_i .$$

Hence

$$e^{\lambda_i} x_i = ax_i + b\lambda_i x_i + c\lambda_i^i x_i + d\lambda_i^3 x_i \tag{2.8}$$

which is a linear system with unknowns a, b, c and d, having a unique solution given by (2.7). If the eigenvalues of A are not distinct then the formula for e^A is very simple. In fact matrix A is always diagonalizable (since iA is Hermitian) and if $\pm iA$ are the eigenvalues of A then it is possible apply the same formula (2.8) to find e^A . In this case e^A is a first degree polynomial in A and proceeding as in Theorem 2.4 we find
If $b_2^2 - 4b_0 = 0$ and $b_2 > 0$,

$$e^A = \cos\mu\, I + \frac{\sin\mu}{\mu} A .$$

If $b_2^2 - 4b_0 = 0$ and $b_2 < 0$,

$$e^A = ch\alpha \ I + \frac{sh\alpha}{\alpha} A..$$

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