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Symmetries of nonlinear telegraph equations in strong fields

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Abstract

In this article the symmetries of a class of nonlinear telegraph equations are examined. These equations represent a physical model describing electro magnetic shock waves. The main interest is concentrated on potential symmetries but Lie point and non classical symmetries are also calculated. With the aid of these symmetries explicit new solutions are derived or given implicitly by defining equations.

1 Introduction

One of the few systematic ways for finding solutions of nonlinear differential equations is a symmetry analysis. With a symmetry analysis a classification of the differential equations and their properties is possible, too. The original method was discovered by the Norwegian S. Lie to determine the point transformations under which the given differential equation is invariant. In course of time the original method due to Lie has been extended to various generalisations. In 1969 Bluman and Cole have proposed a generalisation of Lie's method which they called the non classical method of symmetry reduction or in short the non classical method [10, 13]. Another way for finding new symmetries of a differential equation is to analyse not only the original equation but an extended system of differential equations from which the solutions of the original equation can be be derived. That leads to potential systems and potential symmetries. With these methods new solutions and symmetries of differential equations can be found not obtainable with Lie's original method. Lie's method is also used to decide whether a differential equation can be linearized by an invertible mapping or not. To linearize a differential equation an algorithm was proposed by Bluman [4]. In this article we want to show the use of potential systems and potential symmetries to discover new solutions and to find linearisations. In section 2 Lie's method to determine symmetries of a differential equation is discussed and a formulation based on the Fréchet derivative is introduced. A short overview of the non classical method which is also useful in studying potential systems is given. Section 3 deals with potential systems and potential symmetries. The determination of potential systems and the properties of potential symmetries is presented. These symmetries are treated for a class of nonlinear telegraph equations in section 4. In section 5 a special type of nonlinear telegraph equations included in this class is discussed in detail. The potential symmetries are compared with other symmetries of this equation and explicit solutions are given. The obtained results are summarized in section 6.

2 Symmetries of differential equations

A useful tool for finding new solutions of a system of partial differential equations is the procedure of symmetry analysis. The original method to determine the symmetry group for a system was developed by Lie [1, 2]. This method allows to compute classes of exact solutions which are invariant under the transformations representing the symmetry group. In addition to this property the algorithm proposed by Lie can be used to discover new solutions by constructing them from old ones and to classify the symmetry properties of PDEs.

To exemplify Lie's method let us first discuss the general procedure [3, 4]. The general case of a nonlinear system of PDEs with the order k is represented by differential functions

$$\Delta_\nu(x, u^{(k)}) = \Delta_\nu[u] = 0 \qquad \nu = 1, \ldots, m \tag{1}$$

with n independent variables $x = (x_1 \ldots, x_n)$ and m dependent variables $u = (u^1, \ldots, u^m)$; $u^{(k)}$ denotes all derivatives of the dependent variables u up to the order k. Partial derivatives of the dependent variables are written in the notion of multi–indices $J = (j_1, \ldots, j_p)$ and

$$u^\mu_J = \frac{\partial^p u^\mu}{\partial x_{j_1} \ldots \partial x_{j_p}}.$$

The equations Δ_ν are assumed to be smooth functions of their arguments. Lie's method determines the infinitesimal point transformations under which the solutions of (1) are invariant. The point transformations can be characterized by the infinitesimal generator

$$X = \sum_{\mu=1}^{m} \eta_\mu(x, u^{(1)}) \frac{\partial}{\partial u^\mu} \tag{2}$$

where η_μ is linear in the first derivative of the dependent variables u^μ. According to this condition η_μ can be expressed in the form

$$\eta_\mu = \phi_\mu(x, u) - \sum_{i=1}^{n} \xi_i(x, u) u^\mu_i. \tag{3}$$

with ϕ_{mu} and ξ_i the so–called infinitesimals. These infinitesimals are connected to infinitesimal point transformations of the independent and dependent variables by the relation

$$\begin{aligned} x_i^* &= x_i + \epsilon \xi_i(x, u) + O(\epsilon^2) & (4)\\ (u^\mu)^* &= u^\mu + \epsilon \phi_\mu(x, u) + O(\epsilon^2). & (5) \end{aligned}$$

The invariance criterion for the point transformations with the generator X which map solutions of (1) into other solutions of (1) can be formulated with the Fréchet derivative $\mathbf{D}_\Delta$ of Δ [3, 14]:

$$\sum_{\mu=1}^{m} (\mathbf{D}_\Delta)_{\nu\mu}\, \eta_\mu|_{\Delta=0} = 0. \tag{6}$$

The equation (6) must hold for η if $u(x)$ is a solution of $\Delta = 0$.

The Fréchet derivative $\mathbf{D}_\Delta \eta$ is the differential operator which is defined by the relation [3]

$$\mathbf{D}_\Delta \eta = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \Delta[u + \epsilon \eta]. \tag{7}$$

By evaluating explicitly the expression (7) the Fréchet derivative can be expressed by an $m \times m$ matrix differential operator

$$(\mathbf{D}_\Delta)_{\mu\nu} = \sum_J \left(\frac{\partial \Delta_\mu}{\partial u_J^\nu}\right) D_J \qquad \mu, \nu = 1, \ldots, m \tag{8}$$

where D_i denotes the total derivative

$$D_i = \frac{\partial}{\partial x_i} + \sum_{\mu=1}^{m} u_i^\mu \frac{\partial}{\partial u_i^\mu} + \sum_{\mu=1}^{m} \sum_J u_{J,i}^\mu \frac{\partial}{\partial u_J^\mu}. \tag{9}$$

So $D_J = D_{j_1} \ldots D_{j_p}$ can be composed of single total derivatives.

The determining equations for η resulting from the invariance criterion (6) are a linear over determined system of PDEs in x and u. There exist various programmes in different computer algebra system for calculating the determining equations (and solving them automatically or interactively), e.g. in MATHEMATICA [5], MAPLE [8], REDUCE [7], MACSYMA [6] and in AXIOM [7].

With the calculated infinitesimals ξ_i and ϕ_μ finite symmetry transformations can be calculated by solving the system

$$\frac{d}{d\epsilon} x_i^*(\epsilon) = \xi_i(x^*, u^*) \tag{10}$$

$$\frac{d}{d\epsilon} (u^\mu)^*(\epsilon) = \phi_\mu(x^*, u^*) \tag{11}$$

of ordinary differential equations in ϵ with the initial conditions

$$\begin{aligned} x_i^*(0) &= x_i \\ (u^\mu)^*(0) &= u^\mu. \end{aligned}$$

These point transformations map solutions $u(x)$ of the system (1) into new solutions $u^*(x^*)$. This way one can construct new solutions from known ones by only knowing the symmetry transformations.

Another way for finding new classes of solutions with this method is to determine invariant solutions corresponding to an infinitesimal generator X. Functions $u(x)$ which are invariant under the action of the generator X have to satisfy the characteristic equations or invariant surface condition

$$Xu^\mu = \phi_\mu(x, u) - \sum_{i=1}^{n} \xi_i(x, u) u_i^\mu = 0. \tag{12}$$

Additionally $u(x)$ must solve the original system (1). Furthermore the symmetry analysis due to Lie delivers a criterion to decide whether or not the system Δ can be linearized by an invertible point transformation [4, 16].

A generalisation of Lie's method was proposed by Bluman and Cole [10] which they called the "non classical method of symmetry reduction" or in short the "nonclasscial method". With this method new classes of solutions not obtainable with Lie's method can be derived [11, 12]. When new classes of solutions are found one speaks of non "classical symmetries but no new symmetry transformations are found. The non classical method applies Lie's method to the extended system

$$\begin{aligned} \Delta_\nu &= 0 \qquad \nu = 1, \ldots, m \\ \phi_\mu - \sum_{i=1}^{n} \xi_i u_i^\mu &= 0 \qquad \mu, 1, \ldots, m \end{aligned}$$

which has to be solved by the invariant solutions of the generator X.

The procedure to determine point symmetries can be generalized to discover symmetries which are generated by local symmetries. Local symmetries are characterized by the infinitesimal generator

$$X = \sum_{\mu=1}^{m} \eta_\mu(x, u^{(p)}) \frac{\partial}{\partial u^\mu}$$

which can depend on derivatives up to a fixed order p and has not to be linear in the first derivatives of u. In the formulation of (6) the invariance criterion is the same expression just the form of η has changed.

Up to now the symmetries of Δ are generated by local or point transformations of the dependent and independent variables. These transformations act on the $(x, u^{(p)})$–space, respectively on the (x, u)–space. Symmetries with total new properties can be produced by nonlocal transformations [4, 14]. Nonlocal transformations can't be expressed in the independent and dependent variables x and u and the derivatives of the dependent variables $u^{(p)}$. One has to introduce new variables v which are related to the old variables u through equations which include the original system as a consequence.

Provided one PDE Δ_ν can be expressed as a conservation law there exists a systematic way for finding a class of nonlocal symmetries, so–called potential symmetries, by introducing new potential variables corresponding to a potential system. In the next section, we will show an algorithm to determine potential systems and define potential symmetries.

3 Potential symmetries

If we want to describe nonlocal symmetries it is convenient to introduce new variables $v(x)$ which are related to the old variables $u(x)$ by additional equations. The original system has to be derivable form these equations. In other words, if $(u(x), v(x))$ satisfy the extended equations $u(x)$ has also to be a solution of the system $\Delta = 0$. An auxiliary system with new variables can be introduced if at least one PDE of Δ can be written as a conservations law.

Suppose one PDE of Δ, without loss of generality $\Delta_m = 0$, can be expressed as a conservation law [14]

$$\sum_{i=1}^{n} D_i f^i(x, u^{(k-1)}) = 0.$$

So the system Δ can be written in the form

$$\Delta_\nu(x, u^{(k)}) = 0 \qquad \nu = 1, \ldots, m-1 \tag{13}$$

$$\sum_{i=1}^{n} D_i f^i(x, u^{(k-1)}) = 0. \tag{14}$$

According to the form of (14) it is possible to introduce $n-1$ new auxiliary variables $v(x) = (v^1(x), \ldots, v^n(x))$ to form a new "auxiliary system" or potential system $\Psi(x, u^{(k)}, v^{(1)})$

$$f^1(x, u^{(k-1)}) = v^1_{x_2} \tag{15}$$

$$f^l(x, u^{(k-1)}) = (-1)^l \left[v^l_{l+1} + v^{l-1}_{l+1}\right], \qquad 1 < l < n \tag{16}$$

$$f^n(x, u^{(k-1)}) = (-1)^{n-1} v^{n-1}_{n-1} \tag{17}$$

$$\Delta_\nu(x, u^{(k)}) = 0, \qquad \nu = 1, \ldots, m-1. \tag{18}$$

The systems Δ and Ψ are closely related to each other. A symmetry of the system Ψ is a symmetry of the original system Δ and vice versa. But the same symmetry could have a different character in the two systems. A point symmetry of Ψ could yield a nonlocal symmetry transformation in Δ. Such symmetries of Ψ are called potential symmetries of Δ [4, 14, 15].

A point symmetry of Ψ with the infinitesimal generator

$$X^\Psi = \sum_{\alpha=1}^{m} \left[\phi_\alpha(x, u, v) - \sum_{i=1}^{n} \xi_i(x, u, v) u_i^\alpha\right] \partial_{u^\alpha}$$

$$+\sum_{\beta=1}^{n-1}\left[\chi_\beta(x,u,v)-\sum_{i=1}^{n}\xi_i(x,u,v)v_i^\beta\right]\partial_{v^\beta} \tag{19}$$

is a potential symmetry of Δ if $\phi(x,u,v)$ and $\xi(x,u,v)$ depend essentially on the new auxiliary variables $v(x)$. Otherwise X^Ψ projects onto a point symmetry of Δ corresponding to a generator

$$X=\sum_{\alpha=1}^{m}\left[\phi_\alpha(x,u)-\sum_{i=1}^{n}\xi(x,u)u_i^\alpha\right]\partial_{u^\alpha}.$$

The main problem in determining potential symmetries is to find useful potential systems which allow potential symmetries. But there is not only the possibility to write a PDE of the system (1) $\Delta_\nu=\sum_{i=1}^n D_i f^i$ in a conserved form. Additionally some PDEs can be multiplied by factors to obtain a conservation law [14, 15].

Suppose there exists a set of factors

$$\lambda(x,u)=(\lambda^1(x,u),\dots,\lambda^m(x,u))$$

with at least one $\lambda^\mu\neq 0$ to form the expression

$$\sum_{\nu=1}^{m}\lambda^\nu(x,u))\Delta_\nu=\sum_{i=1}^{n}D_if^i(x,u^{(k)}). \tag{20}$$

Then the system Δ can be replaced by the system $\overline{\Delta}$ which is given by

$$\begin{aligned}\Delta_\nu &= 0 \qquad \nu=1,\dots,\mu-1,\mu+1,\dots,m\\ \sum_{i=1}^{n}D_if^i &= 0\end{aligned} \tag{21}$$

to obtain an auxiliary system. But the factors λ have to be chosen carefully, because solutions of $\Delta=0$ and solutions of the system

$$\begin{aligned}\Delta_\nu &= 0 \qquad \nu=1,\dots,\ \mu-1,\ \mu+1,\dots,m\\ \lambda^\mu(x,u) &= 0\end{aligned}$$

both satisfy the modified system $\overline{\Delta}$. So not all possible integrating factors λ are useful. If the equation

$$\lambda_\mu(x,u)=0$$

has no solutions for any u the system $\overline{\Delta}$ yields a useful potential system. In this case it is assured that the point symmetries of the corresponding potential system are symmetries of the original system, too.

A necessary condition that the factors λ have to satisfy opens a systematic way for finding integrating factors λ [3, 15]. This determining criterion for the integrating factors can be expressed in the form

$$\sum_{\nu=1}^{m} (\mathbf{D}_\Delta)^*_{\mu,\nu} \lambda^\nu \Big|_{\Delta=0} = 0 \tag{22}$$

or in short

$$\mathbf{D}^*_\Delta \lambda|_{\Delta=0} = 0.$$

The adjoint Fréchet derivative $\mathbf{D}^*_\Delta$ is the differential operator which is defined by the relation

$$\int_\Omega V\,\mathbf{D}_\Delta\,W\,dx = \int_\Omega W\,\mathbf{D}^*_\Delta\,V\,dx \tag{23}$$

for any domain $\Omega \subset I\!R^n$ and any smooth functions $V(x) = (V^1(x), \ldots, V^m(x))$, $W(x) = (W^1(x), \ldots, W^m(x))$ with compact support in Ω [3, 14]. In particular the expression

$$V\,\mathbf{D}_\Delta\,W - W\,\mathbf{D}^*_\Delta\,V$$

has to be a divergence expression. It is easy to show — integration by parts — that the formula

$$(\mathbf{D}^*_\Delta)_{\nu\mu} = \sum_J (-D)_J \frac{\partial \Delta_\mu}{\partial u^\nu_J}, \qquad \nu, \mu = 1, \ldots, m, \tag{24}$$

is a matrix representation of the adjoint Fréchet derivative $\mathbf{D}^*_\Delta$.

The procedure to find potential systems can also be applied to a already known potential system. By applying this method step by step it is possible to construct a whole chain of potential systems. Finally the symmetry analysis can be applied to all those systems.

The solutions of the determining condition (22) for the integrating factors could obtain free functions that have to satisfy a PDE or a system of PDEs. These factors are of no use to find new potential systems. They indicate that the original system can be linearized. To use this information given by (22) an algorithm was proposed by Bluman [4, 16] to linearize a scalar PDE or a system of PDEs by an invertible mapping.

With the aid of potential symmetries new solutions of the original system Δ can be found which cannot be derived from local symmetries. The new solutions can be determined by calculating the finite transformations corresponding to a potential symmetry to construct new solutions from old ones or by computing the invariant solution.

Another useful possibility of potential systems is the possibility of linearisation by non–invertible mappings [17]. Sometimes the original system cannot be linearized by an invertible mapping. In contrast to that result a potential system could own an invertible mapping which leads to a linearisation of the potential system.

To see how the different methods work let us discuss the symmetry analysis of a class of nonlinear telegraph equations. For a particular equation the obtained solutions are discussed in detail.

4 Potential symmetries of nonlinear telegraph equations

A general class of nonlinear telegraph equations [18] are described by the system

$$v_t = u_x \tag{25}$$
$$v_x = d(u)u_t + e(u). \tag{26}$$

where $d(u)$ and $e(u)$ are material functions. These equations also govern electro magnetic shock waves [19] as well a pressure waves in a relaxing gas. In the case of electro magnetic shock waves the physical model is described by the Maxwell equations in one spatial direction x

$$E_x = -\mu(H)H_t \tag{27}$$
$$H_x = -\epsilon(E)E_t \tag{28}$$

for the electric field E and the magnetic field H with the magnetic permeabilty μ and the dielectric permittivity ϵ. The model with

$$\epsilon(E) = \epsilon_1 \qquad \mu(H) = \frac{\mu_1}{H^{n+1}}$$

or

$$\epsilon(E) = \frac{\epsilon_1}{E^{n+1}} \qquad \mu(H) = \mu_1$$

leads to the equation

$$u_{xx} + \left(\frac{1}{u^n}\right)_{tt} = 0 \tag{29}$$

where u either is the electric field E or the magnetic field H. The cases with $n = \pm 1$ are excluded because the equation with $n = -1$ is the well known wave equation and $n = 1$ is discussed in detail in section 5

The symmetry analysis of equation (29) due to Lie delivers a four dimensional symmetry group with the infinitesimals

$$\xi_1 = c_2 + c_3 x \tag{30}$$

$$\xi_2 = c_1 + (c_3 - \frac{1}{2}c_4(n+1))t \tag{31}$$

$$\phi_1 = c_4 u. \tag{32}$$

The infinitesimals ξ_1, ξ_2 and ϕ_1 are related to x, t and u and generate space and time translations. Additionally there exist a scaling invariant solution and a special separation ansatz. The symmetry group shows that it is not possible to find an invertible mapping to linearize the equation (29).

A systematic determination of the integrating factors λ by using the determining condition (22) yields the four different factors

$$\lambda_1 = 1, \qquad \lambda_2 = x, \qquad \lambda_3 = t, \quad \text{and} \quad \lambda_4 = xt. \tag{33}$$

For each integrating factor there exists a related potential system. We examined all these systems but only the system related to $\lambda_1 = 1$ is useful for calculating potential symmetries. The corresponding potential system reads

$$v_t^1 - u_x = 0 \tag{34}$$

$$-v_x^1 + \frac{u_t}{u^{n+1}} = 0. \tag{35}$$

The finite dimensional symmetry group of the system (34) and (35) has got beside the point symmetries that project onto point symmetries of the original equation (29) a potential symmetry generated by the infinitesimals

$$\xi_1 = \frac{2tu}{n-1} + \frac{2nxv^1}{1-n} \tag{36}$$

$$\xi_2 = \frac{2x}{(n-1)u^n} + \frac{2tv^1}{n-1} \tag{37}$$

$$\phi_1 = \frac{4v^1 u}{1-n} \tag{38}$$

$$\phi_2 = \frac{4u^{1-n}}{(1-n)^2} + (v^1)^2. \tag{39}$$

Unfortunately the corresponding system of ordinary differential equations

$$\begin{aligned}
\frac{d}{d\epsilon}x(\epsilon) &= \frac{2tu}{n-1} + \frac{2nxv^1}{1-n} \\
\frac{d}{d\epsilon}t(\epsilon) &= \frac{2x}{(n-1)u^n} + \frac{2tv^1}{n-1} \\
\frac{d}{d\epsilon}u(\epsilon) &= \frac{4v^1 u}{1-n} \\
\frac{d}{d\epsilon}v^1(\epsilon) &= \frac{4u^{1-n}}{(1-n)^2} + (v^1)^2
\end{aligned}$$

which serves to compute the finite symmetry transformation cannot be decoupled. So it is also not possible the derive the invariant solution in the most general case.

However in addition to these discrete symmetries there exists an infinite dimensional potential symmetry which is represented by the infinitesimals

$$\xi_1 = f^1(u, v^1) \tag{40}$$

$$\xi_2 = f^2(u, v^1) \tag{41}$$

$$\phi_1 = 0 \tag{42}$$

$$\phi_2 = 0. \tag{43}$$

The free functions $f^1(u, v^1)$ and $f^2(u, v^1)$ have to solve a linear system consisting of two equations with analytical coefficients

$$f^1_u - f^2_{v^1} = 0 \tag{44}$$

$$-f^1_{v^1} + u^{n+1} f^2_u = 0. \tag{45}$$

This continuous potential symmetry indicates that it is possible to linearize the potential system (34) and (35) by a hodograph transformation The linear target system is yet given by the equations (44) and (45). We choose as new independent variables

$$z_1 = u \qquad z_2 = v^1 \tag{46}$$

and the dependent variables as

$$w^1(z_1, z_2) = x \qquad w^2(z_1, z_2) = t. \tag{47}$$

According to the transformation (46) and (47) and the equations (44) and (45) the linearized system reads

$$-w^1_{z_1} + w^2_{z_2} = 0 \tag{48}$$

$$-w^1_{z_2} + z_1^{n+1} w^2_{z_1} = 0. \tag{49}$$

It can be solved by various, well known methods like separation in the independent variables or Laplace transformation etc. A detailed discussion of such solutions for the special case $n = 1$ is given in section 5 .

The equations (48) and (49) are equivalent to a scalar PDE if we introduce a potential representation

$$w^1(z_1, z_2) = g_{z_2} \quad \text{and} \quad w^2(z_2, z_2) = g_{z_1}.$$

which results to

$$g_{z_2 z_2} - z_1^{n+1} g_{z_1 z_1} = 0. \tag{50}$$

The four dimensional symmetry group of equation (50) is

$$\begin{aligned}
\xi_1 &= \frac{2}{1-n}(c_2 + 2c_3 z_2) z_1 \\
\xi_2 &= c_4 + c_2 z_2 + c_3 z_2^2 + \frac{4}{(1-n)^2} c_3 z_1^{1-n} \\
\phi_1 &= c_1 g + \frac{1+n}{1-n} c_3 z_2 g
\end{aligned}$$

where ξ_1, ξ_2 and ϕ_1 are related to the variables z_1, z_2 and g respectively. This representation of the symmetries of equation (50) is a common way to find particular solutions. The invariant solution related to the subgroup $c_2 = 1$ and $c_i = 0 \quad \forall i \neq 2$ with the infinitesimals

$$\xi_1 = \frac{2}{1-n} z_1, \qquad \xi_2 = z_2 \quad \text{and} \quad \phi_1 = 0,$$

yields the expression

$$g(z_1, z_2) = k_1 + k_2 \int^{\zeta} ds \left(4 - s^2(1-n)^2\right)^{\frac{3-n}{2(n-1)}}$$

for $g(z_1, z_2)$ with the similarity variable given by

$$\zeta = z_2 z_1^{\frac{n-1}{2}}.$$

Thus the solutions of the potential system (34) and (35) can be given by the explicit expressions

$$\begin{aligned} w^1(z_1, z_2) &= k_2 z_1^{\frac{n-1}{2}} \left(4 - z_2^2 z_1^{n-1}(1-n)^2\right)^{\frac{3-n}{2(n-1)}} \\ w^2(z_1, z_2) &= k_2 \frac{1}{2}(n-1) z_2 z_1^{\frac{n-3}{2}} \left(4 - z_2^2 z_1^{n-1}(1-n)^2\right)^{\frac{3-n}{2(n-1)}}. \end{aligned}$$

But it is not possible to execute explicitly the back transformations which is defined by the equations

$$\begin{aligned} x &= k_2 u^{\frac{n-1}{2}} \left(4 - (v^1)^2 u^{n-1}(1-n)^2\right)^{\frac{3-n}{2(n-1)}} \\ t &= k_2 \frac{1}{2}(n-1) v^1 u^{\frac{n-3}{2}} \left(4 - (v^1)^2 u^{n-1}(1-n)^2\right)^{\frac{3-n}{2(n-1)}}. \end{aligned}$$

The other non trivial invariant solution corresponding to c_3 cannot be determined because the differential equations connected with the infinitesimals cannot be decoupled (see section 5). The group constant c_4 generates translation in z_2 and c_1 reflects the homogeneity of equation (50).

In the following section, we want to examine a special case of equation (29) with $n = 1$ This situation was excluded in the previous calculations because the group classification shows a singular behaviour for this value of n. This special case describes a physical model of great interest in strong external magnetic fields. There are also further solutions which are not derived in this section.

5 Symmetry analysis for a particular non-linear telegraph equation

The equation

$$u_{xx} + \left(\frac{1}{u}\right)_{tt} = 0 \tag{51}$$

represents the physical model of the Maxwell equations (27) and (28) with [19]

$$\epsilon(E) = \epsilon_1 \quad \text{and} \quad \mu(H) = \frac{\mu_1}{H^2}.$$

The magnetic permeabilty μ and the dielectric permittivity ϵ describe a ferromagnet in a strong external magnetic field.

The point symmetry group of equation (51) is represented by the infinitesimals

$$\xi_1 = c_3 + c_4 x \tag{52}$$

$$\xi_2 = c_1 + c_2 t \tag{53}$$

$$\phi_1 = (c_4 - c_2) u \tag{54}$$

generating space and time translations as well as a scaling invariant solution and a particular separation ansatz in the form

$$u(x,t) = \frac{x}{k_1 + k_2 t}$$

where k_1 and k_2 are integration constants. The symmetry group contains no infinite dimensional subgroup, so there is no possibility to linearize equation (51) by an invertible mapping.

The symmetry analysis with the non classical method delivers another interesting case with infinitesimals

$$\xi_1 = u(x,t) \tag{55}$$

$$\xi_2 = 1 \tag{56}$$

$$\phi_1 = 0. \tag{57}$$

where again ξ_1, ξ_2 and ϕ_1 are related to the variables x, t and u. Here the remarkable fact should be emphasized that the solutions of the invariant surface condition

$$u u_x + u_t = 0 \tag{58}$$

also solve the nonlinear telegraph equations (51). So the problem of solving the PDE (51) is simplified to solve a quasi linear PDE of first order. PDEs of this type can be solved by the method of characteristics. The solution, $u(x,t)$ of (58) obtained by this method is defined by the implicit equation

$$x = G(u) + ut \tag{59}$$

for any arbitrary $G(u)$. Of course equation (59) cannot be solved explicitly for arbitrary $G(u)$. Solutions $u(x,t)$ in an explicit form can be obtained if the defining equation (59) can be expressed as a polynomial in u of maximal degree 4, i.e.

$$G(u) = \delta_0 + \delta_1 u + \delta_2 u^2 + \delta_3 u_3 + \delta_4 u^4$$

or the similar cases

$$G(u) = \delta_0 + \delta_1 u + \frac{\delta_2}{u} + \frac{\delta_3}{u^2} + \frac{\delta_4}{u^3}$$

and

$$G(u) = \delta_0 + \delta_1 u + \delta_2 u^\beta \quad \text{with} \quad \beta = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}.$$

For example the solution of the equation

$$x = \delta_0 + \delta_1 u + tu + \delta_2 u^2$$

— a polynomial of degree 2 — is given by the expression

$$u(x,t) = \frac{1}{2\delta_2}\left(-\delta_1 - t \pm \sqrt{(\delta_1 + t)^2 - 4\delta_2(\delta_0 - x)}\right).$$

The result of a systematic determination of potential systems for equation (51) is the same as for the general equation (29) with u^n and given in (33). The potential system derived from the factors λ_2, λ_3 and λ_4 and further potential systems of these systems do not lead to potential symmetries. As in the previous case potential symmetries exist for the system

$$v_t^1 - u_x = 0 \tag{60}$$

$$\frac{u_t}{u^2} - v_x^1 = 0. \tag{61}$$

which belongs to the factor $\lambda_1 = 1$.

The symmetry analysis of the system (60) and (61) shows beyond the symmetries that project onto the point symmetries of the original equation a potential symmetry generated by the infinitesimals

$$\xi_1 = tu - xv^1 \tag{62}$$

$$\xi_2 = \frac{x}{u} + tv^1 \tag{63}$$

$$\phi_1 = -2uv^1 \tag{64}$$

$$\phi_2 = -2\log(u). \tag{65}$$

Again, in this case it is not possible to calculate explicitly the finite transformations of this infinitesimal transformation. The last two equations (68) and (69) of the corresponding system of ODEs

$$\begin{aligned}\frac{d}{d\epsilon}x(\epsilon) &= tu - xv^1 &(66)\\ \frac{d}{d\epsilon}t(\epsilon) &= \frac{x}{u} + tv^1 &(67)\\ \frac{d}{d\epsilon}u(\epsilon) &= -2uv^1 &(68)\\ \frac{d}{d\epsilon}v^1(\epsilon) &= -2\log(u). &(69)\end{aligned}$$

can be decoupled and a solution can be given explicitly. But the solutions of the remaining equations (66) and (67) cannot be computed. These problems can occur at the calculations of potential symmetries because of the greater number of variables.

The symmetry group of (51) contains also another infinite dimensional subgroup with the infinitesimals

$$\begin{aligned}\xi_1 &= f^1(u, v^1) &(70)\\ \xi_2 &= f^2(u, v^1) &(71)\\ \phi_1 &= 0 &(72)\\ \phi_2 &= 0 &(73)\end{aligned}$$

representing a potential symmetry which leads directly to a linearisation of (51) by a non–invertible mapping. The free functions $f^1(, u, v^1)$ and $f^2(u, v^1)$ have to satisfy the linear system

$$\begin{aligned}f^1_u - f^2_{v^1} &= 0 &(74)\\ -f^1_{v^1} + u^2 f^2_u &= 0. &(75)\end{aligned}$$

It should be noted that the nonlinear telegraph equation (51) could not be linearized by an invertible mapping. But the symmetry (70 - 73) indicates that it is possible to linearize the potential system (60) and (61) by an invertible point transformation. The equations (74) and (75) yet deliver the target system. So the original equation (51) is linearized indirectly since the solutions of (60) and (61) (u, v^1) are also solutions of the equation (51) when v^1 is ignored.

Following the method proposed by Bluman [4] the potential system (60) and (61) is linearized by the hodograph transformations — see the former transformations (46) and (47)

$$z_1 = u, \quad z_2 = v^1 \quad \text{and} \quad w^1 = x, \quad w^2 = t. \tag{76}$$

with the new independent variables z_1, z_2 and the new dependent variables $w^1(z_1, z_2)$ and $w^2(z_1, z_2)$. The transformed equations

$$\begin{aligned} -w^1_{z_1} + w^2_{z_2} &= 0 \qquad (77)\\ -w^1_{z_2} + z_1^2 w^2_{z_1} &= 0 \qquad (78) \end{aligned}$$

are a linear homogeneous system of PDEs. There are various techniques to solve the equations (77) and (78). A simple way to solve this system is to introduce a "potential function" $f(z_1, z_2)$ which has to solve the scalar PDE

$$z_1^2 f_{z_1 z_1} - f_{z_2 z_2} = 0 \tag{79}$$

of second order with

$$w^1 = f_{z_2} \quad \text{and} \quad w^2 = f_{z_1}.$$

A common way to solve the PDE (79) is to make a separation in the independent variables. With the ansatz for $f(z_1, z_2)$ in the form

$$f(z_1, z_2) = F(z_1)G(z_2)$$

one gets $G(z_2)$ as an expression with exponential functions

$$G(z_2) = ae^{\sqrt{k}z_2} + be^{-\sqrt{k}z_2}$$

and $F(z_1)$ can be expressed in powers of z_1

$$F(z_1) = cz_1^{\gamma_1} + dz_1^{\gamma_2}.$$

$F(z_1)$ and $G(z_2)$ contain an arbitrary parameter k and the integration constants a, b, c and d. With the aid of the function $f(z_1, z_2)$ the solutions

$$\begin{aligned} w^1 &= \int dk\sqrt{k}(a(k)e^{\sqrt{k}z_2} - b(k)e^{-\sqrt{k}z_2})(c(k)z_1^{\gamma_1} + d(k)z_1^{\gamma_2}) \qquad (80)\\ w^2 &= \int dk(a(k)e^{\sqrt{k}z_2} + b(k)e^{-\sqrt{k}z_2})(\gamma_1 c(k)z_1^{-\gamma_2} + \gamma_2 d(k)z_1^{-\gamma_1}) \qquad (81) \end{aligned}$$

of the potential system (60) and (61) where the exponents γ_1 and γ_2 are given by

$$\gamma_1 = \frac{1}{2} + \sqrt{1+4k} \quad \text{and} \quad \gamma_2 = \frac{1}{2} - \sqrt{1+4k}$$

are a linear superposition of particular solutions with a continuous parameter k. The range of the parameter k and the choice of the coefficients follow from given boundary and initial values. After the back transformation every solution $u(x,t)$ is a solution of the original nonlinear telegraph equation (51).

For the special choice of the parameters

$$a(k) = b(k) = c(k) = \delta\left(k - \frac{1}{4}\right) \quad \text{and} \quad d(k) = 0$$

with $\delta(k - \frac{1}{4})$ denoting Dirac's δ–function it is possible to execute the back transformation explicitly. The equations (80) and (81) reduce for this case

$$\begin{aligned} w^1 &= \sin\left(\frac{z_2}{2}\right)\sqrt{z_1} \\ w^2 &= \cos\left(\frac{z_2}{2}\right)\frac{1}{\sqrt{z_1}}. \end{aligned}$$

Applying the transformation rules (76) and eliminating v^1 one gets the defining equation

$$x^2 - u + t^2u^2 = 0$$

for $u(x,t)$. The solution $u(x,t)$

$$u(x,t) = \frac{1}{2t^2} \pm \frac{1}{2t^2}\sqrt{1 - 4t^2x^2}, \tag{82}$$

which is easily obtained cannot be derived by other methods presented in the previous sections. This solution of the nonlinear telegraph equation (51) is plotted in figure 1 and figure 2 for the branch with the plus sign.

Another way to treat PDE (79) is the symmetry analysis due to Lie. The result of the analysis is a four dimensional symmetry group represented by the infinitesimals

$$\begin{aligned} \xi_1 &= c_4 z_1 + 2c_2 z_1 z_2 &(83)\\ \xi_2 &= c_3 + 2c_2 \log(z_1) &(84)\\ \phi_1 &= c_1 f + c_2 z_2 f + g(z_1, z_1). &(85) \end{aligned}$$

The free function $g(z_1, z_2)$ reflects the linearity and has to satisfy the original equation (79). The group constant c_1 is connected with the homogeneity of equation (79) and c_3 generates translations in the time t. These group constants are of no use in finding invariant solutions. The invariant solution corresponding to the infinitesimal generator

$$X = [-c_4 z_1 f_{z_1} - c_3 f_{z_2}]\partial_f$$

is a special form of the separation ansatz used in the discussion above. Only the invariant solution of the generator

$$X = [z_2 f - 2z_1 z_2 f_{z_1} - 2\log(z_1) f_{z_2}]\partial_f,$$

created by setting c_2 to unity and the rest of the group constants to zero, leads to a new class of solutions. The invariant surface condition

$$X f = 0$$

is solved by the similarity solution

$$f(z_1, z_2) = \sqrt{z_1}\, w(\zeta) \tag{86}$$

with the similarity variable

$$\zeta = z_2^2 - (\log(z_1))^2.$$

The reduced equation of (79)

$$w(\zeta) + 16w'(\zeta) + 16w''(\zeta) = 0$$

can be solved in terms of Bessel functions

$$w(\zeta) = k_1 \mathrm{J}_0\left(\frac{\sqrt{\zeta}}{2}\right) + k_2 \mathrm{Y}_0\left(\frac{\sqrt{\zeta}}{2}\right).$$

The solutions of equations (77) and (78) read in terms of this expression

$$w^1(z_1, z_2) \;=\; \frac{\sqrt{z_1} z_2}{2\sqrt{z_2^2 - (\log(z_1))^2}} \times$$

$$
\begin{aligned}
& \left(-k_1 \mathrm{J}_1\left(\frac{1}{2}\sqrt{z_2^2-(\log(z_1))^2}\right)-k_2 \mathrm{Y}_1\left(\frac{1}{2}\sqrt{z_2^2-(\log(z_1)^2}\right)\right) && (87)\\
w^2(z_1,z_2) &= \frac{1}{2\sqrt{z_1}}\left(k_1 \mathrm{J}_0\left(\frac{1}{2}\sqrt{z_2^2-(\log(z_1))^2}\right)+k_2 \mathrm{Y}_0\left(\frac{1}{2}\sqrt{z_2^2-(\log(z_1))^2}\right)\right)\\
&+\frac{\sqrt{z_1}\log(z_1)}{2\sqrt{z_2^2-(\log(z_1))^2}}\times\\
&\left(k_1 \mathrm{J}_1\left(\frac{1}{2}\sqrt{z_2^2-(\log(z_1))^2}\right)+k_2 \mathrm{Y}_1\left(\frac{1}{2}\sqrt{z_2^2-(\log(z_1))^2}\right)\right) && (88)
\end{aligned}
$$

which we obtained by differentiating equation (86). The solutions of the potential system (60) and (61) which can be obtained by substituting back the transformation rules (76) are defined the implicit equations

$$
\begin{aligned}
x &= \frac{\sqrt{u}v^1}{2\sqrt{(v^1)^2-(\log(u))^2}}\times\\
&\left(-k_1 \mathrm{J}_1\left(\frac{1}{2}\sqrt{(v^1)^2-(\log(u))^2}\right)-k_2 \mathrm{Y}_1\left(\frac{1}{2}\sqrt{(v^1)^2-(\log(u)^2}\right)\right) && (89)\\
t &= \frac{1}{2\sqrt{u}}\left(k_1 \mathrm{J}_0\left(\frac{1}{2}\sqrt{(v^1)^2-(\log(u))^2}\right)+k_2 \mathrm{Y}_0\left(\frac{1}{2}\sqrt{(v^1)^2-(\log(u))^2}\right)\right)\\
&+\frac{\sqrt{u}\log(u)}{2\sqrt{(v^1)^2-(\log(u))^2}}\times\\
&\left(k_1 \mathrm{J}_1\left(\frac{1}{2}\sqrt{(v^1)^2-(\log(u))^2}\right)+k_2 \mathrm{Y}_1\left(\frac{1}{2}\sqrt{(v^1)^2-(\log(u))^2}\right)\right). && (90)
\end{aligned}
$$

It is obvious that $u(x,t)$ and $v^1(x,t)$ cannot be given in explicit form by these equations. In addition to Lie point symmetries of (79) the non classical method yields no further results.

In the case of the nonlinear telegraph equation (51) it is interesting to look for non classical symmetries of the potential system. The non classical method delivers no new solutions for the infinitesimals unless when ϕ_1 and ϕ_2 are zero. Then the determining equations are satisfied identically for arbitrary $\xi_1(x,t,u,v^1)$. Here $u(x,t)$ and $v^1(x,t)$ have to solve the invariant surface condition

$$
\begin{aligned}
\xi_1 u_x + u_t &= 0 && (91)\\
\xi_1 v_x^1 + v_t^1 &= 0. && (92)\\
& && (93)
\end{aligned}
$$

In addition to the invariant surface condition (u, v^1) have to solve the potential system (60) and (61). The only solutions of this extended system is

$$u(x,t) = k_1 \quad \text{and} \quad v^1(x,t) = k_2.$$

for arbitrary ξ_1. So u and v^1 are trivial solutions and linear dependent. Hence the non classical method for the potential system is of no use to discover new solutions for the nonlinear telegraph equation (51).

6 Conclusions

In this article we have demonstrated the combination of Lie point, non classical and potential symmetries which allow us to discover new solutions. Our main interest was concentrated on potential symmetries which deliver new symmetries of nonlocal character. With the aid of potential symmetries new classes of solutions are also found for a class of nonlinear telegraph equations arising in the field of wave propagation, e.g. at electro magnetic shock waves.

Potential systems are very useful to linearize PDEs by an non invertible mapping. This topic is exemplified at equations treated in this article. The linearisation of the nonlinear telegraph equations listed in section 4 are determined and new solutions calculated with this transformation are pointed out. For the special nonlinear telegraph equation examined in section 5 a broad class of new solutions is given which are derived by the linearisation. Explicit new solutions are calculated, too.

Additionally for the equation of section 5 a remarkable non classical symmetry is presented. Up to now we do not know that the invariant surface condition for the corresponding infinitesimals satisfies the original equation identically. So the invariant solution with respect to the infinitesimals contains an arbitrary function and covers a wide class of special solutions.

In this context the relation of non classical symmetries of the potential system and the non classical symmetries of the original system is discussed.

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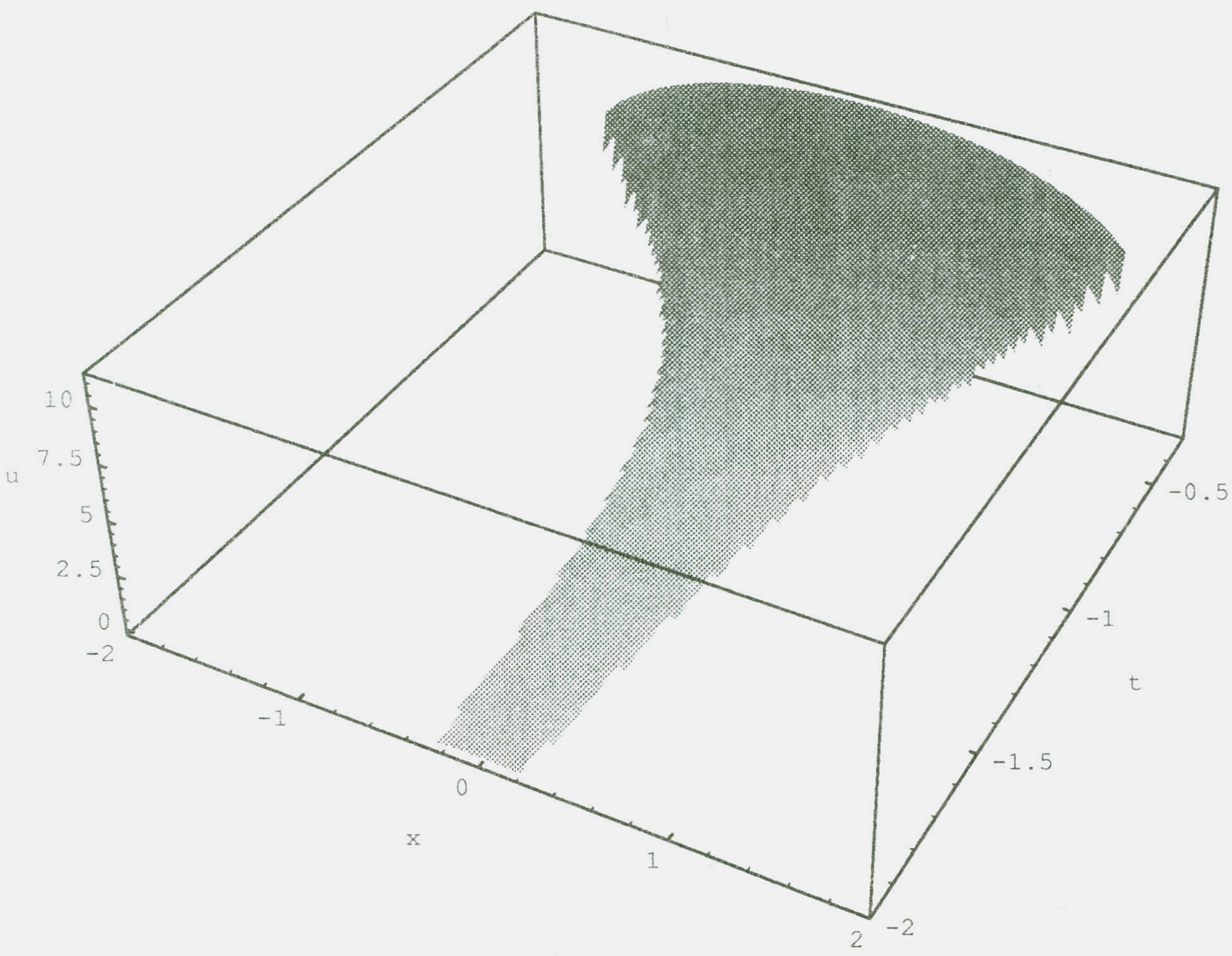

figure 1: Three dimesional plot for the solution $u(x,t)$ (82) of the nonlinear telegraph equation (51).

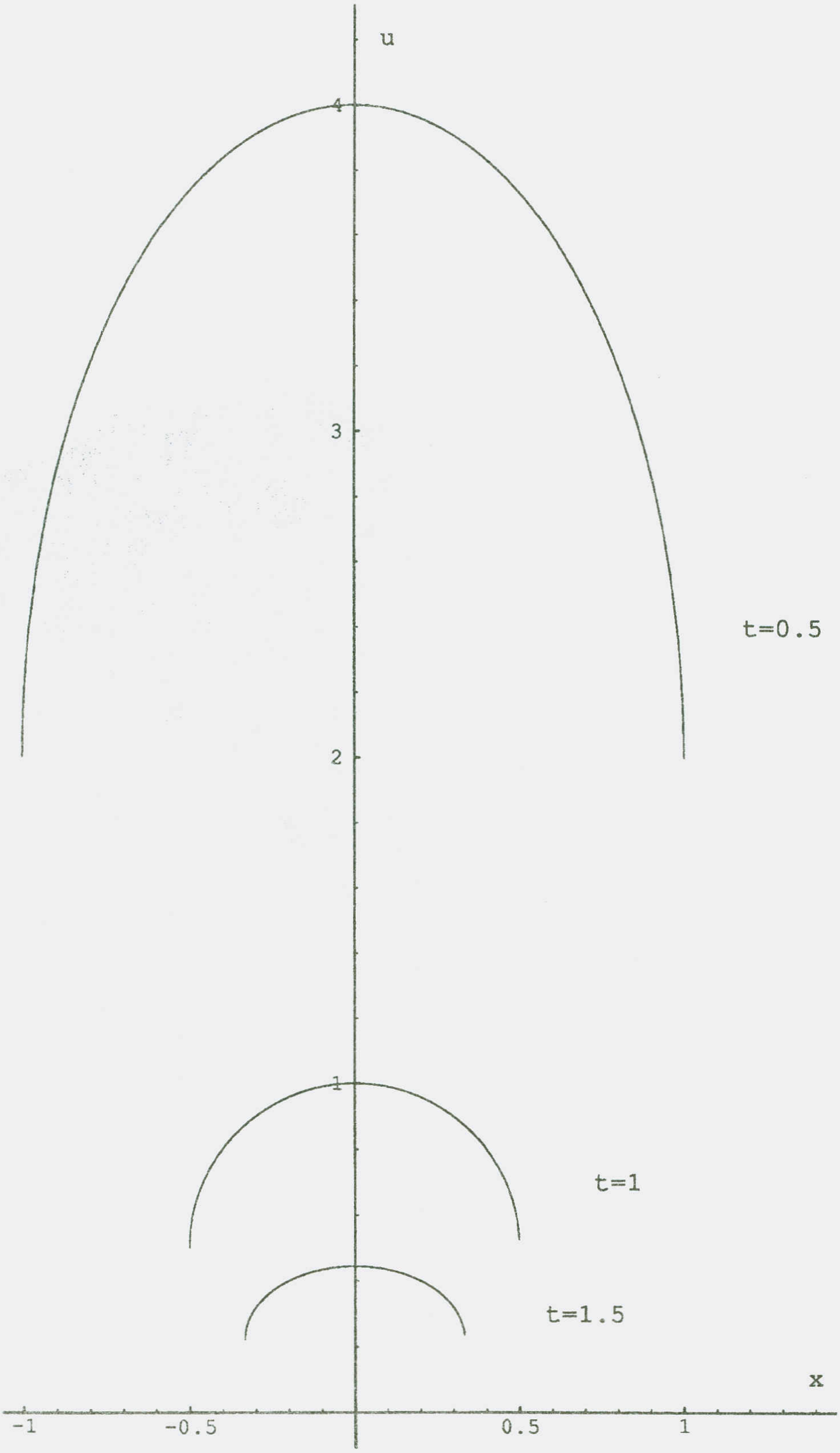

figure 2: Solution $u(x,t)$ (82) of equation (51) plotted as function of x for different times t.