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# NULL GENERALIZED HELICES IN $L^3$ AND $L^4$, 3 AND 4 –DIMENSIONAL LORENTZIAN SPACE

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**Abstract-** In this paper we obtain the harmonic curvatures of a null generalized helix in $L^3$ and $L^4$, 3 and 4-dimensional Lorentzian spaces. Thus we another definitions of null generalized helix in terms of harmonic curvatures. Also we find the relation of between Frenet frame, slope axis and curvature functions $k_1$, $k_2$ of null generalized helix. Finally we characterize the null helices thanks to harmonic curvatures in $L^3$ and $L^4$, 3 and 4- dimensional Lorentzian space.
**Keywords-** Lorentzian space, null helix, harmonic curvature

## 1. INTRODUCTION

The study of generalized helices in 3-dimensional Euclidean space $IR^3$ amounts to 1802 when M.A. Lancret stated that "a necessary and sufficient condition in order to a curve be a generalized helix is that its torsion is a constant multiple of its curvature". Here a generalized helix is a curve of constant slope, that is, a curve whose tangent indicatrix is a planar curve [2].

In $IR^3$, a generalized helix satisfies that its tangent makes a constant angle with a fixed direction (called the axis). In the general case, we must replace "fixed" direction by "parallel vector field". Hayden proved that a curve is a generalized helix is there exists a parallel vector field lying in the osculating space of the curve and making constant angles with the tangent and the principal normals [2].

When the ambient space is a Lorentzian space form, some results have been obtained. For example, in [3] a non-null curve $\gamma$ immersed in $L^3$ is called a generalized helix if its tangent indicatrix is contained in some plane, say $\pi$, of $L^3$ [2].

In the geometry of null curves difficulties arise because the arc length vanishes, so that it is not possible to normalize the tangent vector in the usual way. A method of proceeding is to introduce a new parameter called the pseudo-arc which normalizes the derivative of the tangent vector. Many authors generalize the results of Bonnor, since for a null curve in an n-dimensional Lorentzian space form they introduce a Frenet frame with the minimum number of curvature functions (which call the Cartan frame), and then they study the null helices in those spaces, that is, null curves with constant curvatures [2].

In this paper we use the Duggal's Frenet equations introduced [1] and distinguished Frenet frame with respect to distinguished parameter $t$ to define and study null generalized helices in the Lorentzian space for null curves. First we obtain 1.th Harmonic curvature of a null generalized helix and then we obtain the relation between Frenet frame $\{T, N, W_1\}$ and slope axis $X$ and curvature functions $k_1$, $k_2$. That is we show that

$$\langle T, X\rangle = \sqrt{\frac{k_1}{2k_2}} \text{ and } \langle N, X\rangle = \sqrt{\frac{k_2}{2k_1}},$$

where $k_1$, $k_2$>0. Later we obtain 2.th Harmonic curvatures of null generalized helix in $L^4$. Thus we get the relation of between Frenet frame $\{T, N, W_1, W_2\}$ and slope axis $X$ and Harmonic curvatures $H_1$, $H_2$. Later we give another definitions of null generalized helix in terms of harmonic curvatures. Finally we characterize the null helices in $L^3$ and $L^4$, 3 and 4- dimensional Lorentzian space. That is we show that

a) $\alpha$ is a null helix in $L^3$ $\Leftrightarrow$ $H_1$=constant

b) $\alpha$ is a null helix in $L^4$ $\Leftrightarrow$ $2H_1 + H_2^2$=constant .

## 2. HARMONIC CURVATURE OF A NULL GENERALIZED HELIX IN $L^3$, 3-DIMENSIONAL LORENTZIAN SPACE

Suppose $\alpha$ is a null curve of a 3-dimensional Lorentz manifold $(M, \langle,\rangle)$. Denote by $\nabla$ the Levi-civita connection on $M$ and $\alpha' = T$. In this case, $\{T, N, W_1\}$ is the Frenet frame of $\alpha \subset L^3$, where $T$ and $N$ are null vectors and $W_1$ is a space-like vector. Thus the Frenet equations of a null curve in a 3-dimensional Lorentz manifold write down as follows:

$$\begin{aligned} \nabla T &= hT + k_1 W_1 \\ \nabla N &= -hN + k_2 W_1 \\ \nabla W_1 &= -k_2 T - k_1 N \end{aligned} \tag{2.1}$$

where $h$ and $\{k_1, k_2\}$ are smooth functions and $\{W_1\}$ is a certain orthonormal basis of $\Gamma(S(T\alpha))$. If $h = 0$, then the parameter $t$ is said to be a distinguished parameter. Then $\{T, N, W_1\}$is called a distinguished Frenet frame [1,4] .

**Theorem 2.1-** Assume that $\alpha \subset L^3$ is a null generalized helix given by distinguished Frenet frame $\{T, N, W_1\}$ and curvature functions $k_1$, $k_2$. Let $X$ be a unit and constant vector field (time-like) of $L^3$. Then $Sp\{X\}$being a slope axis,

$$\langle N, X\rangle = H_1 \langle T, X\rangle$$

and, where $H_1 = -\frac{k_2}{k_1}$ = constant, $k_1, k_2 > 0$.

**Proof-** If $\alpha$ is a null helix in $L^3$, then for the unit constant vector field $X$ we can write $\langle T, X\rangle$=constant. Thus by taking the derivative we obtain $\langle \nabla_T T, X\rangle = 0$. Moreover by using the first equation from (2.1) we obtain

$$\langle \nabla_T T, X\rangle = k_1 \langle W_1, X\rangle$$

where, since $\langle \nabla_T T, X\rangle = 0$ we can write

$$0 = k_1 \langle W_1, X\rangle$$

or since we know that $k_1 \neq 0$, we can write

$$0 = \langle W_1, X\rangle . \tag{2.2}$$

The second equation from (2.1) gives us the following equation;

$$\langle \nabla_T N, X \rangle = k_2 \langle W_1, X \rangle .$$

If we use (2.2) here, we obtain

$$\langle \nabla_T N, X \rangle = 0 \Rightarrow \langle N, X \rangle = constant . \tag{2.3}$$

Using the third equation from (2.1) we obtain

$$\langle \nabla_T W_1, X \rangle = -k_2 \langle T, X \rangle - k_1 \langle N, X \rangle$$

where since $\langle W_1, X \rangle = 0$, we can write

$$0 = -k_2 \langle T, X \rangle - k_1 \langle N, X \rangle$$

or

$$\langle N, X \rangle = -\frac{k_2}{k_1} \langle T, X \rangle$$

or

$$\langle N, X \rangle = H_1 \langle T, X \rangle . \tag{2.4}$$

Since $\langle T, X \rangle = constant$ and $\langle N, X \rangle = constant$, we have

$$H_1 = -\frac{k_2}{k_1} = constant . \tag{2.5}$$

Finally, since $X \in Sp\{T, N, W_1\}$, it is not difficult to show that:

$$X = \langle N, X \rangle T + \langle T, X \rangle N \tag{2.6}$$

where since $X$ is a time-like vector field, then $\langle X, X \rangle = -1$. Thus from (2.6) we have

$$-1 = \langle X, X \rangle = 2 \langle N, X \rangle \langle T, X \rangle$$

$$-1 = 2H_1 \langle T, X \rangle^2$$

$$-\frac{1}{2H_1} = \langle T, X \rangle^2$$

and since $H_1 = -\frac{k_2}{k_1}$, we obtain $\langle T, X \rangle^2 = \frac{k_1}{2k_2}$, $k_1, k_2 > 0$ or

$$\langle T, X \rangle = \pm \frac{\sqrt{2}}{2} \sqrt{\frac{k_1}{k_2}} = constant$$

from this

$$\langle N, X \rangle = \pm \frac{\sqrt{2}}{2} \sqrt{\frac{k_2}{k_1}} = constant .$$

Conversely if we use (2.4) and (2.5) we can show that $X$ is a constant vector field. For this we take the derivative of (2.6) with respect to $T$. Thus we obtain

$$\nabla_T X = \langle \nabla_T N, X \rangle T + \langle N, X \rangle \nabla_T T + \langle T, X \rangle \nabla_T N + \langle \nabla_T T, X \rangle N$$

or since $\langle \nabla_T N, X \rangle = \langle \nabla_T T, X \rangle = 0$, we obtain

$$\nabla_T X = \langle N, X \rangle \nabla_T T + \langle T, X \rangle \nabla_T N .$$

If we use the Frenet equations in (2.1) we have

$$\nabla_T X = \langle N, X\rangle(k_1W_1) + \langle T, X\rangle(k_2W_1)$$
$$= \{k_1\langle N, X\rangle + k_2\langle T, X\rangle\}W_1$$

or from (2.4) and (2.5) we have

$$\nabla_T X = 0.$$

That is, $X$ is a constant vector field of $L^3$.

**Definition 2.1-** A null curve $\alpha : I \to L^3$ is said to be a generalized helix if there exist a non-zero unit constant vector $X$ such that the products $\langle \alpha'(t), X\rangle \neq 0, \langle N(t), X\rangle \neq 0$, are constant and $\langle W_1, X\rangle = 0$. Where $X$ is a time-like vector field. Also, if 1.th and 2.th curvature functions of $\alpha$ are $k_1$ and $k_2$ then we obtain

$$\langle \alpha'(t), X\rangle = \pm\sqrt{\frac{k_1}{2k_2}} \text{ and } \langle N(t), X\rangle = \pm\sqrt{\frac{k_2}{2k_1}}.$$

## 3. HARMONIC CURVATURE OF A NULL GENERALIZED HELIX IN $L^4$, 4-DIMENSIONAL LORENTZIAN SPACE

Suppose $\alpha$ is a null curve of a 4-dimensional Lorentz manifold $(M, \langle,\rangle)$. Denote by $\nabla$ the Levi-Civita connection on $M$ and $\alpha' = T$. In this case, $\{T, N, W_1, W_2\}$ is the Frenet frame of $\alpha \subset L^4$, where $T$ and $N$ are null vectors and $W_1, W_2$ are space-like vectors. Thus the Frenet equations of a null curve in a 4-dimensional Lorentz manifold write down as follows:

$$\begin{aligned} \nabla T &= hT + k_1W_1 \\ \nabla N &= -hN + k_2W_1 + k_3W_2 \\ \nabla W_1 &= -k_2T - k_1N + k_4W_2 \\ \nabla W_2 &= -k_3T - k_4W_1 \end{aligned} \tag{3.1}$$

where $h$ and $\{k_1, k_2, k_3, k_4\}$ are smooth functions and $\{W_1, W_2\}$ is a certain orthonormal basis of $\Gamma(S(T\alpha))$. If $h = 0$, then the parameter $t$ is said to be a distinguished parameter. Moreover, if the last curvature $k_4$ vanishes, then $\{T, N, W_1, W_2\}$ is called a distinguished Frenet frame [1,3]. Thus we have

$$\langle T, T\rangle = \langle T, W_1\rangle = \langle T, W_2\rangle = \langle W_1, W_2\rangle = 0$$
$$\langle N, N\rangle = \langle N, W_1\rangle = \langle N, W_2\rangle = 0$$
$$\langle W_1, W_1\rangle = \langle W_2, W_2\rangle = 1 \quad and \quad \langle T, N\rangle = 1.$$

**Theorem 3.1-** Assume that $\alpha \subset L^4$ is a null generalized helix given by distinguished Frenet frame $\{T, N, W_1, W_2\}$ and curvature functions $k_1, k_2, k_3$. Let $X$ be a unit and constant vector field (time-like or space-like) of $L^4$. Then $Sp\{X\}$ being a slope axis,

$$\langle W_2, X\rangle = H_2\langle T, X\rangle$$

and, where $H_2 = \dfrac{H_1'}{k_3}$, moreover $H_2' = -k_3$.

**Proof-** Since $\alpha$ is a null helix in $L^4$,we can write $\langle T, X\rangle = constant$. Moreover, from (3.1) we have $\langle W_1, X\rangle = 0, \langle N, X\rangle = H_1\langle T, X\rangle$ and $H_1 = -\dfrac{k_2}{k_1}$. The second equation from (3.1) gives us the following equation;

$$k_3\langle W_2, X\rangle = \langle \nabla_T N, X\rangle$$

or

$$\langle W_2, X\rangle = \frac{1}{k_3} H_1^{'}\langle T, X\rangle \tag{3.2}$$

where, $H_2 = \dfrac{H_1^{'}}{k_3}$ . Therefore we can write

$$\langle W_2, X\rangle = H_2\langle T, X\rangle . \tag{3.3}$$

If we take the derivative of (3.3) with respect to $T$ we have

$$\langle \nabla_T W_2, X\rangle = H_2^{'}\langle T, X\rangle . \tag{3.4}$$

Using the last equation from (3.1) we obtain

$$\langle \nabla_T W_2, X\rangle = -k_3\langle T, X\rangle . \tag{3.5}$$

Thus from (3.4) and (3.5) we have

$$H_2^{'} = -k_3 \ . \tag{3.6}$$

Finally, since $X \in \{T, N, W_1, W_2\}$, it is not difficult to show that:

$$X = \langle N, X\rangle T + \langle T, X\rangle N + \langle W_2, X\rangle W_2 \tag{3.7}$$

where since $X$ is a time-like or space-like vector field, then $\langle X, X\rangle = \pm 1$. Thus from (3.7)we have,

$$\begin{aligned} \pm 1 &= \langle X, X\rangle = 2\langle N, X\rangle\langle T, X\rangle + \langle W_2, X\rangle^2 \\ \pm 1 &= \left(2H_1 + H_2^2\right)\langle T, X\rangle^2 \\ \pm\frac{1}{\langle T, X\rangle^2} &= 2H_1 + H_2^2 = constant. \end{aligned} \tag{3.8}$$

If we take the derivative of (3.8) we obtain

$$0 = 2H_1^{'} + 2H_2 H_2^{'}$$

or

$$0 = 2H_1^{'} + 2H_2(-k_3)$$

or

$$H_2 = \frac{H_1^{'}}{k_3} .$$

**Definition 3.1-** Assume that $\alpha \subset L^4$ is a null generalized helix given by curvature functions $k_1, k_2, k_3$. Then the harmonic curvatures of $\alpha$ in $L^4$ write-down as follows:

$$H_i = \begin{cases} -\dfrac{k_2}{k_1}, & i = 1 \\ \dfrac{H_1^{'}}{k_3}, & i = 2 \end{cases}$$

**Definition 3.2-**A null curve $\alpha : I \to L^4$ is said to be a generalized helix if there exist a non-zero unit constant vector $X$ such that $\langle \alpha^{'}(t), X \rangle = cons\tan t$. Then $Sp\{X\}$ is called slope axis and for the Frenet frame $\{T, N, W_1, W_2\}$ we have $\langle N(t), X \rangle = H_1 \langle T, X \rangle$, $\langle W_1, X \rangle = 0$, $\langle W_2, X \rangle = H_2 \langle T, X \rangle$.

**Definition 3.3-**A null curve $\alpha : I \to L^4$ is said to be a generalized helix if there exist harmonic curvatures $H_1$ and $H_2$ such that

$$H_1^{'} + H_2 H_2^{'} = 0.$$

**Corollary 3.1-**A null curve $\alpha : I \to L^4$ is said to be a generalized helix if there exist harmonic curvatures $H_1$ and $H_2$ such that

$$2H_1 + H_2^2 = cons\tan t \left( = \pm \frac{1}{\langle T, X \rangle^2} \right).$$

## 4. THE CHARACTERIZATION OF NULL HELIX WITH EVERY CONSTANT CURVATURES IN $L^4$

**Theorem 4.1-**Assume that $\alpha \subset L^4$ is a null generalized helix given by distinguished Frenet frame $\{T, N, W_1, W_2\}$ and curvature functions $k_1, k_2, k_3 (k_4 = 0)$. If $k_1 = 1$ and $k_2, k_3$ are both constants, then

$$\nabla_T^4 T - 2H_1 \nabla_T^2 T = 0. \tag{4.1}$$

This is the characterization of null generalized helix with every constant curvatures.

**Proof-**Since $k_1 = 1$, from (3.1) we have

$$\nabla_T^2 T = \nabla_T W_1 \Rightarrow \nabla_T^3 T = \nabla_T^2 W_1 \Rightarrow \nabla_T^4 T = \nabla_T^3 W_1. \tag{4.2}$$

Since $\nabla_T^2 W_1 = -k_2 \nabla_T T - \nabla_T N$, we have

$$\nabla_T^3 W_1 = -k_2 \nabla_T^2 T - \nabla_T^2 N \tag{4.3}$$

and from (3.1) we have

$$\nabla_T^2 N = k_2 \nabla_T W_1 + k_3 \nabla_T W_2 . \tag{4.4}$$

Thus from (3.1), (4.3) and (4.4) we can

$$\nabla_T^3 W_1 = -k_2 \nabla_T^2 T + k_3^2 T - k_2 \nabla_T W_1 . \tag{4.5}$$

From (4.2) and (4.5) we can write;

$$\nabla_T^4 T = -2k_2 \nabla_T^2 T + k_3^2 T$$

or

$$\nabla_T^4 T + 2k_2 \nabla_T^2 T - k_3^2 T = 0 .$$

where, since $H_1 = cons\tan t$, $H_2 = 0$ that is $k_3 = 0$. Thus we have

$$\nabla_T^4 T + 2k_2 \nabla_T^2 T = 0$$

or since $k_2 = -H_1$ we obtain

$$\nabla_T^4 T - 2H_1 \nabla_T^2 T = 0 .$$

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