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Impact of Thermal Radiation and Heat Source/Sink on MHD Time-Dependent Thin-Film Flow of Oldroyd-B, Maxwell, and Jeffery Fluids over a Stretching Surface

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Abstract: In this study paper, we examined the magnetohydrodynamic (MHD) flow of three combined fluids, Maxwell, Jeffery, and Oldroyd-B fluids, with variable heat transmission under the influence of thermal radiation embedded in a permeable medium over a time-dependent stretching sheet. The fluid flow of liquid films was assumed in two dimensions. The fundamental leading equations were changed to a set of differential nonlinear and coupled equations. For this conversion, suitable similarity variables were used. An optimal tactic was used to acquire the solution of the modeled problems. The convergence of the technique has been shown numerically. The obtained analytical and numerical consequences are associated graphically and tabulated. An excellent agreement was obtained between the homotopy analysis method (HAM) and numerical methods. The variation of the skin friction and Nusslet number and their influence on the temperature and concentration profiles were scrutinized. The influence of the thermal radiation, unsteadiness effect, and MHD were the main focus of this study. Furthermore, for conception to be physically demonstrated, the entrenched parameters are discussed graphically in detail along with their effect on liquid film flow.

Keywords: Jeffrey, Maxwell, Oldroyd-B fluids; unsteady stretching surface; magnetic field; homotopy analysis method (HAM)

1. Introduction

In the last few years, thin film flow problems have received great attention because of the importance of thin film flows in various technologies. The investigation of thin film flow has obtained importance by its vast application and uses in engineering, technology, and industries. Cable, fiber undercoat, striating of foodstuff, extrusion of metal and polymer, constant forming, fluidization of the device, elastic sheets drawing, chemical treating tools, and exchanges are several uses. In surveillance of these applications, researchers have paid attention to cultivating the examination of liquid film on stretching surface. Emslie et al. [1] investigated thin film flow with applications. During the disk rotating process, they considered the balance between centrifugal and viscous forces. They simplified the Navier–Stokes equations and concluded that the film is uniformly maintained with its continuous thinning property. Higgins [2] considered the influence of the film inertia over a rotating disk. A liquid film over a rotating plate with constant angular velocity was analyzed by Dorfman [3]. The fluid film rotation on an accelerating disk was analyzed by Wang et al. [4]. For the thin and thick film parameter and small accelerating parameters, the asymptotic solutions were obtained. Andersson et al. [5]

asymptotically and numerically examined the magnetohydrodynamics (MHD) liquid thin film due on a rotating disk. Over a rotating disk, Dandapat and Singh [6] examined the two layer film flow. The heat transfer flow of thin film flow of nonfluids was deliberated on by Sandeep et al. [7]. Recently, researchers [8–15] examined non-Newtonian nanofluid thin films using different models in different geometries and obtained useful results.

Jeffrey, Maxwell, and Oldroyd-B nanofluids have certain importance in the area of fluid mechanics because of the stress relaxation possessions. Kartini et al. [16] studied the two-dimensional MHD flow and heat transfer of Jeffrey nanofluid over an exponentially stretched plate. Hayat et al., [17] studied the convection of heat transfer in two-dimensional flow of Oldroyd-B nanofluid over a stretching sheet under radiation, and they also found that, with the escalation in radiation parameter, the temperature profile of the nanofluid escalates. Raju et al. [18] debated the presence of a homogenous–heterogeneous reaction in the nonlinear thermal radiation effect on Jeffrey nanofluid flow. Hayat et al. [19] presented a series solution of MHD flow of Maxwell nanofluid over a permeable stretching sheet with suction and injection impacts. Sandeep et al. [20] studied the unsteady mixed convection flow of micropolar fluid over a stretching surface with a non-uniform heat source. Nadeem et al. [21] discussed the heat and mass transfer in peristaltic motion of Jeffrey nanofluid in an annulus. Sheikholeslami [22,23] discussed the hydrothermal behavior of nanofluids flow due to external heated plates. Shah et al. [24–29] investigated MHD nanofluid flow heat transfer, and they mathematically analyzed it with Darcy–Forchheimer phenomena and Cattaneo–Christov flux impacts. Nanofluid flow was studied between parallel plates and stretching sheets. They used analytical and numerical approaches and obtained excellent results. Dawar et al. [30] analyzed the MHD carbon nanotubes (CNTs) Casson nanofluid in rotating channels. Khan et al. [31] studied the 3-D Williamson nanofluid flow over a linear stretching surface.

Lee [32] was the first to investigate the flow with a high Reynolds number (boundary layer concept) over a thin body with variable thickness. In engineering science, the variable thickness extending sheet has many more applications compared to the flat sheet. For instance, for a straightly-extending sheet of an incompressible material, if the extending speed is directly relative to the separation from the slot, the sheet thickness diminishes straightly with the separation. For different materials with various extensibility, the plate/sheet thickness may change as indicated by different profiles. An extending plate/sheet with variable thickness is more realistic. Peristaltic flow of two layers of power law fluids was studied in Usha et al. [33]. Eegunjobi and Makinde [34] mutually derived the effect of buoyancy force and Navier slip in a channel of vertical pores on entropy generation. Numerical analyses of the buoyancy force's influence on the unsteady flow of hydromagnetic by a porous channel with injection/suction were presented by Makinde et al. [35].

Many mathematical problems in the field of engineering and science are complicated, and it is often impossible to solve these types of problems exactly. To find an approximate solution, numerical and analytical techniques are widely implemented in the literature. Liao in 1992 [36,37] investigated the solution of such types of problems by implementing a new proposed technique. This technique was named the homotopy analysis method. He further discussed the convergence of the newly implemented method. A solution is a function of a single variable in the form of a series. Due to fast convergence and strong results, many researchers [38–45] used the homotopy analysis method. Another powerful analytical tool to solve differential equations is the Lie algebra method [46–48].

The goal of the present study was to inspect the MHD flow of three combined nanofluids (Maxwell, Oldroyd-B, and Jeffrey) over a linear stretching surface. The fundamental leading equations were changed to a set of differential nonlinear equations with the support of appropriate correspondence variables. An optimal tactic was used to achieve the solution of the modeled equations, which is nonlinear. The effects of all embedding parameters were studied graphically. Boundary layer methodology was used in the mathematical expansion. The influences of the skin friction, Nusselt number, and Sherwood number on the velocity profile, temperature profile, and concentration profile, respectively, were studied.

2. Problem Formulation

Consider a time-dependent and electric-conducting thin film flow of magnetohydrodynamic Jeffrey, Maxwell, and Oldroyd-B thin film liquids while extending a plate. The flexible sheet starts from an inhibiting slit, which is immovable at the descent of the accommodating system. The Cartesian accommodating system oxy is adjusted in a well-known manner where ox is equivalent to the plate and oy is smooth to the sheet. The surface of the flow is stretched, applying two equivalent and reversed forces along the x -axis and keeping the origin stationary. The x -axis is taking along the extending surface with stressed velocity $U_w(x, t) = \frac{bx}{1-\alpha t}$, in which α, b are constant and the y -axis is perpendicular to it. $T_w(x, t) = T_0 + T_r \left(\frac{bx^2}{2v_f} \right) (1-\alpha t)^{-1.5}$ is the wall temperature of the fluid, v_f represents the kinematic viscosity of the fluid, and T_0, T_r are slits. An exterior magnetic ground $B(t) = B_0(1-\alpha t)^{-0.5}$ is given normally to the extending sheet (as shown in Figure 1). All the body forces are ignored in the flow field.

Figure 1. Geometry of the problem.

Considering these hypotheses, the continuity equation, the elementary boundary governing equation, and the heat transfer and concentration equations can be stated as:

$$u_x + u_y = 0 \quad (1)$$

$$\rho_f \left[u_t + uu_x + vv_y + \lambda_1 (u^2 u_{xx} + v^2 u_{yy} + 2uvu_{xy}) \right] = \frac{\mu_f(T)}{1+\lambda_2} \left[u_{yy} + \lambda_3 (u_{tyy} + uu_{xyy} + vv_{yyy} - u_x u_{yy} + u_y u_{xy}) \right] - \sigma_{nf} B^2(t) u, \quad (2)$$

$$T_t + uT_x + vT_y = \frac{1}{\rho c_p} T_{yy} + \frac{Q}{\rho c_p} T_s - T_0 - \frac{1}{\rho c_p} q_y, \quad (3)$$

The applicable boundary conditions are:

$$u = U_w, v = 0, T = T_w \text{ at } y = 0, \quad (4)$$

$$u_x = 0, T_x = 0, \text{ at } y = h \quad (5)$$

where ϑ_{rd} is:

$$\vartheta_{rd} = -\frac{16\varphi}{3\hat{k}} T_y^4 \quad (6)$$

Applying Taylor series expansion on T^4 , we get:

$$T^4 = T_0^4 + 4T_0^3(T - T_0)^2 + \dots \quad (7)$$

By ignoring higher-order terms:

$$T^4 = 4TT_0^3 - 3T_0^4 \quad (8)$$

on which u, v are the velocity constituents alongside the x and y -axes, respectively, σ is the electric conductive parameter, μ, ρ_f, ν_f are the dynamic viscosity, density, and kinematic viscosity, respectively and $\gamma_1, \gamma_2, \gamma_3$ represents the relaxation and retardation time ratios. $B(t)$ characterizes the applied magnetic pitch, T_s represents the temperature of the fluid, and T_w characterizes the temperature. Equation (2) manages distinctive fluid models dependent on the below conditions [7]:

- I. The fluid should be Jeffrey fluid when $\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 \neq 0$;
- II. The fluid should be Maxwell fluid when $\lambda_1 \neq 0, \lambda_2 = 0, \lambda_3 = 0$;
- III. The fluid should be Oldroyd-B fluid when $\lambda_1 \neq 0, \lambda_3 \neq 0, \lambda_2 = 0$.

Considering the above similarity transformations:

$$\begin{aligned} \psi &= x \sqrt{\frac{vb}{1-at}} f(\eta), u = \psi_y = \frac{bx}{(1-at)} f'(\eta), \\ v &= -\psi_x = -\sqrt{\frac{vb}{1-at}} f(\eta), \eta = \sqrt{\frac{b}{v(1-at)}} y, y = \sqrt{\frac{v(1-at)}{b}} \phi(\eta) \\ \phi(\eta) &= \frac{T-T_0}{T_w-T_0}, \end{aligned} \quad (9)$$

where prime specifies the derivative with respect to η and ψ indicates the stream function, $h(t)$ specifies the liquid film thickness, and $\nu = \frac{\mu}{\rho}$ is the kinematics viscosity. The dimensionless film thickness $\beta = \sqrt{\frac{b}{v(1-at)}} h(t)$, which gives $h_t = -\frac{\alpha\beta}{2} \sqrt{\frac{v}{b(1-at)}}$. Substituting Equation (9) into Equations (1)–(6) gives L

$$\begin{aligned} &\frac{1}{(1+\lambda_2)} \left[f''' + \lambda_3 \left\{ (f'')^2 - f f^{iv} + S \left(2f''' + \frac{\eta}{2} f^{iv} \right) - M f' \right\} \right] + \\ &(1 + \epsilon\kappa) \left[f f'' - (f')^2 - S \left(2f' + \frac{\eta}{2} f'' \right) - \lambda_1 \left\{ (f')^2 f''' - 2f f' f'' \right\} \right] = 0 \end{aligned} \quad (10)$$

$$(1 + Rd)\theta'' - Pr \left\{ \frac{S}{2} (3\theta' + \eta\theta') + (2f' - q)\theta - \theta' f \right\} = 0$$

The boundary constraints are:

$$\begin{aligned} f(0) &= 0, f'(0) = 1, \theta(0) = 1, \\ f(\beta) &= \frac{S\beta}{2}, f''(\beta) = 0, \theta'(\beta) = 0 \end{aligned}$$

Skin friction is defined as $C_f = \frac{(\tilde{S}_{xy})_{y=0}}{\rho u_w^2}$,

$$C_f = \frac{1 + \lambda_1}{1 + \lambda_2} f''(0)$$

The Nusselt number is:

$$\begin{aligned} u &= \frac{Q_w}{\hat{k}(T - T_h)}, \\ u &= \frac{1}{2} (1 - at)^{-\frac{1}{2}} - \theta'(0) Re^{-\frac{1}{2}} f_x, \end{aligned} \quad (11)$$

After generalization, the physical constraints are as follows: $\lambda_1, \lambda_2, \lambda_3$ are the parameters of relaxation and retardation ratios, $S = \frac{a}{b}$ is the measure of unsteadiness of the nondimensional, $M = \frac{\sigma_f B_0^2}{b\rho_f}$ denotes the magnetic field parameter, and $Pr = \frac{\rho v c_p}{k}$ denotes the Prandtl number.

3. Solution by HAM

Equations (8)–(10), considering the boundary condition in Equation (11), are solved with HAM. The solutions encircled the auxiliary parameter h , which normalizes and switches to a conjunction of the solutions.

The following is the initial guess:

$$f_0(\eta) = \frac{(2-S)\eta^3 + (3S-6)\beta\eta^2 + 4\beta^2\eta}{4\beta^2}, \quad \theta(\eta) = 1 \quad (12)$$

L_f, L_θ represent linear operators:

$$L_f(f) = f^{iv}, \quad L_\theta(\theta) = \theta' \quad (13)$$

where:

$$L_f\left(\frac{e_1}{6}\eta^3 + \frac{e_2}{2}\eta^2 + e_3\eta + e_4\right) = 0, \quad L_\theta(e_5 + e_6\eta) = 0 \quad (14)$$

Non-linear operators N_f, N_θ are given as:

$$N_f[f(\eta;r)] = \frac{1}{(1+\lambda_2)} \left[\frac{\partial^3 f(\eta;r)}{\partial \eta^3} + \lambda_3 \left(\frac{\partial^2 f(\eta;r)}{\partial \eta^2} \right)^2 - f(\eta;r) \frac{\partial^4 f(\eta;r)}{\partial \eta^4} + S \left(2 \frac{\partial^3 f(\eta;r)}{\partial \eta^3} + \frac{\eta}{2} \frac{\partial^4 f(\eta;r)}{\partial \eta^4} \right) - M \frac{\partial f(\eta;r)}{\partial \eta} \right] + (1 + \varepsilon\kappa) \left[f(\eta;r) \frac{\partial^2 f(\eta;r)}{\partial \eta^2} - \left(\frac{\partial f(\eta;r)}{\partial \eta} \right)^2 - S \left(2 \frac{\partial f(\eta;r)}{\partial \eta} + \frac{\eta}{2} \frac{\partial^2 f(\eta;r)}{\partial \eta^2} \right) - \lambda_1 \left((f(\eta;r))^2 \frac{\partial^3 f(\eta;r)}{\partial \eta^3} - 2f(\eta;r) \frac{\partial f(\eta;r)}{\partial \eta} \frac{\partial^2 f(\eta;r)}{\partial \eta^2} \right) \right] \quad (15)$$

$$N_\theta[f(\eta;r), \theta(\eta;r)] = (1 + Rd) \frac{\partial^2 \theta(\eta;r)}{\partial \eta^2} - Pr \left\{ \frac{S}{2} \left(3\theta(\eta;r) + \eta \frac{\partial \theta(\eta;r)}{\partial \eta} \right) + \left(2 \frac{\partial f(\eta;r)}{\partial \eta} - q \right) \theta(\eta;r) - f(\eta;r) \frac{\partial \theta(\eta;r)}{\partial \eta} \right\} \quad (16)$$

The 0th-order problem is defined as:

$$L_f[f(\eta;r) - f_0(\eta)](1-r) = r_f N_f[f(\eta;r)] \quad (17)$$

$$L_\theta[\theta(\eta;r) - \theta_0(\eta)](1-r) = r_\theta N_\theta[f(\eta;r), \theta(\eta;r)] \quad (18)$$

The correspondent boundary constraints are:

$$f(\eta;r)|_{\eta=0} = 0, \quad \frac{\partial f(\eta;r)}{\partial \zeta}|_{\eta=0} = 1, \quad \frac{\partial^2 f(\zeta;r)}{\partial \eta^2}|_{\eta=\beta} = 0, \quad f(\eta;r)|_{\eta=\beta} = 0 \\ \vartheta(\eta;r)|_{\eta=0} = 1, \quad \frac{\partial \theta(\eta;r)}{\partial \eta}|_{\eta=\beta} = 0, \quad (19)$$

In the case of $r = 0, r = 1$:

$$f(\eta;1) = f(\eta), \quad \theta(\eta;1) = \theta(\eta) \quad (20)$$

After applying Taylor's series:

$$\begin{aligned} f(\eta; r) &= f_0(\eta) + \sum_{n=1}^{\infty} f_n(\eta) r^n, \\ \theta(\eta; r) &= \theta_0(\eta) + \sum_{n=1}^{\infty} \theta_n(\eta) r^n, \end{aligned} \quad (21)$$

where:

$$f_n(\eta) = \frac{1}{n!} \left. \frac{\partial f(\eta; r)}{\partial \eta} \right|_{r=0}, \quad \theta_n(\eta) = \frac{1}{n!} \left. \frac{\partial \theta(\eta; r)}{\partial \eta} \right|_{r=0} \quad (22)$$

The subordinate limitations $\hbar_f, \hbar_\theta$ are selected such that the series in Equation (23) converges at $r = 1$. Switching $r = 1$ in Equation (23), we obtain:

$$\begin{aligned} f(\eta) &= f_0(\eta) + \sum_{n=1}^{\infty} f_n(\eta), \\ \theta(\eta) &= \theta_0(\eta) + \sum_{n=1}^{\infty} \theta_n(\eta), \end{aligned} \quad (23)$$

The n th-order problem:

$$\begin{aligned} L_f [f_n(\eta) - \chi_n f_{n-1}(\eta)] &= \hbar_f R_n^f(\eta), \\ L_\theta [\theta_n(\eta) - \chi_n \theta_{n-1}(\eta)] &= \hbar_\theta R_n^\theta(\eta), \end{aligned} \quad (24)$$

The consistent boundary conditions are:

$$\begin{aligned} f_n(0) &= f'_n(0) = \theta_n(0) = 0 \\ f_n(\beta) &= f''_n(\beta) = \theta'_n(\beta) = 0 \end{aligned} \quad (25)$$

and:

$$\chi_n = \begin{cases} 0, & \text{if } r \leq 1 \\ 1, & \text{if } r > 1 \end{cases} \quad (26)$$

HAM Solution Convergence

In this section, the convergence of the problem of the modeled and solved equations is discussed. Figure 2 presents the h -curves of the combined temperature and velocity functions. A valid region of convergent is observed in Figure 2. In Table 1, numerical convergence is presented. A fast convergence of HAM is shown in Table 1.

Figure 2. The combined h -curve graph of the velocity and temperature profiles.

Table 1. Fast convergence of HAM (homotopy analysis method).

Solution Approximations	$f''(0)$	$\theta'(0)$
1	0.448400	0.122519
4	0.858574	−0.282846
7	0.912994	−0.356251
10	0.920043	−0.369590
13	0.920946	−0.372026
16	0.921062	−0.372472
19	0.921077	−0.372554
22	0.921080	−0.372569
25	0.921080	−0.372572
28	0.921080	−0.372572

4. Results and Discussion

This work investigated thin particle flow (Maxwell, Jeffry, and Oldroyd-B) fluids considering the influence of MHD and radiation with a time-dependent porous extending surface. This subsection presents the physical impact of dissimilar implanting parameters over velocity distributions $f(\eta)$ and temperature distribution $\Theta(\eta)$ as exemplified in Figures 2–12.

Figure 3 shows the effect of κ on velocity field $f(\eta)$. There is a direct relation between the cohesive and adhesive forces and viscosity. The overall fluid motion is reduced due to the rising in κ , which generates strong cohesive and adhesive forces that cause resistance.

Figure 3. Variations in velocity field $f(\eta)$ for different values of κ .

Figure 4 shows the effect of β during the motion of the flow. Increasing β decreases the flow velocity, as it is a reducing function of the velocity and thickness of the liquid film. Substantially, the viscous forces rising with higher numbers of β cause the fluid motion. Thus, fluid velocity drops.

Figure 4. Fluctuations in velocity field $f(\eta)$ for various values of β .

Figure 5 shows the impact of thin film thickness on the temperature profile. It is observed that an increase in β decreases temperature profile $\theta(\eta)$.

Figure 5. Variations in the temperature gradient $\theta(\eta)$ for the various measures of β .

Figure 6 describes the features of M on (η) . The velocity profile reduces due to the rise in M . The object of this phenomenon is Lorentz force. This force is responsible for the reduction in the motion of fluid.

Figure 6. Dissimilarities in $f(\eta)$ for different numbers of M .

Figure 7 shows that with different values of M , the concentration field increases because an increase in oblique magnetic pitch on the fluid augments the Lorentz force. Lorentz force is the combination of electric and magnetic force on a point charge due to electromagnetic fields. This force is lower at the center of the channel and at the boundaries, but up near the walls. Velocity increases with the unsteady constraint S .

Figure 7. Changes in temperature gradient $\theta(\eta)$ for diverse measures of M .

Figure 8 demonstrates the performance of the unsteadiness constraint S on $f(\eta)$. It is concluded that $f(\eta)$ is directly proportional to S . The fluid motion increases due to an increase in S . Thus, the solution is dependent on S , S .

Figure 8. Fluctuations in $f(\eta)$ for several numbers of S .

Figure 9 shows the effect of S on the heat profile $\theta(\eta)$. It was concluded that $\theta(\eta)$ is directly proportional to S . An increase in the unsteadiness parameter S increases the temperature which, in turn, augments the kinetic energy of the fluid flow; thus, the thin film moment increases.

Figure 9. Changes in temperature gradient $\theta(\eta)$ for diverse measures of S .

Figure 10 shows the effect of q . Increasing q increases the temperature field. Actually, raising q increases the kinetic energy of thin film, which results in increasing the internal heat.

Figure 10. Changes in temperature gradient $\theta(\eta)$ for diverse measures of q .

The influence of radioactivity parameter Rd on $\theta(\eta)$ is shown in Figure 11. Radioactivity is an important part on the inclusive surface heat transmission when the coefficient of convection heat transmission is small. When thermal radiation is increased, the thermal radiation augments the temperature in the boundary layer area in the fluid layer. This increase leads to a drop in the rate of cooling for liquid film flow.

Figure 11. Changes in temperature gradient $\theta(\eta)$ for diverse measures of Rd .

The influence of Pr on the temperature distributions $\theta(\eta)$ is shown in Figure 12. The temperature distribution varies inversely with Pr . Clearly, the temperature distribution decreases for large values of Pr and increases for small values of Pr . Physically, the fluids having a small Pr have larger thermal diffusivity and vice versa. Thus, large Pr causes the thermal boundary layer to decrease.

Figure 12. Changes in temperature gradient $\theta(\eta)$ for diverse measures of Pr .

Figures 13 and 14 illustrate the comparison of HAM and numerical solution for velocities and temperature functions. An excellent agreement is found here.

Figure 13. Comparison graph of HAM and numerical solution for $f(\eta)$.

Figure 14. Comparison graph of HAM and numerical solution for $\theta(\eta)$.

Tables 2 and 3 present a comparison of HAM and numerical solution for velocities and temperature functions. We have observed an excellent covenant between the analytical and numerical approach.

The numerical values of surface temperature $\theta(\beta)$ for different values of M , Rd and k are given in Table 2. Increasing values of M , Rd , and k increase surface temperature $\theta(\beta)$, whereas the opposite effect was found for Pr , that is, large values of Pr reduce surface temperature $\theta(\beta)$.

Table 2. Comparison of HAM and numerical solution for $f(\eta)$.

η	HAM Solution $f(\eta)$	Numerical Solution $f(\eta)$	Absolute Error
0.0	1.00000	1.00000	0.00000
0.1	0.907327	0.911301	3.9×10^{-3}
0.2	0.829165	0.835332	6.2×10^{-3}
0.3	0.764141	0.770975	6.8×10^{-3}
0.4	0.710984	0.717248	6.3×10^{-3}
0.5	0.668524	0.673301	4.8×10^{-3}
0.6	0.635691	0.638412	2.7×10^{-3}
0.7	0.611519	0.611979	4.5×10^{-4}
0.8	0.595141	0.593511	1.8×10^{-3}
0.9	0.585791	0.582627	7.7×10^{-3}
1.0	0.582805	0.579045	3.8×10^{-3}

Table 3. Comparison of HAM and numerical solution for $\theta(\eta)$.

η	HAM Solution $\theta(\eta)$	Numerical Solution $\theta(\eta)$	Absolute Error
0.0	1.0000	1.00000	0.00000
0.1	1.05078	1.04882	1.9×10^{-3}
0.2	1.09412	1.09106	3.1×10^{-3}
0.3	1.13081	1.12742	3.3×10^{-3}
0.4	1.16156	1.15845	3.1×10^{-3}
0.5	1.18696	1.18458	2.3×10^{-3}
0.6	1.20745	1.20608	1.3×10^{-3}
0.7	1.22335	1.22308	2.7×10^{-4}
0.8	1.23481	1.23555	7.3×10^{-3}
0.9	1.24184	1.24331	1.4×10^{-3}
1.0	1.24425	1.2460	1.7×10^{-3}

The numerical value of the gradient of wall temperature $\theta'(0)$ for different values of embedded parameters Rd, β, Pr, S is shown in Tables 4 and 5. Larger values of thermal radiation Rd, β , and Pr decrease the wall temperature, whereas S increases the wall temperature gradient $\theta'(0)$.

Table 4. Numerical values for the skin friction coefficient for several physical parameters when: $\lambda_1 = 0.1, \lambda_2 = 1.0, \lambda_3 = 0.1, S = 0.5, M = 0.6, \kappa = 0.7, Pr = 1.0, \beta = 1$.

S	κ	M	β	$\frac{1+\lambda_1}{1+\lambda_2} f''(0)$ Jeffrey Fluid	$\frac{1+\lambda_1}{1+\lambda_2} f''(0)$ Maxwell Fluid	$\frac{1+\lambda_1}{1+\lambda_2} f''(0)$ Oldroyd-B Fluid
0.5	0.5	0.6	1.0	−0.705657	−1.40417	−1.29373
0.6				−0.690279	−1.34383	−1.25575
0.7				−0.676699	−2.19921	−1.23210
0.5	0.5			−0.705657	−1.40417	−1.29373
	0.6			−0.719191	−1.43646	−1.32213
	0.7			−0.732514	−1.46831	−1.35012
	0.5	0.6		−0.705657	−1.40417	−1.29373
		0.7		−0.725666	−1.44510	−1.33013
		0.8		−0.745259	−1.48521	−1.36548
		0.6	1.0	−0.705657	−1.40417	−1.29373
			1.1	−0.725279	−1.469866	−1.33341
			1.2	−0.740967	−1.54394	−0.268147
			1.0	−0.705657	−1.40417	−1.29373

Table 5. Numerical values of the local Nusselt number for several physical parameters, when: $\lambda_1 = 0.1, \lambda_2 = 1.0, \lambda_3 = 0.1, S = 0.5, M = 0.6, \kappa = 0.7, Pr = 1.0, \beta = 1, Rd = 0.5, q = 0.5$.

S	κ	M	β	Rd	q	$\theta'(0)$ Jeffrey Fluid	$\theta'(0)$ Maxwell Fluid	$\theta'(0)$ Oldroyd-B Fluid
0.5	0.5	0.6	1.0	0.5	0.5	−0.550388	−0.590996	−0.616292
0.6						−0.545322	−0.590885	−0.611008
0.7						−0.539440	−0.683523	−0.605811
0.5	0.5					−0.550388	−0.590996	−0.616292
	0.6					−0.543228	−0.582409	−0.608402
	0.7					−0.536265	−0.574031	−0.600680
	0.5	0.6				−0.550388	−0.590996	−0.616292
		0.7				−0.538941	−0.580121	−0.605984
		0.8				−0.527911	−0.569593	−0.595950
		0.6	1.0			−0.550388	−0.590996	−0.616292
			1.1			−0.551698	−0.589585	−0.619001
			1.2			−0.547039	−0.580929	−0.376664
			1.0	0.5		−0.550388	−0.590996	−0.616292
				0.6		−0.517715	−0.555700	−0.579345
				0.7		−0.488677	−0.524350	−0.546542
				0.5	0.5	−0.550388	−0.590996	−0.616292

5. Conclusions

The MHD flow of three combined fluids, Maxwell, Jeffrey, and Oldroyd-B fluid, with variable heat transmission under the influence of thermal radiation was investigated. The fundamental leading equations were changed to a set of differential nonlinear and coupled equations. An optimal and numerical tactic was used to acquire the solution of the modeled problems.

The key conclusions of this work are as follows:

- The velocity profile shows an increase with increasing values of the unsteadiness parameter St , while increasing values of the magnetic parameter cause the decline of the velocity profile of the nanofluid film.
- Rd augmented heat flux
- Higher values of Pr increase the surface temperature.
- Fast convergence is shown numerically and graphically.
- The liquid film flow effects on the flow of three fluids (Jeffrey, Maxwell, and Oldroyd-B fluids) are presented using graphs and tables.
- It was found that the coefficient of skin friction rises with the larger rates of the magnetic parameter M and the unsteadiness parameter St , whereas the coefficient of skin friction decreases with the higher values of the stretching and thickness parameters.
- It was found that liquid film flow is affected by the Lorentz's force.
- The obtained analytical and numerical consequences are presented graphically and tabulated. An excellent agreement was obtained between HAM and the numerical method.

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Nomenclature

J Current density	α_1, α_2 Material constants
μ Coefficient of viscosity	μ_0 Magnetic permeability
T Cauchy stress tensor	M Magnetic parameter
σ Electric conductivity	α non-Newtonian parameter
E Electric field	Ω Pressure gradient parameter
ω Frequency parameter	A_1 and A_2 Rivlin–Ericksen tensor
p pressure	t Time variable
I Identity tensor chord	Rd Radiation parameter
St Unsteadiness parameter	
Pr Prandtl number	Nu Nusselt number
C_f Skin friction	

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