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THE FRENET AND DARBOUX INSTANTANEOUS ROTATION VECTORS OF CURVES ON TIME-LIKE SURFACE

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ABSTRACT

In this paper, depending on the Darboux instantaneous rotation vector of a solid perpendicular trihedron in the Minkowski 3-space $R_1^3 = [R^3, (+,+,-)]$ the Frenet instantaneous rotation vector was stated for a space-like curve (c) with the principal normal **n** being a time-like vector. The Darboux instantaneous rotation vector for the Darboux trihedron was found when the curve (c) is on a time-like surface. Some theorems and results giving the relations between two frames were stated and proved.

1. INTRODUCTION

In Euclidean Space R^3, a solid perpendicular trihedrons' Darboux instantaneous rotation vector and Darboux and Frenet instantaneous rotation vectors of a curve on the surface are known. Let φ be the angle between principal normal **n** and surface normal **N** on a point P of the curve. For the radii of geodesic torsion T_g, normal curvature R_n and geodesic curvature R_g, some relations are given in [1].

Instead of space R^3, Let us consider the Minkowski 3-space R_1^3 provided with Lorentzian inner product

$$\langle \mathbf{a}, \mathbf{b} \rangle = a_1 b_1 + a_2 b_2 - a_3 b_3 \tag{1.1}$$

with $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3) \in R^3$. In this condition the following definitions can be given.

Definition1.1 Let $\mathbf{a} = (a_1, a_2, a_3) \in R_1^3$ If;

i) $\langle \mathbf{a}, \mathbf{a} \rangle > 0$ then **a** is space-like vector,

ii) $\langle \mathbf{a}, \mathbf{a} \rangle < 0$ then **a** is time-like vector,

iii) $\langle \mathbf{a}, \mathbf{a} \rangle = 0$ then **a** is light-like(null) vector.

If $\sqrt{a_1^2 + a_2^2} < a_3$ ($\sqrt{a_1^2 + a_2^2} > a_3$) then **a** is future pointing (past pointing) vector [2].

Definition1.2 Let (c) be a curve in space R_1^3. $c'(t)$ is the tangent vector for

$\forall t \in I \subset R$ then; if

i) $\langle c'(t), c'(t) \rangle > 0$ then (c) is space-like curve,

ii) $\langle c'(t), c'(t) \rangle < 0$ then (c) is time -like curve,

iii) $\langle c'(t), c'(t) \rangle = 0$ then (c) is light-like(null) curve,

[3].

Definition1.3 Let $\mathbf{a} = (a_1,a_2,a_3)$ and $\mathbf{b} = (b_1,b_2,b_3)$ be vectors in space R_1^3. The vectoral product of **a** and **b** is given by

$$\mathbf{a} \wedge \mathbf{b} = (a_3b_2 - a_2b_3 \ , a_1b_3 - a_3b_1 \ , a_1b_2 - a_2b_1) \tag{1.2}$$

[4].

Definition1.4 Let $\mathbf{y} = \mathbf{y}(u,v)$ be a surface in space R_1^3. If $\forall$ $p \in \mathbf{y}(u,v)$ and $\langle , \rangle |_y$ is a Lorentzian metric then $\mathbf{y}(u,v)$ is time-like surface [5].

Definition1.5 Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2) \in R_1^2$ be future-pointing (past-pointing) time-like vectors. The number $\theta \in \mathbf{R}$ in equality

$$\begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{1.3}$$

in this equality $\theta \in R$ number is an hyperbolic angle from **a** to **b** and θ is shown by $\theta = (\mathbf{a}, \mathbf{b})$ [2].

Lemma 1.6 Let **a** and **b** be future pointing time-like unit vectors. If θ is an hyperbolic angle from **a** and **b**, then

$$\cosh\theta = -\langle \mathbf{a}, \mathbf{b} \rangle \tag{1.4}$$

[2].

Let $[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]$ be a solid perpendicular trihedron in space R_1^3. In this situation the following theorem can be given:

Theorem.1.7 If a solid perpendicular trihedrons' unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ $\mathbf{e}_3$ are changing relative to t parameter, then

$$\frac{d\mathbf{e}_i}{dt} = \mathbf{w} \wedge \mathbf{e}_i \qquad i=1,2,3 \tag{1.5}$$

where, $\mathbf{e}_1$ and $\mathbf{e}_2$ are space-like vectors, $\mathbf{e}_3$ is a time-like vector and Darboux instantaneous rotation vector is

$$\mathbf{w} = a\,\mathbf{e}_1 - b\,\mathbf{e}_2 - c\,\mathbf{e}_3 \tag{1.6}$$

[6].

2. FRENET TRIHEDRON FOR A SPACE-LIKE SPACE CURVE WITH TIME-LIKE PRINCIPAL NORMAL

Let us consider $c = c(s)$ space-like curve. For any parameter s on all points on this curve , we can construct Frenet trihedron [**t** , **n** ,**b**], here,**t** ,**n** and **b** tangent ,principal normal and binormal unit vectors, respectively. In this trihedron, **n** is time-like unit vector; **t** and **b** are space-like unit vectors. For that,

$$\langle \mathbf{t}, \mathbf{t} \rangle = \langle \mathbf{b}, \mathbf{b} \rangle = 1 \ , \qquad \langle \mathbf{n}, \mathbf{n} \rangle = -1 \tag{2.1}$$

$$\langle \mathbf{t}, \mathbf{n} \rangle = \langle \mathbf{b}, \mathbf{t} \rangle = \langle \mathbf{n}, \mathbf{n} \rangle = 0 \tag{2.2}$$

can be written. For Frenet trihedron's vectors, the vectoral product is given by

$$\mathbf{t} \wedge \mathbf{n} = -\mathbf{b} \ , \ \mathbf{n} \wedge \mathbf{b} = -\mathbf{t} \ , \ \mathbf{b} \wedge \mathbf{t} = \mathbf{n} \tag{2.3}$$

It can be written

$$\begin{aligned} \frac{d\mathbf{t}}{ds} &= a_{11}\,\mathbf{t} + a_{12}\,\mathbf{n} + a_{13}\,\mathbf{b} \\ \frac{d\mathbf{n}}{ds} &= a_{21}\,\mathbf{t} + a_{22}\,\mathbf{n} + a_{23}\,\mathbf{b} \\ \frac{d\mathbf{b}}{ds} &= a_{31}\,\mathbf{t} + a_{32}\,\mathbf{n} + a_{33}\,\mathbf{b} \end{aligned} \tag{2.4}$$

for a parameter's specific value $s = s_0$ with $\mathbf{t} = \mathbf{t}(s)$, $\mathbf{n} = \mathbf{n}(s)$,and $\mathbf{b} = \mathbf{b}(s)$.

If we take derivatives of eqs. (2.1) and (2.2) with respect to arc s ,then we have

$$\begin{aligned} &\langle \mathbf{t}, \frac{d\mathbf{t}}{ds} \rangle = \langle \mathbf{b}, \frac{d\mathbf{b}}{ds} \rangle = \langle \mathbf{n}, \frac{d\mathbf{n}}{ds} \rangle = 0 \\ &\langle \frac{d\mathbf{t}}{ds}, \mathbf{n} \rangle + \langle \mathbf{t}, \frac{d\mathbf{n}}{ds} \rangle = \langle \frac{d\mathbf{t}}{ds}, \mathbf{b} \rangle + \langle \mathbf{t}, \frac{d\mathbf{b}}{ds} \rangle = \langle \frac{d\mathbf{n}}{ds}, \mathbf{b} \rangle + \langle \mathbf{n}, \frac{d\mathbf{b}}{ds} \rangle = 0 \end{aligned} \tag{2.5}$$

and

$$a_{11} = a_{22} = a_{33} = 0 \ .$$

Assuming that

$$a_{23} = a_{32} = a \ , \ a_{21} = a_{12} = c \quad \text{and} \quad a_{13} = -a_{31} = b$$

we obtain the following formulas for derivatives :

$$\begin{aligned} \frac{d\mathbf{t}}{ds} &= c.\mathbf{n} + b.\mathbf{b} \\ \frac{d\mathbf{n}}{ds} &= c.\mathbf{t} + a.\mathbf{b} \\ \frac{d\mathbf{b}}{ds} &= -b.\mathbf{t} + a.\mathbf{n} \end{aligned} \tag{2.6}$$

On a space-like curve $c = c(s)$ given a point P, if the radius of curvature is R and radius of torsion is T then Frenet formulae are obtained as following [7].

$$\begin{aligned} \frac{dt}{ds} &= \frac{1}{R}.\mathbf{n} \\ \frac{d\mathbf{n}}{ds} &= \frac{1}{R}.\mathbf{t} + \frac{1}{T}.\mathbf{b} \\ \frac{d\mathbf{b}}{ds} &= \frac{1}{T}.\mathbf{n} \end{aligned} \tag{2.7}$$

If (2.6) and (2.7) are compared, then we obtain

$c = 1 / R$, $b = 0$, $a = 1 / T$. For that, for any perpendicular trihedron , the Darboux vector is

$$\mathbf{w} = a.\mathbf{e}_1 - b.\mathbf{e}_2 - c.\mathbf{e}_3$$

and for Frenet trihedron we find

$$\mathbf{f} = -\frac{\mathbf{t}}{T} + \frac{\mathbf{b}}{R} . \tag{2.8}$$

In this situation, if we apply the formula (1.5) obtained for a general perpendicular trihedron to Frenet trihedron, we can write

$$\begin{aligned} \frac{dt}{ds} &= \mathbf{f} \wedge \mathbf{t} \\ \frac{d\mathbf{n}}{ds} &= \mathbf{f} \wedge \mathbf{n} . \\ \frac{d\mathbf{b}}{ds} &= \mathbf{f} \wedge \mathbf{b} \end{aligned} \tag{2.9}$$

Then the matrix form of (2.7) is

$$\frac{d}{ds}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & 1/R & 0 \\ 1/R & 0 & 1/T \\ 0 & 1/T & 0 \end{bmatrix}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}.$$

3. DARBOUX TRIHEDRON FOR A SPACE-LIKE CURVE WITH TIME-LIKE GEODESIC NORMAL

Let us consider time-like surface $\mathbf{y} = \mathbf{y}(u,v)$. For a space-like curve (c) on $\mathbf{y} = \mathbf{y}(u,v)$ there exists the Frenet trihedron $[\mathbf{t} , \mathbf{n} , \mathbf{b}]$ at all points of (c) .There is also a second trihedron because the curve (c) is on surface $\mathbf{y} = \mathbf{y}(u,v)$. Let us denote the tangent space-like unit vector with $\mathbf{t}$, the normal space-like unit vector with $\mathbf{N}$ at the point P . In this case , if we take the time-like vector $\mathbf{g}$ defined by

$$\mathbf{t} \wedge \mathbf{g} = -\mathbf{N} \qquad \mathbf{g} \wedge \mathbf{N} = -\mathbf{t} \qquad \mathbf{N} \wedge \mathbf{t} = \mathbf{g} \tag{3.1}$$

then we obtain a new trihedron [$\mathbf{t}$, $\mathbf{g}$, $\mathbf{N}$] .

Let θ is the hyperbolic angle between the time-like vectors $\mathbf{N}$ and $\mathbf{g}$ (Figure 1).

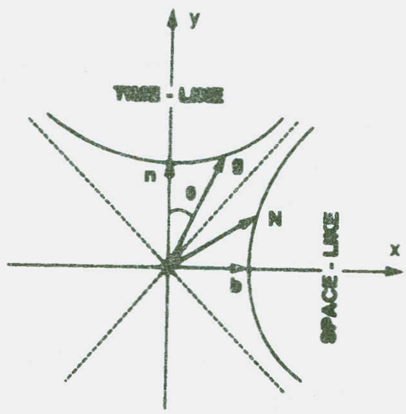

Figure 1

Then we can write

$$\mathbf{N} = \mathbf{n} \sinh\theta + \mathbf{b} \cosh\theta \qquad \mathbf{g} = \mathbf{n} \cosh\theta + \mathbf{b} \sinh\theta \tag{3.2}$$

If we take the derivatives of $\mathbf{t}$, $\mathbf{g}$ and $\mathbf{N}$ according to the arc s of curve (c) , we can find

$$\frac{d\mathbf{t}}{ds} = \rho \ \mathbf{g} \cosh\theta - \rho \mathbf{N} \sinh\theta$$

$$\frac{d\mathbf{g}}{ds} = \rho \ \mathbf{t} \cosh\theta + (\tau + \frac{d\theta}{ds})\mathbf{N}$$

$$\frac{d\mathbf{N}}{ds} = \rho \ \mathbf{t} \sinh\theta + (\tau + \frac{d\theta}{ds})\mathbf{g}$$

$$\rho \cosh\theta = \frac{\cosh\theta}{R} = \frac{1}{R_g} = \rho_g$$

$$\rho \sinh\theta = \frac{\sinh\theta}{\mathrm{R}} = \frac{1}{\mathrm{R_n}} = \rho_n$$

$$\tau + \frac{d\theta}{ds} = \frac{1}{\mathrm{T}} + \frac{d\theta}{ds} = \frac{1}{\mathrm{T_g}} = \tau_g$$

then the Darboux derivative formulae are given by

$$\begin{aligned} \frac{d\mathbf{t}}{ds} &= \rho_g\, \mathbf{g} - \rho_n\, \mathbf{N} \\ \frac{d\mathbf{g}}{ds} &= \rho_g\, \mathbf{t} + \tau_g\, \mathbf{N} \ , \\ \frac{d\mathbf{N}}{ds} &= \rho_n\, \mathbf{t} + \tau_g\, \mathbf{g} \end{aligned} \tag{3.3}$$

where ρ_g is geodesic curvature, ρ_n is normal curvature and τ_g is geodesic torsion. The matrix form of (3.3) is

$$\frac{d}{ds}\begin{bmatrix} \mathbf{t} \\ \mathbf{g} \\ \mathbf{N} \end{bmatrix} = \begin{bmatrix} 0 & \rho_g & -\rho_n \\ \rho_g & 0 & \tau_g \\ \rho_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{g} \\ \mathbf{N} \end{bmatrix}.$$

We can write the Darboux instantaneous rotation vector of Darboux trihedron

$$\mathbf{w} = -\frac{\mathbf{t}}{\mathrm{T_g}} - \frac{\mathbf{g}}{\mathrm{R_n}} + \frac{\mathbf{N}}{\mathrm{R_g}} \ . \tag{3.4}$$

Darboux derivative formulae (3.3) can be given with Darboux vector as below

$$\begin{aligned} \frac{d\mathbf{t}}{ds} &= \mathbf{w} \wedge \mathbf{t} \\ \frac{d\mathbf{g}}{ds} &= \mathbf{w} \wedge \mathbf{g} \ . \\ \frac{d\mathbf{N}}{ds} &= \mathbf{w} \wedge \mathbf{N} \end{aligned} \tag{3.5}$$

Theorem 3.1 If the radius of torsion of the space-like (c) drawn on time-like surface $\mathbf{y} = \mathbf{y}(u,v)$ is T and the hyperbolic angle between time-like unit vectors $\mathbf{n}$ and $\mathbf{g}$ is θ, then we have

$$\frac{1}{T_g} = \frac{1}{\mathrm{T}} + \frac{d\theta}{ds} \ . \tag{3.6}$$

Proof: If we take derivative of both sides of equation $\langle\, \mathbf{n}\, ,\, \mathbf{g}\, \rangle = -\cosh\theta$ and use the (2.7) and (3.3) we obtain

$$\frac{dn}{ds}\,g + n\,\frac{dg}{ds} = -\sinh\theta\,\frac{d\theta}{ds}$$

$$\left(\frac{1}{R}\,t + \frac{1}{T}\,b\right)g + n\left(\frac{1}{R_g}\,t + \frac{1}{T_g}\,N\right) = -\sinh\theta\,\frac{d\theta}{ds}$$

$$\frac{1}{R}\langle t, g\rangle + \frac{1}{T}\langle b, g\rangle + \frac{1}{R_g}\langle n, t\rangle + \frac{1}{T_g}\langle n, N\rangle = -\sinh\theta\,\frac{d\theta}{ds}.$$

Since $\langle \mathbf{n}, \mathbf{N}\rangle = -\sinh\theta$, we have

$$\frac{1}{T}\sinh\theta - \frac{1}{T_g}\sinh\theta = -\sinh\theta\,\frac{d\theta}{ds}$$

$$\frac{1}{T_g} = \frac{1}{T} + \frac{d\theta}{ds}$$

Theorem.3.2 If the radius of curvature of the space-like curve (c) on time-like surface $\mathbf{y} = \mathbf{y}(u,v)$ is R and the hyperbolic angle between time-like unit vectors $\mathbf{n}$ and $\mathbf{g}$ is θ. then we have

$$\frac{1}{R_n} = \frac{\sinh\theta}{R} \qquad\qquad (3.7)$$

$$\frac{1}{R_g} = \frac{\cosh\theta}{R}$$

Proof: From (2.9) and (3.5),

$$(\mathbf{f} - \mathbf{w}) \wedge \mathbf{t} = 0 .$$

If the values of vectors **f** and **w** are written,

$$\frac{1}{R}(\mathbf{b} \wedge \mathbf{t}) + \frac{1}{R_n}(\mathbf{g} \wedge \mathbf{t}) - \frac{1}{R_g}(\mathbf{N} \wedge \mathbf{t}) = 0$$

are obtained. Here, since

$$\mathbf{b} \wedge \mathbf{t} = \mathbf{n} \ , \ \mathbf{g} \wedge \mathbf{t} = \mathbf{N} \ , \ \mathbf{N} \wedge \mathbf{t} = \mathbf{g}$$

we find

$$\frac{\mathbf{n}}{R} = -\frac{\mathbf{N}}{R_n} + \frac{\mathbf{g}}{R_g} .$$

If the both sides of this equation are scalarly multiplied with the vectors **N** and **g** and considered the equalities

$$\langle \mathbf{N}, \mathbf{g}\rangle = 0 \ , \ \langle \mathbf{n}, \mathbf{g}\rangle = -\cosh\theta \ , \ \langle \mathbf{g}, \mathbf{g}\rangle = -1 ,$$

the proof is completed .

Corollary 3.3. There is a relation between Frenet and Darboux vectors as follows :

$$\mathbf{w} = \mathbf{f} - \frac{d\theta}{ds}\,\mathbf{t}$$

Proof: From equations (2.9) and (3.5) we can write

$$\mathbf{w} = \mathbf{f} + \lambda\,\mathbf{t} \quad , \quad \lambda \in \mathbf{R}.$$

From (2.8) and (3.4)

$$-\frac{1}{T_g} = -\frac{1}{T} + \lambda$$

is obtained . If we consider the formula (3.6) we have

$$\lambda = -\frac{d\theta}{ds}.$$

This completes the proof.

Corollary 3.4. Instantaneous rotation velocity vector **w** of the Darboux trihedron is consists of two components. One of them coincides with Frenet trihedrons' instantaneous rotation velocity vector. The other is a component in the opponent direction of tangent and equals to $d\theta/ds$. For this reason, when a point P of space-like curve (c) on the time-like surface moves on this curve Darboux trihedron moves with radial velocity $d\theta / ds$ according to Frenet trihedron in the opposite direction to the tangent at any time.

Corollary 3.5 If the hyperbolic angle between the principal normal of a space-like curve on a time-like surface and the vector **g** at the same of point of the surface of g vector becomes always constant , then

$$\frac{1}{T_g} = \frac{1}{T} \quad , \quad \mathbf{w} = \mathbf{f}$$

Thus, the torsion of the curve at each point equals to the geodesic torsion of the surface at that point.

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