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THE GROUP OF TWIST KNOTS

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ABSTRACT. In this paper some interesting invariants of twist knots are calculated. Especially the first homotopy and the first homology groups of all three sheeted branched covering spaces of the three dimensional sphere branched over twist knots are given.

1.INTRODUCTION. In this paper the invertibility, non-amphicheirality (expect two of them), the Alexander matrix, the Alexander polynominal, the genus, the knot groups and all three sheeted branched covering spaces of twist knots are presented.

Twist knots are in a certain sense generalizations of the trefoil knot (5), the figure-eight knot (6, 10) and the Stevedore's knot (10). These knots are listed as 3_1, 4_1 and 6_1 knots in the table given at the end of Knothentheorie by Reidemeiser (12). In that table, the knots listed as 5_2, 7_2, 8_1 and 9_2 are also twist knots.

A twist knot with n half-twists is denoted by T_n. See figure la. Twist knots are alternating knots and the number of half-twists determine them. Namely, T_n and T_m are equivalent if and only if $n = m$.

Twist knots are also known as Whitehead doubles of the trival knot, the circle (15). Whitehead, using Seifert's method (14) and his original method of calculation gives the following results for the Alexander polynomial of doubled knots.

$$\pm\Delta(t) = \rho t^2 - (2\rho + 1)t + \rho \qquad \text{if } \rho \neq 0$$
$$\Delta(t) = 1 \qquad \text{if } \rho = 0$$

where the integer ρ denotes the complete twists in the doubled knots. Comparing these with the Alexander polynomials obtained below one can see that the Whitehead doubles of the trival knot with $\rho>0$ and $\rho<0$ complete twists corresponds to $T_{2\rho}$ and $T_{-2\rho-1}$ respectively. This can be seen by Reidemeister's moves (12) on the normal diagrams of T_n also.

Bing and Martin (4), studied twist knots and showed by an algebraic method that twist knots satisfy "the Poincare property".

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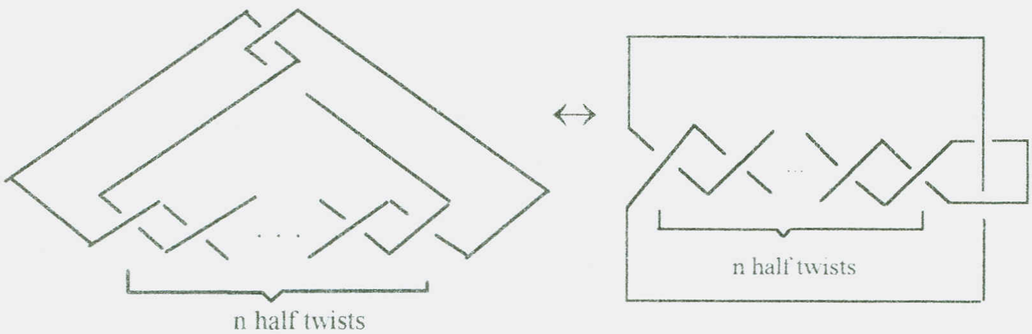
We published some invariants of an equivalent class of knots under the name "Shepherd's knots" in Turkish (2, 3). This name is given to these knots because of their normal diagrams resembling a shepherd who wears acape. See Figure 1b. In (2) it was shown that the three sphere S^3, has a three sheeted cyclic branched covering space branched over every
twist knots. As it is expected, the first homology groups of these covering spaces are direct doubles of a cyclic group (11).

In (7), it was shown that S^3 has athree sheeted irregular covering spaces branched over T_n only for $n = 6m + 1$ and $n = 6m + 4$ $(m = 0,1,2,...)$ since twist knots are with 2-bridges all these covering spaces of S^3 which are also called three sheeted dihedral covering spaces, are again just three dimensional spheres (13, 8). Here some detailed proofs are given.



a. The twist khot T_n b. The Shepherd's knot.

Figure 1.

2. SOME BASIC DEFINITIONS IN KNOT THEORY

Definition 1. A homeomorphic image (an embedding) K of the unit circle $\{(x, y) : x^2+y^2 = 1\}$ into S^3 is called a knot (10). Namely a knot is a simple closed curve in S^3.

Definition 2. Two oriented knots K and L are called equivalent if there exists an orientation preserving homeomorphism of S^3 onto itself which maps K onto L (10).

Definition 3. A knot K is called invertible if there exists an orientation preserving homeomorphism $f: S^3 \to S^3$, such that the restriction $f \mid K$ is an orientation reversing homeomorphism of K onto itself (10).

All twist knots are invertible. One has only to turn them over.

Definition 4. A knot K is called amphicheiral if it is equivalent to its mirror image.

Definition 5. The fundamental group (the first homotopy group) of S^3 - K is called the knot group of the knot K (10).

Definition 6. Let $(x_1, x_2, \ldots x_n : r_1, r_2, \ldots r_m)$ be a presentation of a knot group $G=\pi_1(S^3-K)$. The matrix (a_{ij}) defined by

$$a_{ij} = \alpha\phi\left(\frac{\partial r_i}{\partial x_j}\right), \quad i=1,\ldots,m; \quad j=1,\ldots,n$$

is called the Alexander matrix of G (or of K), where α is Abelianizer of G and $\partial/\partial x_j$ are Fox's free derivatives (10).

Definition 7. For any integer $k \geq 0$ the k th knot polynomial of a finite presentation $G=(x_1, x_2, \ldots x_n : r_1, r_2, \ldots r_m)$ of a knot group is the greatest common divisor of the determinants of all (n-k)x(n-k) submatrices of the Alexander matrix of G. The first knot polynomial $\Delta(t)$ is called the Alexander polynomial of the knot group (or the knot) (10).

Definition 8. A surface with only one boundary which takes a given knot as its boundary is called a spanning surface of the knot (11, 14).

Definition 9. The minimum of genera of oriented surfaces which span a knot is called the genus of the knot (11, 14).

3. SOME INTERESTING PROPERTIES OF TWIST KNOTS

Theorem 1. All twist knots are invertible.

Theorem 2. All twist knots expect T_0 and T_2 are not amphicherial.

Proof. Every twist knot is a knot with two bridges. Namely T_{2k} and T_{2k+1} are equivalent to (4k + 1, 2k + 1) and (4k + 3, 2k + 1) - two bridge knots respecttively. According to Schubert (13), (α, β)-two bridge knot is amphicheiral if $\beta^2 \equiv (-1) \pmod{2\alpha}$.

This congruence for T_{2k} becomes

$$(2k + 1)^2 \equiv (-1) \pmod{(8k + 2)}$$

and it holds only when k = 0 or 1. Namely only T_0 and T_2 are amphicheiral. For T_{2k+1} the above congruence becomes

$$(2k + 1)^2 \equiv (-1) \pmod{(8k + 6)}$$

and this can not be satisfied for any natural number k. Thus except T_0 and T_2 all twist knots are not amphicheiral.

Theorem 3. The Alexander matrices of T_{2n} and T_{2n+1} are different.

Proof. The Alexander matrix of T_n is denoted Mn. M_{2n} and M_{2n+1} are calculated by Alexander's original method in (2)and are given below.

$$M_{2n} = \begin{vmatrix} -1 & x & 0 & 1 & -x & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -x & x & 1 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & x & -x & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 1 & 0 & -1 & -x & \cdots & 0 & 0 & 0 & 0 \\ \cdot & & & & & & & & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & & & & & & & & \\ 0 & 0 & 1 & x & 0 & 0 & 0 & \cdots & -x & -1 & 0 & 0 \\ 0 & 0 & x & 1 & 0 & 0 & 0 & \cdots & 0 & -1 & -x & 0 \\ 0 & 0 & 1 & x & 0 & 0 & 0 & \cdots & 0 & 0 & -x & -1 \\ -x & 0 & x & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \end{vmatrix}$$

$$M_{2n+1} = \begin{vmatrix} -x & 1 & 0 & x & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -x & x & 1 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 1 & -1 & -x & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & x & 0 & -x & -1 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & & & & & & & & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & & & & & & & & \\ 0 & 0 & 1 & x & 0 & 0 & 0 & \cdots & -x & -1 & 0 & 0 \\ 0 & 0 & x & 1 & 0 & 0 & 0 & \cdots & 0 & -1 & -x & 0 \\ 0 & 0 & 1 & x & 0 & 0 & 0 & \cdots & 0 & 0 & -x & -1 \\ -x & 0 & x & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \end{vmatrix}$$

Alexander's original method is given in (1).

Theorem 4. The Alexander polynominals of T_{2n} and T_{2n+1} are different and are as follows respectively:

$$\Delta(t) = nt^2 - (2n+1)t + n$$

$$\Delta(t) = (n+1)t^2 - (2n+1)t + (n+1).$$

Proof. These results follow as in (2) and (3), from the definition 7 and the theorem 3.

Theorem 5. The genus of any twist knot T_n, for $n \neq 0$, is equal to one.

Proof. This theorem is a corollary of the theorem 4 and follows from a theorem of Crowell (9), which states "The genus of an alternating knot is equal to the half of the degree of its Alexander polynominal".

A geometric method (cut and paste) is used to prove theorem 5 in (3).

4. THE PRESENTATION OF THE GROUP OF TWIST KNOTS

The group of twist knot Tn, $\pi_1(S^3-T_n)$, is denoted by G_n.

Theorem 6. G_{2n} and G_{2n+1} are not isomorphic.

Proof. The groups G_{2n} and G_{2n+1} are obtained by Wirtinger's method in (4). Their calculations are justified by Dehn's method in (2). These are as follows :

$$G_{2n} = \left| b, c : \bar{c}(\bar{b}c)^n b(\bar{b}c)^{-n} b(\bar{b}c)^n \bar{b}(\bar{b}c)^{-n} = 1 \right|$$

$$G_{2n+1} = \left| b, c : \bar{c}(\bar{b}c)^{n+1} \bar{c}(\bar{b}c)^{-n-1} b(\bar{b}c)^{n+1} c(\bar{b}c)^{-n-1} = 1 \right|$$

Here and later $\bar{x}$ means x^{-1}. Since the lengths of the relations in G_{2n} and G_{2n+1} are different and no contractions occur these groups are obviously not isomorphic.

5. THREE SHEETED COVERING SPACES OF TWIST KNOTS

In this section the following notations are used.

S_3, the symetric gpoup of order six.

Σ_n, the three sheeted (cyclic in 5.1 and 5.2, irreguler in 5.3 and 5.4) branched covering space of S^3-T_n, branched over T_n.

σ_n,the branch curve in Σ_n lying over T_n.

$A_n = \pi_1(\Sigma_n - \sigma_n)$, the first homotopy group of $(\Sigma_n - \sigma_n)$.

$B_n = \pi_1(\Sigma_n)$, the first homotopy group of, Σ_n

$H_1(\Sigma_n)$, the first integral homology group of Σ_n.

All twist knot groups accept at least one representation in S_3. Namely the cyclic representation. $f: G_n \to S_3$, f(b)=f(c)=(123). But only the groups of T_{6m+1} and T_{6m+4} (m=0,1,2,...) accept a arepresentation onto S_3. Namely for n=6m+1 and n=6m+4

$h: G_n \to S_3$, h(b)=(12), h(c)=(23).

According to the results obtained by Fox's algorithm (11) the three sheeted covering spaces of S^3 branched over twist knots (simply called covering space of twist knots) can be devided into four classes.

5.1. THE THREE SHEETED CYCLIC COVERING SPACE OF T_{2n}

According to the fundamental theorem of covering spaces (11) the cyclic representation $f: G_{2n} \to S_3$, $f(b) = f(c) = (123)$ corresponds to a three sheeted regular covering space of $S^3 - T_{2n}$. Since $f(\bar{b}) = f(\bar{c}) = (132)$ the following table 1 shows that f takes the relation of G_{2n} onto the identity permutation in S_3.

1	3	23 … 23	1	31 … 31	2	12 … 12	1	31 … 31
2	1	31 … 31	2	12 … 12	3	23 … 23	2	12 … 12
3	2	12 … 12	3	23 … 23	1	31 … 31	3	23 … 23
		n		n		n		n

Table 1.

By Fox's algorithm which is equivalent to Reidemeister-Schreier's Theorem (11, pages 146-148) one obtains the following free product $A_{2n} * F_2$.

$$A_{2n} * F_2 = \left| \begin{array}{ll} b_1, b_2 & \bar{c}_3 (\bar{b}_2 c_2)^n b_3 (\bar{c}_3 b_3)^n b_1 (\bar{b}_1 c_1)^n \bar{b}_1 (\bar{c}_3 b_3)^n = 1 \\ b_3, c_1 : & \bar{c}_1 (\bar{b}_3 c_3)^n b_1 (\bar{c}_1 b_1)^n b_2 (\bar{b}_2 c_2)^n \bar{b}_2 (\bar{c}_1 b_1)^n = 1 \\ c_2, c_3 & \bar{c}_2 (\bar{b}_1 c_1)^n b_2 (\bar{c}_2 b_2)^n b_3 (\bar{b}_3 c_3)^n \bar{b}_3 (\bar{c}_2 b_2)^n = 1 \end{array} \right|$$

Where F_2 is a free group of rank 2. As generators of F_2 one can choose c_2 and c_3. Adjoining $c_2 = 1$, $c_3 = 1$ to $A_{2n} * F_2$ one eliminates F_2 and obtains A_{2n}.

$$A_{2n} = \left| \begin{array}{ll} b_1, b_2 & b_2^{-n} b_3^{n+1} b_1 (b_1^{-1} c_1)^n b_1^{-1} b_3^n = 1 \\ c_1, b_3 & c_1^{-1} b_3^{-n} b_1 b_2^{n+1} b_3^{-n} b_2^{n+1} b_3^{-n} b_2^n = 1 \end{array} \right|$$

The branch relations are $b_1 b_2 b_3 = 1$, $c_1 c_2 c_3 = 1$ which reduce to $c_1 = 1$ and $b_3 = (b_1 b_2)^{-1}$. Adjoining these to A_{2n} and denoting b_1 and b_2 with x and y respectively one obtains

$$B_{2n} = \left| \begin{array}{ll} x & y^{-n} (xy)^{-n-1} x^{-n} (xy)^{-n} = 1 \\ y & (xy)^n xy^{n+1} (xy)^n y^{n+1} (xy)^n y^n = 1 \end{array} \right.\ : \left. \right|$$

Hence, $H_1(\sum_{2n}) = Z_{3n+1} \oplus Z_{3n+1}$.

5.2 THE THREE SHEETED CYCLIC COVERING SPACE OF T_{2n+1}

The cyclic representation $h{:}G\, h{:}G_{2n+1} \to S_3$, $h(b)=h(c)=(123)$corresponds to a three sheeted regular covering space of $S^3 - T_{2n+1}$. Since $h(\bar{b})=h(\bar{c})=(132)$ the following tabel 2shows that h takes the relation of G_{2n+1} onto the identitiy permu - tation in S_3.

$$\begin{array}{cccccccccc}
1 & 3 & 23 \ldots 23 & 2 & 12 \ldots 12 & 3 & 23 \ldots 23 & 1 & 31 \ldots 31 \\
2 & 1 & 31 \ldots 31 & 3 & 23 \ldots 23 & 1 & 31 \ldots 31 & 2 & 12 \ldots 12 \\
3 & 2 & \underline{12 \ldots 12} & 1 & \underline{31 \ldots 31} & 2 & \underline{12 \ldots 12} & 3 & \underline{23 \ldots 23} \\
 & & n+1 & & n+1 & & n+1 & & n+1
\end{array}$$

Table 2.

Again by Fox's algorithm one has

$$A_{2n+1} * F_2 = \left| \begin{array}{ll} b_1, b_2 & \bar{c}_3(\bar{b}_2 c_2)^{n+1} \bar{c}_2(\bar{c}_1 b_1)^{n+1} b_2(\bar{b}_2 c_2)^{n+1} c_3(\bar{c}_2 b_3)^{n+1} = 1 \\ b_3, c_1 : & \bar{c}_1(\bar{b}_3 c_3)^{n+1} \bar{c}_3(\bar{c}_2 b_2)^{n+1} b_3(\bar{b}_3 c_3)^{n+1} c_1(\bar{c}_1 b_1)^{n+1} = 1 \\ c_2, c_3 & \bar{c}_2(\bar{b}_1 c_1)^{n+1} \bar{c}_1(\bar{c}_3 b_3)^{n+1} b_1(\bar{b}_1 c_1)^{n+1} c_2(\bar{c}_2 b_2)^{n+1} = 1 \end{array} \right|$$

As generatos of F_2 one can choose c_2 and c_3. Adjoining $c_2 = 1$, $c_3 = 1$ to $A_{2n+1} * F_2$ one eliminates F_2 and obtains A_{2n+1} as follows,

$$A_{2n+1} = \left| \begin{array}{ll} b_1, b_2 & b_2^{-n-1}(\bar{c}_1 b_1)^{n+1} b_2^{-n} b_3^{n+1} = 1 \\ & : \quad c_1 b_3^{-n-1} b_2^{n+1} b_3^{-n} c_1 (\bar{c}_1 b_1)^{n+1} = 1 \\ c_1, b_3 & (\bar{b}_1 c_1)^{n+1} c_1^{-1} b_3^{n+1} b_1 (\bar{b}_1 c_1)^{n+1} b_2^{n+1} = 1 \end{array} \right|$$

After adjoining the branch relations $b_1 b_2 b_3 = 1$, $c_1 c_2 c_3 = 1$ which reduce to $c_1 = 1$,$b_3 = (b_1 b_2)^{-1}$ to A_{2n+1} and denoting b_1 and b_2 with x and y respectively one obtains B_{2n+1}.

$$B_{2n+1} = \left| \begin{array}{ll} x & y^{-n-1} x^{n+1} y^{-n} (xy)^{-n-1} = 1 \\ \quad : & \\ y & x^{n+1} y^{-n-1} x^{n} (xy)^{2n+2} y^{n} x^{-n-1} y^{n+1} = 1 \end{array} \right| .$$

Hence, $H_1(\sum_{2n+1}) = Z_{3n+2} \oplus Z_{3n+2}$.

5.3 THREE SHEETED IRREGULAR COVERING SPACE OF T_{6m+1}

The presentation of the group of T_{6m+1} is

$G_{6m+1} = \left| x, y{:} \bar{y}(\bar{x}y)^{3m+1} y^{-1} (\bar{x}y)^{-3m-1} x (\bar{x}y)^{3m+1} y (\bar{x}y)^{-3m-1} = 1 \right|$..

The mapping $h{:}G_{6m+1} \to S_3$, $h(x)=(12)$, $h(y)=(23)$ is a representation of T_{6m+1} ontoS_3.This representation corresponds to a three sheeted covering space of

T_{6m+1}. The groups A_{6m+1} and B_{6m+1} are again found by Fox's method. One can see that $\sum_{6m+1}$ is a homotopy sphere.

$$A_{6m+1} = \left(\begin{array}{ll} a,b & x^{-1}(\bar{a}\bar{b}yx)^m a^{-1}y^{-1}(\bar{y}ba\bar{x})^m y^{-1}b^2(\bar{b}yx\bar{a})^m b^{-1}y(a\bar{x}\bar{y}b)^m a = 1 \\ : & y^{-1}(\bar{b}yx\bar{a})^m b^{-1}y(a\bar{x}\bar{y}b)^m a(x\bar{a}\bar{b}y)^m x^2(\bar{x}\bar{y}ba)^m x^{-1} = 1 \\ x,y & (x\bar{a}\bar{b}y)^m(\bar{x}\bar{y}ba)^m x^{-1}a(\bar{a}\bar{b}yx)^m a^{-1}y(\bar{y}ba\bar{x})^m y^{-1}b = 1 \end{array} \right)$$

$B_{6m+1} = \pi_1(\sum_{6m+1}) = \{1\}$.

Hence $H_1(\sum_{6m+1}) = \{0\}$.

Theorem 7. The three sheeted irregular branched covering space $\sum_{6m+1}$ of T_{6m+1} is a three sphere.

Proof. Since the twist knot T_{6m+1} is (12m+3,6m+1) two bridge knot and three sheeted irregular covering space of (α,β) two bridge knot is S^3 by Burde's theorem (8), the theorem follows.

5.4. THE THREE SHEETED IRREGULAR COVERING SPACE OF T_{6m+4}

The presentation of the group of T_{6m+4} is

$G_{6m+4} = \left| x,y : \bar{y}(\bar{x}y)^{3m+2}x(\bar{x}y)^{-3m-2}x(\bar{x}y)^{3m+2}x^{-1}(\bar{x}y)^{-3m-2} = 1 \right|$.

The mapping $f: G_{6m+4} \to S_3$, $f(x) = (12)$, $f(y) = (23)$ is a representation of G_{6m+4} onto S_3. This representation corresponds to a three sheeted irregular covering space of T_{6m+4}. The groups A_{6m+4} and B_{6m+4} are obtained by similar calculations.

$$A_{6m+1} = \left(\begin{array}{ll} a,b & (\bar{a}\bar{b}yx)^{m+1}\bar{x}a(\bar{x}\bar{y}ba)^{m+1}\bar{a}b(\bar{b}yx\bar{a})^{m+1}(\bar{y}ba\bar{x})^{m+1} = 1 \\ : & (\bar{b}yx\bar{a})^{m+1}a(\bar{y}ba\bar{x})^{m+1}x(x\bar{a}\bar{b}y)^m x\bar{a}\bar{y}(a\bar{x}\bar{y}b)^m a\bar{x} = 1 \\ x,y & (x\bar{a}\bar{b}y)^m x\bar{a}y(a\bar{x}\bar{y}b)^m a\bar{x}a(\bar{a}\bar{b}yx)^m a\bar{b}y(\bar{x}\bar{y}ba)^{m+1}\bar{a}\bar{y} = 1 \end{array} \right)$$

$B_{6m+4} = \{1\}$.

Hence $H_1(\sum_{6m+4}) = \{0\}$. Thus $\sum_{6m+4}$ is a homotopy sphere.

Theorem 8. The three sheeted irregular branched covering space of T_{6m+4} is a three sphere.

Proof. Since the twist knot T_{6m+4} is (12m+9 , 6m+5) two bridge knot the theorem follows from Burde's theorem (8).

The results obtained here classify all three sheeted branched covering spaces of all twist knots.

Is there a geometric proof of theorem 7 or 8 ?

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