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SATISFACTION OF BOUNDARY CONDITIONS ON COMPLEX CHANNEL WALLS WITH QUADRATIC MINIMIZATION METHOD

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Abstract- Numerical analysis of cellular two-dimensional Stokes flow induced by rotation of a circular cylinder in a V-shaped channel bounded by a cylindrical surface has been made. In the numerical calculations, series in terms of polar coordinates are used to represent the stream function and quadratic minimization method is employed to satisfy the outer boundary conditions on the channel walls. Numerical results are presented in the form of streamline patterns.

1. INTRODUCTION

In this work, cellular two-dimensional Stokes flow induced by rotation of a circular cylinder in a V-shaped channel bounded by a cylindrical surface is analyzed numerically. In recent literature, much work has been devoted to the study of cavity flow, to separation phenomena and to viscous cells. The most studied case is naturally a cavity of a rectangular shape. The work done on this subject and many practical applications encountered have been provided by Shen and Floryan[1], Higdon[2], Rybicki and Floryan[3].

In the closed rectangular cavity, the fluid motion is forced by the translation of the upper boundary with uniform velocity in its own plane. This is the well-known driven cavity problem which has been subjected to many numerical investigations, such as finite differences, finite elements, false transients, spectral methods, multigrid methods, etc. Some of these include Burggraf[4], Tuann and Olson[5], Ghia, Ghia and Shin[6], Schreiber and Keller[7], Gustafson and Halasi[8], Napolitano and Pascazio[9], Shyy, Thakur and Wrigth[10], Iliev, Makarov and Vassilevski[11] and Cortes and Miller[12]. Shankar[13] investigated the motion in the cavity by an analytical and semi-analytical method.

Although the rectangular cavity flow has been investigated extensively,publications are very scarce on the subject of the triangular cavity. In fact, the triangular shape is more common in practice. The practical applications of the triangular grooves are described by Ribbens, Watson and Wang[14] and in a related work by Savvides and Gerrard[15].

In this paper, numerical analysis of cellular two-dimensional Stokes flow induced by rotation of a circular cylinder in a V-shaped channel bounded by a cylindrical surface has been made. In the calculations, series in terms of polar coordinates are used to represent the stream function and quadratic minimization method is employed to satisfy the outer boundary conditions on the channel walls. The streamline patterns are obtained for different wedge lengths. The mathematical formulation of the problem is given in the next section. The application of quadratic minimization method is explained in detail in Section 3, which outlines the numerical procedure used. The results of the numerical calculations are given in the form of streamlines in Section 4.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

Let us consider an infinitely long vertical circular cylinder of radius R_1, placed in a V-shaped channel bounded by a cylindrical surface of radius R_2, as shown in Fig.1. The rotating cylinder is positioned at the center of the cylindrical surface O. As can be seen from Fig.1, this special geometry of the outer boundary of the channel has only one sharp corner at P.

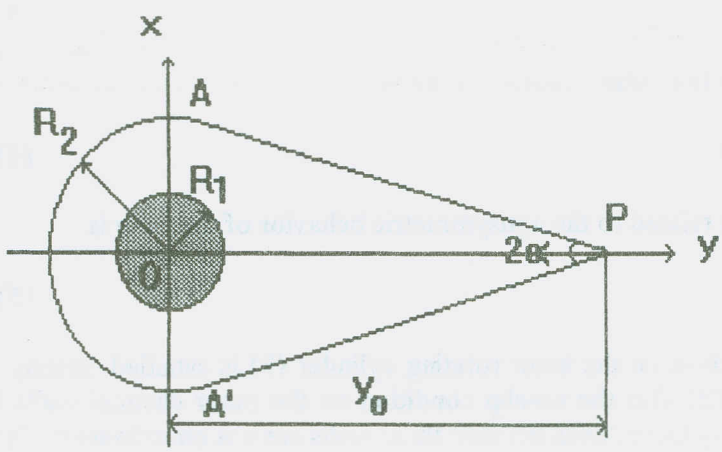

Fig.1- The geometry of the problem.

The cylindrical surface and wedge are matched at A and A' such that there are no sharp corners at A and A'. The distance between the center of the cylinder and the corner is denoted by y_o. The wedge angle is 2α.

This channel is filled with a highly viscous fluid of constant physical properties; its kinematic viscosity is denoted by ν. The motion is obtained by a very slow rotation of the inner cylinder with a uniform angular velocity ω_o. Thus the Reynolds number of the flow, defined by $Re=(\omega_o R_1^2)/\nu$, is supposed to be sufficiently small to ensure the validity of the Stokes regime hypothesis. Under these conditions, the equation of motion in the horizontal cross-section is

$$\Delta(\Delta\psi)=\nabla^4\psi=0$$
(1)

where ψ is the stream function and ∇^2 is a second-order differential operator.

The solution of (1), expressed in polar coordinates is

$$\psi(r,\theta) = a_o + b_o \ln(r) + c_o r^2 + d_o r^2 \ln(r)$$
$$+ [a_1 r^{-1} + b_1 r \ln(r) + c_1 r + d_1 r^3] \cos\theta$$
$$+ [e_1 r^{-1} + f_1 r \ln(r) + g_1 r + h_1 r^3] \sin\theta$$
$$+ \sum_{n=2}^{\infty} [a_n r^{-n} + b_n r^{-n+2} + c_n r^n + d_n r^{n+2}] \cos(n\theta)$$
$$+ \sum_{n=2}^{\infty} [e_n r^{-n} + f_n r^{-n+2} + g_n r^n + h_n r^{n+2}] \sin(n\theta) \qquad (2)$$

where a_n , b_n ... are arbitrary coefficients to be determined by the boundary conditions.

If the coordinate r and the radial and tangential velocity components V_r and V_θ are normalized by R_1 and $\omega_o R_1$ respectively, the boundary conditions on the cylinder are written as follows

$$V_r = 0, \qquad V_\theta = 1; \qquad (3)$$

The boundary conditions on the outer channel walls are

$$V_r = 0, \qquad V_\theta = 0; \qquad (4)$$

and the symmetry condition related to the antisymmetric behavior of the flow is

$$\psi(r,\theta) = \psi(r,-\theta) \qquad (5)$$

The no-slip boundary condition on the inner rotating cylinder (3) is satisfied exactly by the polar coordinate series (2). But the no-slip condition on the outer channel walls (4) cannot be satisfied exactly by these series because these walls are not coordinate surfaces of the frame that we have used to define the stream function. Therefore, the latter condition has been satisfied optimally by using the quadratic minimization method. Applying the boundary conditions (3), (4), (5) to (2) we will determine the coefficients of the series (2). The numerical procedure will be explained in the next section.

3. NUMERICAL PROCEDURE

The satisfaction of the boundary conditions on the outer channel walls with quadratic minimization method consists of minimizing the quadratic difference between the imposed velocity on the boundary and the velocity deduced from the series (2). This corresponds to the minimization of the integral

$$I = \int_{\Gamma} (V_r^2 + V_\theta^2)\, dS \qquad (6)$$

where Γ is the outer boundary. For convenience of programming, we can write the expressions for the velocity components and stream function calculated from (2), in the linear form

$$V_r = \sum_{j=1}^{4N+1} A_j F_j \tag{7}$$

$$V_\theta = r + \sum_{j=1}^{4N+1} A_j G_j \tag{8}$$

$$\Psi = -\frac{1}{2} r^2 + \sum_{j=1}^{4N+1} A_j H_j \tag{9}$$

where A_j represents the coefficients b_o , b_n , d_n ,.....; N the number of terms retained in the series (2).

The minimization of I can be written in the form

$$\frac{\partial}{\partial A_i} \int_\Gamma (V_r^2 + V_\theta^2) dS = 0 \qquad i = 1,2,\ldots,4N+1 \tag{10}$$

Then, the minimization yields the linear system

$$\sum_{j=1}^{4N+1} A_j \sum_{N_P} (F_i F_j + G_i G_j) = -r \sum_{N_P} G_i \qquad i = j = 1,2,\ldots,4N+1 \tag{11}$$

In this system the integrals are replaced by a simple summation on an adequate number N_P of points, regularly spaced on the outer boundary Γ.

This linear system yields the unknown A_j coefficients from which we shall calculate velocity components and stream function. To check the accuracy of the results, the mean values of the velocity on the channel walls are calculated and the numerical results are presented in the next section.

4. THE NUMERICAL RESULTS

In our calculations, the number of terms N retained in the series (2) is 40. Initially, we tried with 30, 40 and 50 terms in the series, and saw that this would not sensibly affect the accuracy of the result, so we decided to use 40 terms in the calculations. Thus, using 161 coefficients A_j of the series and 361 minimization points N_P, we have obtained streamline patterns for different values of y_o. For the numerical calculations, confining aspect ratio R_1/R_2 is 0.5. For convenience, R_1 is set to 1. y_o is variable and taken as a parameter. The mean values of the velocity V_R on the channel walls are calculated and tabulated in Table 1 for different values of y_o.

Table-1. The mean values of velocity on the channel walls and torque ratio.

y_o	2α	V_R	b_o
2.5	86°	4.082×10^{-6}	1.525
3	70°	3.776×10^{-5}	1.434
4	60°	5.380×10^{-4}	1.365
5	46°	1.909×10^{-3}	1.341
6	40°	9.946×10^{-4}	1.329
7	32°	1.442×10^{-3}	1.323
7.5	30°	2.073×10^{-3}	1.320

In this table, the values of b_o that represents the ratio of the torque experienced by the cylinder in the channel to the torque experienced by this cylinder in an infinite medium are also indicated. We have obtained excellent accuracy for small y_o values. As can be seen from this table, the accuracy begins to decrease when y_o takes large values. It is due to the presence of the positive exponents in the series which increase with increasing distance. This prevents to detect the cellular flow in the vicinity of the corner only at large y_o.

The streamline patterns are presented in Fig.2 for y_o=3.0.

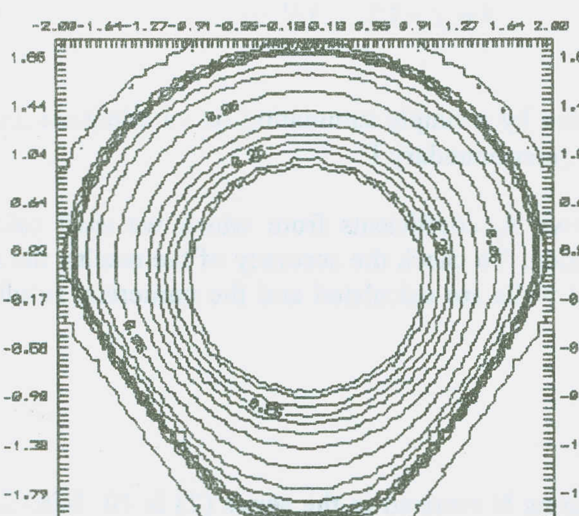

Fig.2- The streamline patterns for y_o =3.0.

It must be worthwhile to note that these numerical results are in excellent agreement with the corresponding visualization photographs obtained by the author, Kent[16] experimentally by means of flow visualization technique for the same geometry.

5. CONCLUSION

Numerical analysis of cellular two-dimensional Stokes flow induced by rotation of a cylinder in a wedge-shaped channel bounded by a cylindrical surface has been made. The satisfaction of the boundary conditions on the outer channel walls with quadratic minimization method is explained in detail. Numerical results are presented in the form of streamline patterns.

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