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THE FRENET AND DARBOUX INSTANTANEOUS ROTATION VECTORS FOR CURVES ON SPACE-LIKE SURFACE

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ABSTRACT

In this paper , considering the Darboux instantaneous rotation vector of a solid perpendicular trihedron in the Minkowski 3-space R_1^3 , the Frenet instantaneous rotation vector was stated for the Frenet trihedron of a space -like space curve (c) with the binormal **b** being a time-like vector. The Darboux derivative formulas and the Darboux instantaneous rotation vector were found when the curve (c) is on a space -like surface . A fundamental relation , as a base for the geometry of space-like surfaces, was obtained among the Darboux vectors of the parameter curves (c_1) , (c_2) and an arbitrary curve (c) on a space-like surface.

1. INTRODUCTION

In Euclidean space R^3 a solid perpendicular trihedrons' Darboux instantaneous rotation vector and Darboux and Frenet instantaneous rotation vectors of a curve on the surface are known. Let φ be the angle between principal normal **n** and surface normal **N** on a point P of a curve . For the Radii of geodesic torsion T_g, normal curvature R_n and geodesic curvature R_g some relations are given in [1].

Instead of space R^3 with $\mathbf{a} = (a_1,a_2,a_3)$ and $\mathbf{b} = (b_1,b_2,b_3) \in R^3$,

let us consider the Minkowski 3-space R_1^3 provided with Lorentzian inner pruduct .

$$\langle \mathbf{a} ,\mathbf{b}\rangle = a_1b_1 + a_2b_2 - a_3b_3 \qquad (1.1)$$

In this condition the following definitions can be given.

Let $\mathbf{a} = (a_1,a_2,a_3) \in R^3_1$ $\langle \mathbf{a}, \mathbf{a} \rangle > 0$ then $\mathbf{a}$ is space-like vector, $\langle \mathbf{a}, \mathbf{a} \rangle < 0$ then $\mathbf{a}$ is time-like vector, $\langle \mathbf{a}, \mathbf{a} \rangle = 0$ then $\mathbf{a}$ is light-like(null) vector. If $\sqrt{a_1^2 + a_2^2} < a_3$ (or $\sqrt{a_1^2 + a_2^2} > a_3$) then $\mathbf{a}$ is future pointing (or past pointing) vector [2].

Let (c) be a curve in space R^3_1. $c'(t)$ is the tangent vector for $\forall\ t \in I \subset R$ then if

$\langle c'(t), c'(t) \rangle > 0$ then (c) is space-like curve,

$\langle c'(t), c'(t) \rangle < 0$ then (c) is time-like curve ,

$\langle c'(t), c'(t) \rangle = 0$ then (c) is light-like (null) curve

[3].

Let $\mathbf{a}$ and $\mathbf{b}$ be a surface in space R^3_1 .The vectoral product of $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{a} \wedge \mathbf{b} = (a_3b_2 - a_2b_3\ ,\ a_1b_3 - a_3b_1\ ,\ a_1b_2 - a_2b_1) \qquad (1.2)$$

[4].

Definition 1.1. Let $y = y(u,v)$ be a surface in space R^3_1.if $\forall$ $p \in y(u,v)$ and $\langle , \rangle|_y$ is a Lorentzian metric then $y = y(u,v)$ is space-like surface [5].

Definition 1.2. Let $\mathbf{a} = (a_1,a_2)$ and $\mathbf{b} = (b_1,b_2) \in R^2_1$ be future pointing (past pointing)time -like vectors. The number $\theta \in R$ in equality

$$\begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad (1.3)$$

is an hyperbolic angle from $\mathbf{a}$ to $\mathbf{b}$ and θ is shown by $(\mathbf{a},\mathbf{b})$ [2].

Lemma 1.3. Let $\mathbf{a}$ and $\mathbf{b}$ be, future pointing time-like unit vectors. If θ is an hyperbolic angle from $\mathbf{a}$ and $\mathbf{b}$, then

$$\cosh\theta = -\langle \mathbf{a}, \mathbf{b} \rangle \qquad (1.4)$$

[2].

Let $[e_1,e_2,e_3]$ be a solid perpendicular trihedron in space R^3_1.In this situation the following theorem can be given :

Theorem 1.4. If A solid perpendicular trihedrons' unit vectors e_1, e_2 e_3 are changing relative to t parameter , then

$$\frac{de_i}{dt} = w \wedge e_i \quad , \quad i = 1,2,3 \qquad (1.5)$$

where, e_1 and e_2 are space-like vectors, e_3 is time-like vector and Darboux instantaneous rotation vector is

$$w = ae_1 - be_2 - ce_3 \qquad (1.6)$$

[6].

2.FRENET TRIHEDRON FOR A SPACE-LIKE SPACE CURVE WITH THE BINORMAL TIME-LIKE VECTOR

Let us consider c = c(s) space-like curve. For any parameter s on all points on this curve , we can construct Frenet trihedron [**t** , **n** , **b**] , here, **t** , **n** and **b** tangent, principal normal and binormal unit vectors, respectively. In this trihedron , **b** is time-like unit vector ; **t** and **n** are space-like unit vectors. For that,

$$\langle \mathbf{t}, \mathbf{t} \rangle = \langle \mathbf{n}, \mathbf{n} \rangle = 1 \quad , \quad \langle \mathbf{b}, \mathbf{b} \rangle = -1 \tag{2.1}$$

$$\langle \mathbf{t}, \mathbf{n} \rangle = \langle \mathbf{n}, \mathbf{b} \rangle = \langle \mathbf{b}, \mathbf{t} \rangle = 0 \tag{2.2}$$

can be written . For Frenet trihedrons' vectors ,the vectoral product is given by

$$\mathbf{t} \wedge \mathbf{n} = \mathbf{b} \qquad \mathbf{n} \wedge \mathbf{b} = -\,\mathbf{t} \qquad \mathbf{b} \wedge \mathbf{t} = -\,\mathbf{n} \tag{2.3}$$

It can written

$$\begin{aligned} \frac{d\mathbf{t}}{ds} &= a_{11}\mathbf{t} + a_{12}\mathbf{n} + a_{13}\mathbf{b} \\ \frac{d\mathbf{n}}{ds} &= a_{21}\mathbf{t} + a_{22}\mathbf{n} + a_{23}\mathbf{b} \\ \frac{d\mathbf{b}}{ds} &= a_{31}\mathbf{t} + a_{32}\mathbf{n} + a_{33}\mathbf{b} \end{aligned} \tag{2.4}$$

for a parameter's specific value s = s_0 with **t** = **t**(s) , **n** = **n**(s) , and **b** = **b**(s). If we take derivatives of equivalence (2.1) and (2.2) with respect to arc s , then we have

$$\begin{aligned} &\langle \mathbf{t}, \frac{d\mathbf{t}}{ds} \rangle = \langle \mathbf{n}, \frac{d\mathbf{n}}{ds} \rangle = \langle \mathbf{b}, \frac{d\mathbf{b}}{ds} \rangle = 0 \\ &\langle \frac{d\mathbf{t}}{ds}, \mathbf{n} \rangle + \langle \mathbf{t}, \frac{d\mathbf{n}}{ds} \rangle = \langle \frac{d\mathbf{n}}{ds}, \mathbf{b} \rangle + \langle \mathbf{n}, \frac{d\mathbf{b}}{ds} \rangle = \langle \frac{d\mathbf{b}}{ds}, \mathbf{t} \rangle + \langle \mathbf{b}, \frac{d\mathbf{t}}{ds} \rangle = 0 \end{aligned} \tag{2.5}$$

Assuming that $a_{12} = -\,a_{21} = c$, $a_{32} = a_{23} = a$ and $a_{31} = a_{13} = b$

we obtain the following formulas for derivatives :

$$\begin{aligned} \frac{d\mathbf{t}}{ds} &= c\mathbf{n} + b\mathbf{b} \\ \frac{d\mathbf{n}}{ds} &= -c\mathbf{t} + a\mathbf{b} \\ \frac{d\mathbf{b}}{ds} &= b\mathbf{t} + a\mathbf{n} \end{aligned} \tag{2.6}$$

On a space-like curve $c = c(s)$ given a point P, if the radius of curvature is R and radius of torsion is T then Frenet formulae are obtained as following [7].

$$\frac{d\mathbf{t}}{ds} = \frac{1}{R}\mathbf{n}$$
$$\frac{d\mathbf{n}}{ds} = -\frac{1}{R}\mathbf{t} + \frac{1}{T}\mathbf{b} \qquad (2.7)$$
$$\frac{d\mathbf{b}}{ds} = \frac{1}{T}\mathbf{n}$$

If (2.6) and (2.7) are compared , then we obtain

$$a = \frac{1}{T} \ , \quad b = 0 \ , \quad c = \frac{1}{R}.$$

For that , for any perpendicular trihedron , the Darboux vector is

$$\mathbf{w} = a\mathbf{e}_1 - b\mathbf{e}_2 - c\mathbf{e}_3$$

and for Frenet trihedron we find

$$\mathbf{f} = \frac{1}{T}\mathbf{t} - \frac{1}{R}\mathbf{b} \ . \qquad (2.8)$$

In this situation , if we apply the formula (1.5) obtained for a general perpendicular trihedron to Frenet trihedron , we can write

$$\frac{d\mathbf{t}}{ds} = \mathbf{f} \wedge \mathbf{t}$$
$$\frac{d\mathbf{n}}{ds} = \mathbf{f} \wedge \mathbf{n} \qquad (2.9)$$
$$\frac{d\mathbf{b}}{ds} = \mathbf{f} \wedge \mathbf{b}$$

Then the matrix form of (2.7) is

$$\frac{d}{ds}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & 1/R & 0 \\ -1/R & 0 & 1/T \\ 0 & 1/T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} . \qquad (2.10)$$

3. DARBOUX TRIHEDRON FOR A SPACE-LIKE CURVE WITH TIME-LIKE NORMAL

Let us consider space-like $y = y(u,v)$ surface. For space-like curve (c) on $y = y(u,v)$ there exists the Frenet trihedron $[\mathbf{t},\mathbf{n},\mathbf{b}]$ at all points of (c) .There is also exist a second trihedron because the curve (c) is on surface $y = y(u,v)$. Let us denote the tangent space -like unit vector with $\mathbf{t}$, the normal space-like unit vector with $\mathbf{g}$ at the point P . In this case, If we take the time-like vector $\mathbf{N}$ defined by

$$\mathbf{t} \wedge \mathbf{g} = \mathbf{N} \qquad \mathbf{g} \wedge \mathbf{N} = -\mathbf{t} \qquad \mathbf{N} \wedge \mathbf{t} = -\mathbf{g} \tag{3.1}$$

Then we obtain a new trihedron [$\mathbf{t}$, $\mathbf{g}$, $\mathbf{N}$]. Let θ is the hyperbolic angle between the time-like vectors $\mathbf{b}$ and $\mathbf{N}$ (figure. 1.)

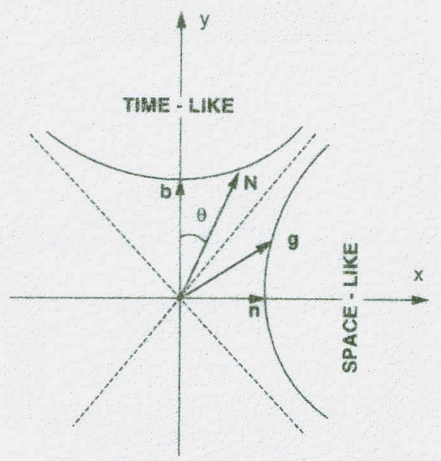

Figure. 1

Then we can write

$$\begin{aligned} \mathbf{g} &= \mathbf{n}\cosh\theta + \mathbf{b}\sinh\theta \\ \mathbf{N} &= \mathbf{n}\sinh\theta + \mathbf{b}\cosh\theta \end{aligned} \tag{3.2}$$

If we take derivatives of . $\mathbf{t}$, $\mathbf{g}$ and $\mathbf{N}$ according to the arc s of curve (c) we can find

$$\begin{aligned} \frac{d\mathbf{t}}{ds} &= \rho\cosh\theta\,\mathbf{g} - \rho\sinh\theta\,\mathbf{N} \\ \frac{d\mathbf{g}}{ds} &= -\rho\cosh\theta\,\mathbf{t} + \left(\tau + \frac{d\theta}{ds}\right)\mathbf{N} \\ \frac{d\mathbf{N}}{ds} &= -\rho\sinh\theta\,\mathbf{t} + \left(\tau + \frac{d\theta}{ds}\right)\mathbf{g} \end{aligned}$$

here , if we replace

$$\rho\cosh\theta = \frac{\cosh\theta}{R} = \frac{1}{R_g} = \rho_g$$

$$-\rho\sinh\theta = -\frac{\sinh\theta}{R} = \frac{1}{R_n} = \rho_n$$

$$\tau + \frac{d\theta}{ds} = \frac{1}{T} + \frac{d\theta}{ds} = \frac{1}{T_g} = \tau_g$$

then the Darboux derivative formulae are given by

$$\begin{aligned} \frac{d\mathbf{t}}{ds} &= \rho_g \mathbf{g} + \rho_n \mathbf{N} \\ \frac{d\mathbf{g}}{ds} &= -\rho_g \mathbf{t} + \tau_g \mathbf{N}, \\ \frac{d\mathbf{N}}{ds} &= \rho_n \mathbf{t} + \tau_g \mathbf{g} \end{aligned} \tag{3.3}$$

where ρ_g is geodesic curvature, , ρ_n is normal curvature and τ_g is geodesic torsion. The matrix form of (3.3) is

$$\frac{d}{ds}\begin{bmatrix} \mathbf{t} \\ \mathbf{g} \\ \mathbf{N} \end{bmatrix} = \begin{bmatrix} 0 & 1/R_g & 1/R_n \\ -1/R_g & 0 & 1/T_g \\ 1/R_n & 1/T_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{g} \\ \mathbf{N} \end{bmatrix}. \tag{3.4}$$

We can write the Darboux instantaneous rotation vector of Darboux trihedron

$$\mathbf{w} = \frac{\mathbf{t}}{T_g} - \frac{\mathbf{g}}{R_n} - \frac{\mathbf{N}}{R_g}. \tag{3.5}$$

Darboux derivative formulae (3.3) can be given with Darboux vector as below

$$\begin{aligned} \frac{d\mathbf{t}}{ds} &= \mathbf{w} \wedge \mathbf{t} \\ \frac{d\mathbf{g}}{ds} &= \mathbf{w} \wedge \mathbf{g} \quad . \\ \frac{d\mathbf{N}}{ds} &= \mathbf{w} \wedge \mathbf{N} \end{aligned} \tag{3.6}$$

Theorem 3.1. If the radius of torsion of space-like curve (c) drawn on space-like surface y = y(u,v) is and the hyperbolic angle between time-like unit vectors **b** and **N** is θ , then we have

$$\frac{1}{T_g} = \frac{1}{T} + \frac{d\theta}{ds} \tag{3.7}$$

Proof: If we take derivative of both sides of equation $\langle \mathbf{b} , \mathbf{N} \rangle = -\cosh\theta$ and use the (2.7) and (3.3) we obtain

$$\frac{1}{T}\langle \mathbf{n} , \mathbf{N}\rangle + \langle \mathbf{b} , \left(\frac{1}{R_n}\mathbf{t} + \frac{1}{T_g}\mathbf{g}\right) \rangle = -\sinh\theta\frac{d\theta}{ds}$$

$$\frac{1}{T}\langle \mathbf{n} , \mathbf{N}\rangle + \frac{1}{T_g}\langle \mathbf{b} , \mathbf{g}\rangle = -\sinh\theta\frac{d\theta}{ds}$$

since $\langle \mathbf{n} , \mathbf{N} \rangle = \sinh\theta$, $\langle \mathbf{b} , \mathbf{g} \rangle = -\sinh\theta$, we have

$$\frac{1}{T}\sinh\theta - \frac{1}{T_g}\sinh\theta = -\sinh\theta\frac{d\theta}{ds}$$

$$\frac{1}{T_g} = \frac{1}{T} + \frac{d\theta}{ds}$$

Theorem 3.2. If the radius of curvature of the space-like curve (c) on space-like surface $\mathbf{y} = \mathbf{y}(u,v)$ is $\mathbf{R}$ and the hyperbolic angle between time-like unit vectors $\mathbf{b}$ and $\mathbf{N}$ is θ then we have

$$\frac{1}{R_n} = -\frac{\sinh\theta}{R}$$
$$\frac{1}{R_g} = \frac{\cosh\theta}{R} \qquad (3.8)$$

Proof: From (2.9) and (3.5),

$$\mathbf{f} \wedge \mathbf{t} = \mathbf{w} \wedge \mathbf{t}$$
$$(\mathbf{f} \wedge \mathbf{t}) - (\mathbf{w} \wedge \mathbf{t}) = 0$$
$$(\mathbf{f} - \mathbf{w}) \wedge \mathbf{t} = 0$$

If the values of vectors $\mathbf{f}$ and $\mathbf{w}$ are written

$$\left[\left(\frac{1}{T}\mathbf{t} - \frac{1}{R}\mathbf{b}\right) - \left(\frac{\mathbf{t}}{T_g} - \frac{\mathbf{g}}{R_n} - \frac{\mathbf{N}}{R_g}\right)\right] \wedge \mathbf{t} = 0$$

$$-\frac{1}{R}(\mathbf{b} \wedge \mathbf{t}) + \frac{1}{R_n}(\mathbf{g} \wedge \mathbf{t}) + \frac{1}{R_g}(\mathbf{N} \wedge \mathbf{t}) = 0$$

are obtained. Here , since $\mathbf{b} \wedge \mathbf{t} = -\mathbf{n}$, $\mathbf{g} \wedge \mathbf{t} = -\mathbf{N}$, $\mathbf{N} \wedge \mathbf{t} = -\mathbf{g}$

we find

$$\frac{1}{R}\mathbf{n}=\frac{1}{R_n}\mathbf{N}+\frac{1}{R_g}\mathbf{g}.$$

If the both sides of this equation are scalarly multiplied with the vectors **N** and **g** and considered the equalities

$$\langle \mathbf{n}, \mathbf{N} \rangle = \sinh\theta \ , \langle \mathbf{N}, \mathbf{N} \rangle = -1 \ , \langle \mathbf{n}, \mathbf{g} \rangle = \cosh\theta \ , \ \langle \mathbf{g}, \mathbf{g} \rangle = 1,$$

the proof is completed.

Corollary 3.3. There is a relation between Frenet and Darboux vectors like below:

$$\mathbf{w}=\mathbf{f}+\frac{d\theta}{ds}\mathbf{t} \qquad (3.9)$$

Proof: From (2.8) and (3.4) equations

$$\mathbf{w}=\mathbf{f}+\lambda\ \mathbf{t} \quad , \quad \lambda \in R$$

can be written with $\lambda \in R$.

$$\frac{\mathbf{t}}{T_g}-\frac{\mathbf{g}}{R_n}-\frac{\mathbf{N}}{R_g}=\frac{1}{T}\mathbf{t}-\frac{1}{R}\mathbf{b}+\lambda\,\mathbf{t}$$

$$\frac{1}{T_g}=\frac{1}{T}+\lambda$$

is obtained .If we considered the formula (3.7) , we have

$$\lambda=\frac{d\theta}{ds}.$$

This completes the proof.

Corollary 3.4. Instantaneous rotation velocity vector **w** of the Darboux trihedron is consists of two components. One of them coincides with Frenet trihedrons' instantaneous rotation velocity vector. The other component in the opposite direction of tangent and equal to $\frac{d\theta}{ds}$. For this reason, when a point P of space-like curve (c) on the time-like surface moves, on this curve the Darboux trihedron moves with radial velocity $\frac{d\theta}{ds}$ according to Frenet trihedron in the opposite direction to the tangent at any time.

Corollary 3.5. If the hyperbolic angle between the principal normal of a space-like curve on a time-like surface and at the same of point of the surface of g vector becomes always constant then

$$\frac{1}{T_g} = \frac{1}{T} \quad , \quad w = f$$

Thus , the torsion of the curve at each point equals to the geodesic torsion of the surface at that point.

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