

Article

Open Python-Based Simulation and MOPSO Multiobjective Optimization of a Rod Mill–Hydrocyclone–Ball Mill Circuit

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Abstract

Comminution–classification circuits are difficult to optimize because hydraulic, granulometric, energy, and economic responses are nonlinearly coupled, while circuit simulation, equipment sizing, simulator benchmarking, and operating optimization are often treated separately. This study aimed to develop an open Python framework for steady-state simulation and five-objective optimization of a rod mill–hydrocyclone–ball mill circuit processing a gold ore. The framework integrates solid and water balances, Rosin–Rammler particle-size reconstruction, comminution and hydrocyclone models, preliminary equipment sizing, explicit feasibility constraints, and Multiobjective Particle Swarm Optimization (MOPSO). Its novelty lies in coupling complete-circuit simulation, simulator-to-simulator benchmarking against USIM PAC[®], model-based sizing, convergence diagnostics, and Pareto optimization within one transparent workflow. The benchmark produced zero or below $10^{-3}\%$ errors in solid balances and sizing differences of 1.07%, 8.21%, and 0.00% for the rod mill, ball mill, and hydrocyclone, respectively. Relative to the base case, the joint minimum-water, minimum-energy, and minimum-cost solution reduced specific water consumption by 24.50%, specific grinding energy by 4.24%, specific operating cost by 10.19%, and mass recirculation by 8.80%, while useful recovery decreased slightly from 82.67% to 81.78%. The maximum-recovery solution increased useful recovery to 84.45%, with higher water, energy, and operating-cost requirements. The framework supports reproducible evaluation of resource–recovery trade-offs in grinding–classification circuits.

Keywords: comminution; grinding–classification circuit; hydrocyclone; process simulation; multiobjective optimization; MOPSO; Pareto front



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1. Introduction

The mineral processing stage serves as the operational foundation of extractive metallurgy. It establishes the physical conditions needed for concentration (separating valuable minerals from waste) and recovery (extracting these minerals for use) processes. Within these operations, comminution (breaking ore into smaller pieces) and classification (sorting particles by size) are central. They control particle size, mineralogical liberation, and the quality of material entering later stages [1,2]. Because comminution is highly energy-intensive, effective design is critical for metallurgical and energy efficiency.

For gold ores, the mineral's response to grinding forces is determined by its internal structure, micro-defects, and the energy needed to generate new surfaces [3]. This natural

heterogeneity affects both the liberation size and the particle size distribution from comminution. As a result, precise control of operating variables is necessary. This prevents insufficient liberation, overgrinding, excessive fines, and energy inefficiency.

A typical grinding and classification circuit uses a sequence of equipment: a rod mill (which reduces coarse feed to finer particles), a hydrocyclone (which separates particles by size), and a ball mill (which further grinds the material). This setup creates a highly coupled system. Cut size (the particle size at which the separating device divides the feed), circulating load (the amount of material repeatedly cycled through the mills), energy consumption, solids percentage (the concentration of solids in the slurry), and breakage kinetics (the rate and pattern at which particles break down) all interact nonlinearly [4]. Due to these interdependencies, circuit design and analysis require tools to predict steady-state behavior and assess sensitivity to operational variations [5].

The mathematical models used to describe these systems rely on selection and breakage functions [6], energy equations to estimate the specific grinding energy consumption [7], and partition curves to represent hydrocyclone performance [8]. Together, these approaches provide the basis for steady-state simulation and equipment sizing in comminution circuits [4,9]. Commercial simulators, however, often impose closed structures. These structures limit the review and modification of internal equations, restrict automation of large-scale simulations, and complicate the integration of modern optimization techniques.

Open models offer a flexible alternative to overcome these limitations by enabling the direct implementation of circuit equations, mass-balance calculations, particle-size distribution reconstruction, automated simulation, and straightforward coupling with multiobjective optimization algorithms.

The optimization of comminution–classification circuits requires the simultaneous treatment of conflicting objectives, operational constraints, and nonlinear process responses. Consequently, researchers have used data-driven models, hybrid models, and evolutionary strategies, coupled with phenomenological simulators, to reduce the computational cost of the search process and explore feasible operating regions [10–12].

Recent studies have demonstrated the potential of advanced modeling and multiobjective optimization in grinding and classification processes. Wang et al. [13] applied PBM, response surface modeling, and NSGA-II to an industrial grinding circuit, increasing mill throughput by 3.83 t/h and the fraction of particles finer than 0.074 mm in the hydrocyclone overflow by 2.53%. Yang et al. [14] developed a hybrid mechanism–data-driven model whose industrial validation produced deviations of 4.04% for overflow particle size and 1.62% for overflow flow rate, together with a mean-square error of 3.40% for overflow solids concentration. At the unit-operation level, Tan et al. [15] simultaneously optimized four hydrocyclone performance objectives and compared six multiobjective metaheuristic algorithms. More recently, E et al. [16] evaluated 25 feed particle-size distribution scenarios and reported reductions of 17–27% in cut size, improvements of 14–25% in separation sharpness, and increases of 38–95% in throughput relative to a static operating condition.

Among these methods, Multiobjective Particle Swarm Optimization (MOPSO) provides a suitable alternative for problems whose evaluations depend on numerical simulators. Unlike gradient-based methods, MOPSO explores the search space through a population of particles that evolve toward a set of non-dominated solutions, which enables the approximation of the Pareto front without requiring derivative information from the model [17,18]. In mineral processing, researchers have applied this type of strategy to optimize grinding and classification processes while balancing productivity, energy consumption, and particle-size control [13,19].

Despite these advances, recent studies have primarily emphasized hybrid and data-driven modeling of grinding–classification processes [14], optimization of industrial grind-

ing circuits using a limited number of objectives [13], or multiobjective optimization and adaptive control of individual unit operations such as hydrocyclones [15,16]. Based on the reviewed literature, previous studies have not jointly addressed complete-circuit steady-state simulation, preliminary model-based equipment sizing, comparison against a commercial simulator, explicit granulometric and geometric feasibility constraints, and five-objective optimization within a unified computational framework.

This study develops and compares an open Python-based model for steady-state simulation, model-based equipment sizing, and multiobjective optimization of a rod mill–hydrocyclone–ball mill circuit applied to a gold ore. The study benchmarks the model against USIM PAC[®]. It uses mass balances, particle-size distributions, characteristic (d_{80}) values, mass recirculation ratio, hydrocyclone partition curves, and equipment dimensions for this purpose. After comparison, the simulator is coupled with MOPSO. This combination explores the decision space globally and identifies operating configurations that can reduce specific water consumption, specific grinding energy, and the specific cost associated with energy and water. The resulting solutions allow the associated trade-offs with useful recovery and hydraulic stability to be evaluated. Accordingly, the novelty of this work lies in integrating complete-circuit steady-state simulation, preliminary model-based equipment sizing, explicit granulometric and geometric feasibility constraints, comparison with USIM PAC[®], five-objective MOPSO, and convergence diagnostics within a unified and transparent Python-based computational framework.

Therefore, the research problem addressed in this study is the fragmented treatment of complete-circuit simulation, equipment sizing, model comparison, operational feasibility, and multiobjective decision-making in grinding–classification circuits.

The proposed approach differs from previous studies by integrating complete-circuit steady-state simulation, preliminary model-based equipment sizing, explicit granulometric and geometric feasibility constraints, comparison with USIM PAC[®], five-objective MOPSO, and convergence diagnostics within a unified computational workflow.

2. Methodology

The methodology comprised three stages: (i) reference circuit definition in USIM PAC[®] version 3.3.0.10, (ii) development of an equivalent Python model including steady-state comminution simulation and equipment sizing, followed by comparison with the reference simulation, and (iii) coupling with a MOPSO-based multiobjective optimization scheme.

As shown in Figure 1, the procedure begins by simulating the reference grinding–classification circuit in USIM PAC[®]. This stage defines main process inputs: feed particle-size distribution, Work Index, Abrasion Index, solid concentrations, and target d_{80} values for grinding and classification. After solving the steady-state balance, size reduction, and material classification, a scanning algorithm determines the sizes of the main equipment. An equivalent open simulator in Python version 3.12.11 then reproduces the main responses of the reference circuit. The comparison examines mass balances, particle-size distributions, partition curves, and equipment dimensions. Finally, the Python simulator is coupled with MOPSO to identify operating solutions under multiple, potentially conflicting objectives.

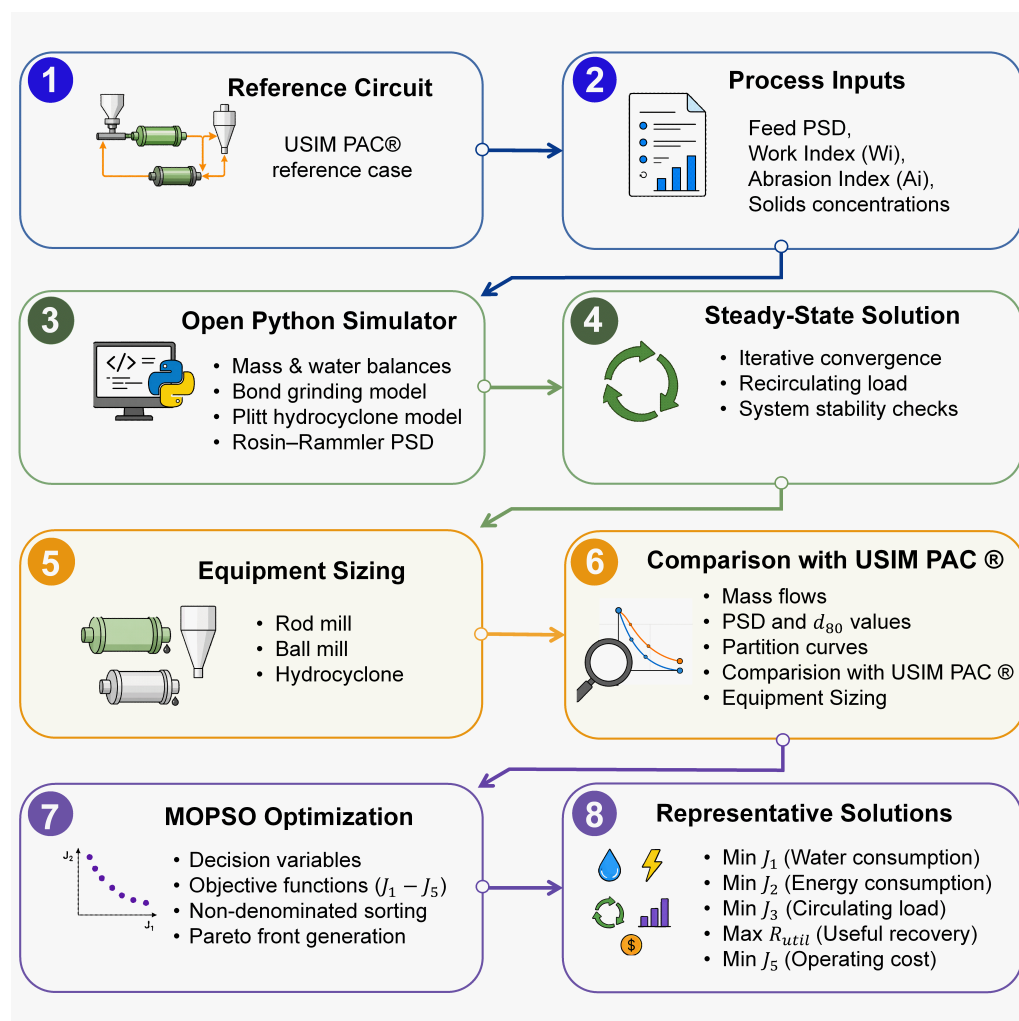


Figure 1. General methodological workflow for the development, comparison, model-based equipment sizing, and multiobjective optimization of the model implemented in Python.

2.1. Description of the Grinding–Classification Circuit and Feed

The case study examines a closed rod mill–hydrocyclone–ball mill circuit (see Figure 2). Fresh ore and make-up water mix in the feeder, then flow into the rod mill. The rod mill discharge enters the hydrocyclone, where water is added to regulate solids. The hydrocyclone splits the slurry into an overflow (the fine product) and an underflow. The underflow receives more water to adjust solids before entering the ball mill. The ball mill discharge returns to the hydrocyclone feed, closing the circuit and creating a recirculating stream that influences hydraulic stability, energy use, and particle size. Numbered streams indicate simulator calculation points for balance and variable determination.

The reference configuration established in USIM PAC® specifies a fresh feed rate of 550 t/d, specific gravity of 3.0, Work Index of 15 kWh/st, and Abrasion Index of 0.147. Set operating conditions are 70% solids in the rod mill feed, 40% in the hydrocyclone feed, and 55% in the ball mill feed.

Figure 3 shows the fresh feed particle-size distribution used in the model. The Rosin–Rammler distribution, also referred to as the Rosin–Rammler–Bennett distribution in the particle-size literature, was used to represent the cumulative particle-size distribution [20] and is given by Equation (1).

$$P(x) = 100 \left[1 - \exp \left(- \left(\frac{x}{x_c} \right)^n \right) \right]. \quad (1)$$

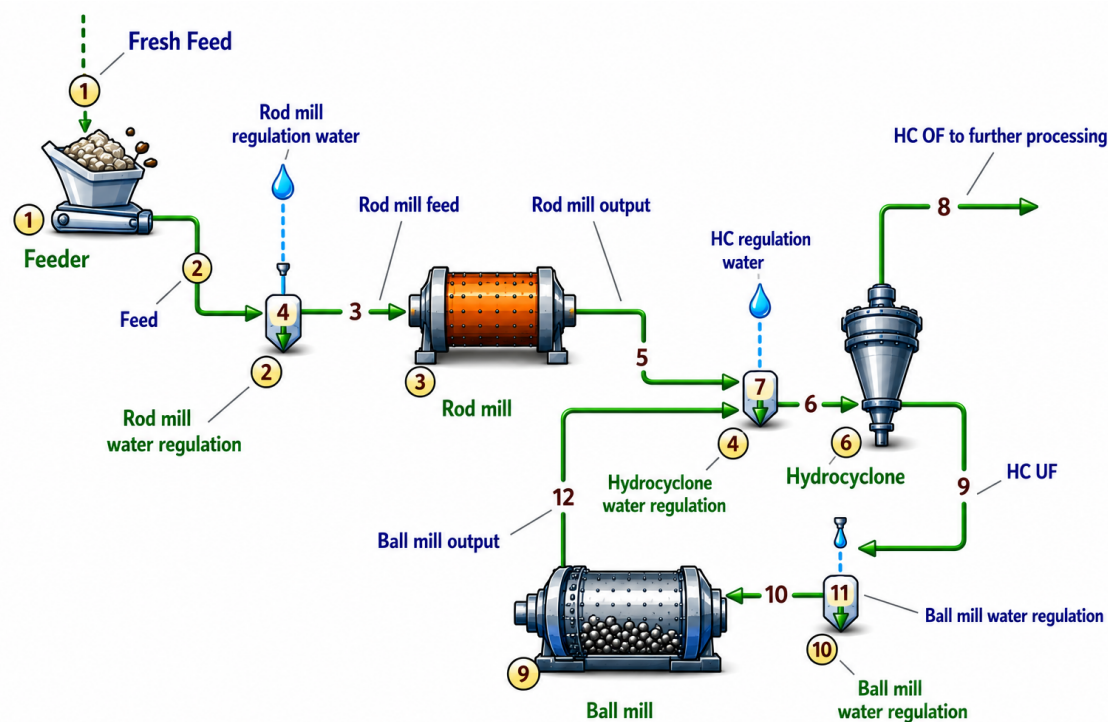


Figure 2. Simplified flowsheet of the closed rod mill–hydrocyclone–ball mill grinding–classification circuit used as the reference case.

The term Rosin–Rammmler is retained throughout this work for consistency with the reference USIM PAC[®] implementation. Here, $P(x)$ denotes the cumulative passing percentage, x represents the particle size, x_c corresponds to the characteristic size ($d_{63.2}$), and n defines the uniformity index. The fitted parameters and statistical indicators obtained from the 13 sieve data points are summarized in Table 1.

Table 1. Fitted parameters and statistical indicators of the Rosin–Rammmler particle-size distribution.

Parameter or Statistical Indicator	Value
Number of sieve data points, N	13
Uniformity index, n	1.262062
Characteristic size, $x_c = d_{63.2}$ (mm)	5.867
Particle size, d_{80} (mm)	8.554
Root mean square error, RMSE (%)	3.56
Coefficient of determination, R^2	0.9861

The statistical indicators summarized in Table 1 indicate that the fitted distribution adequately captures the cumulative trend of the feed and supports its use as the initial particle-size distribution of the model. The remaining error mainly reflects the fitting of 13 discrete sieve measurements with an idealized two-parameter continuous distribution. Small local deviations are therefore expected, particularly in the coarsest and finest size classes; nevertheless, the overall agreement is sufficient for initializing the steady-state model.

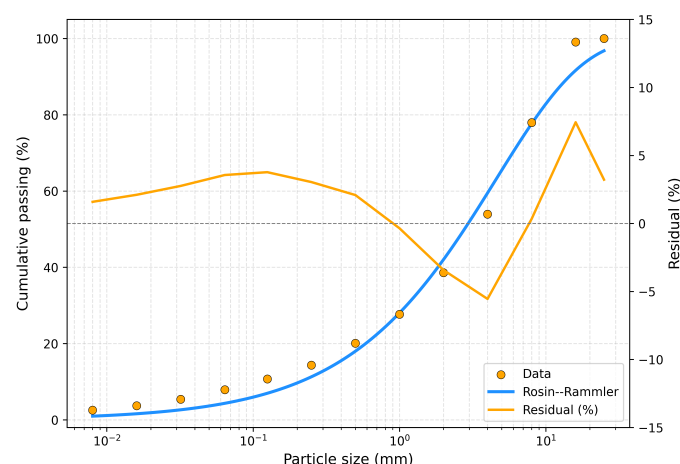


Figure 3. Cumulative particle-size distribution of the fresh feed and Rosin–Rammler model fit.

2.2. Steady-State Circuit Model

We implemented the steady-state model in Python to solve the hydraulic and particle-size behavior of the closed circuit. The computational structure followed documented formulations for the USIM PAC[®] performance models, which served as a reference to represent mixing, density regulation, grinding, and classification [21,22].

The circuit calculation relied on solid and water balances. Mixing nodes combined the inlet flow rates; regulators calculated the water addition required to reach the specified solids contents; the mills generated Rosin–Rammler distributions defined by the discharge (d_{80}); and the hydrocyclone separated the feed using the corrected Plitt partition curve [8], expressed by Equation (2):

$$Y(X) = H + (100 - H) \left[1 - \exp \left(-0.693 \left(\frac{X}{d_{50c}} \right)^M \right) \right], \quad (2)$$

where ($Y(X)$) denotes the fraction of particles of size (X) that reports to the underflow, (H) represents the bypass fraction, (d_{50c}) is the corrected cut size, and (M) controls the classification sharpness.

The algorithm followed a deterministic procedure. For each operating condition, the simulator calculated the characteristic sizes by interpolation and updated the ball mill–hydrocyclone recirculating stream until the closed loop reached convergence. At the end of the calculation, the simulator reported flow rates, solids content, pulp density, mass recirculation ratio, particle-size distributions, and characteristic particle size (d_{80}) values for the main streams.

2.3. Model-Based Sizing Using a Scanning Search

The equipment sizing routine used a geometric-variable search based on criteria from the USIM PAC[®] level-1 models. For each item, the routine evaluated options and selected the one that minimized error relative to the design criterion.

For the rod and ball mills, the corrected Bond equation was used to estimate the specific grinding energy according to Equation (3), including the Rowland–Allis Chalmers efficiency correction factors [7,23–25].

$$E_{\text{corr}} = 10W_i \left(\frac{1}{\sqrt{P_{80}}} - \frac{1}{\sqrt{F_{80}}} \right) \prod_{i \in \mathcal{I}_m} EF_i, \quad (3)$$

where $\mathcal{I}_m$ denotes the set of correction factors applied to the specific mill type.

Here, (E_{corr}) is the corrected specific energy, (W_i) the Work Index, (F_{80}) and (P_{80}) the feed and product sizes, and (EF_i) the correction factors versus standard Bond conditions. The ball mill calculation excluded (EF_6) and (EF_8), while the rod mill calculation excluded (EF_2), (EF_5), (EF_7), and (EF_8) [23–25].

For each mill, the scanning routine checked possible diameters and, for each candidate, calculated the matching length, the required power, and the available power using the Rowland approach. It selected the geometry in which the required and available powers were closest in value. The routine excluded edge-case solutions and ensured compatibility with the specified particle-size reduction [26].

The hydrocyclone sizing routine used Plitt-type formulas that consider equipment shape and feed properties. During scanning, the routine changed the internal diameter but kept the same proportions for the inlet, vortex finder, apex, and height. For each diameter, the model estimated the corrected cut size, sharpness, pressure drop, and flow split. The selected design minimized the difference between the calculated overflow (d_{80}) and the target d_{80} in the fine product. This process matches recent research on improving hydrocyclone performance [27].

The multiobjective optimization stage later used the dimensions, estimated powers, and relative errors obtained during the scanning search as technical feasibility criteria.

3. Multiobjective Optimization Using MOPSO

This study coupled the model with a Multiobjective Particle Swarm Optimization (MOPSO) algorithm. This approach explored the operating conditions of a rod mill–hydrocyclone–ball mill circuit under simultaneous criteria: specific water consumption, specific grinding energy, mass recirculation ratio, useful recovery, and the specific cost associated with energy and water. The circuit’s response is nonlinear, constrained, and depends on an iterative simulation. Thus, MOPSO provides a suitable strategy for approximating the Pareto front without gradient information. In this implementation, each particle represents a candidate operating condition. Its evolution depends on the particle’s historical best position and a leader chosen from an external archive of non-dominated solutions.

Figure 4 summarizes the complete computational sequence of the MOPSO implementation coupled with the Python steady-state simulator.

The procedure includes swarm initialization, particle evaluation, steady-state simulation, objective calculation, feasibility verification, personal-best and external-archive updates, leader selection, particle movement, Gaussian mutation, and the termination criterion.

3.1. Decision Variables and Objective Functions

Each candidate solution is represented by the decision vector defined in Equation (4):

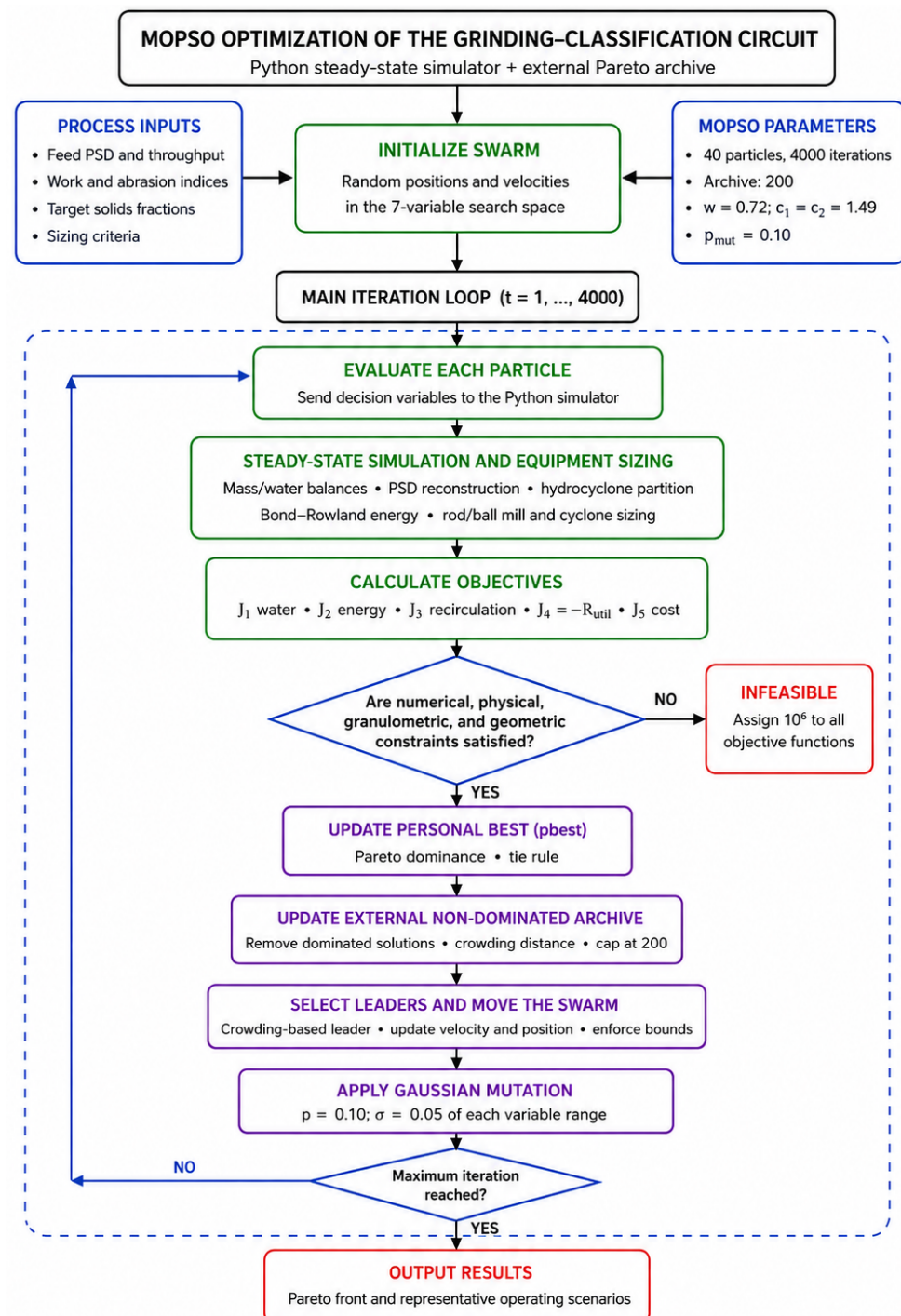
$$\mathbf{x} = [reg_{rod}, reg_{hc}, reg_{ball}, \alpha, d_{80_{rod}}, d_{80_{ball}}, d_{80_{of}}], \quad (4)$$

Here, (reg_{rod}), (reg_{hc}), and (reg_{ball}) represent the solids contents in the rod mill feed, hydrocyclone feed, and ball mill feed, respectively. The variable (α) corresponds to the scaling factor for the fresh solid feed. The parameters ($d_{80_{rod}}$), ($d_{80_{ball}}$), and ($d_{80_{of}}$) denote the target characteristic discharge sizes of the rod mill, ball mill, and hydrocyclone overflow, respectively.

The study defined the search intervals based on the base case and physically admissible operating limits. Table 2 summarizes the domains considered during the optimization.

Table 2. Decision variables and search domain used during the multiobjective optimization.

Variable	Description	Lower Bound	Upper Bound
reg_{rod}	Solids content in the rod mill feed (%)	10	80
reg_{hc}	Solids content in the hydrocyclone feed (%)	10	80
reg_{ball}	Solids content in the ball mill feed (%)	10	80
α	Factor applied to the fresh feed	0.7	1.3
$d_{80,rod}$	Rod mill discharge d_{80} (mm)	0.54	0.66
$d_{80,ball}$	Ball mill discharge d_{80} (mm)	0.09	0.11
$d_{80,of}$	Hydrocyclone overflow d_{80} (mm)	0.058	0.062

**Figure 4.** Detailed workflow of the MOPSO implementation coupled with the Python steady-state simulator and the external Pareto archive.

The five objective functions were derived from the main steady-state outputs and operational priorities of the circuit. J_1 quantifies the total water added per unit of fresh solids, J_2 represents the combined rod and ball mill power per unit of fresh solids, and J_3 measures the internal solids recirculation between the hydrocyclone and the ball mill. The objective $J_4 = -R_{\text{util}}$ expresses useful fine-product recovery as a minimization problem, whereas J_5 combines the costs of electrical energy and process water. No weighting factors were imposed because MOPSO identifies the trade-offs among these objectives through Pareto dominance.

The multiobjective problem used the following formulation, based on the MOPSO framework proposed by Coello et al. [17] and presented in Equation (5):

$$\min_{\mathbf{x} \in \Omega} \mathbf{J}(\mathbf{x}) = [J_1, J_2, J_3, J_4, J_5], \quad (5)$$

where (Ω) represents the feasible decision space. The objective functions are defined by Equations (6)–(10) and describe different aspects of the circuit operating performance. The first three objective functions were defined in this study as the specific water consumption, specific grinding energy, and mass recirculation ratio, respectively:

$$J_1 = \frac{\dot{m}_{w,add}}{\dot{m}_{s,fresh}}, \quad (6)$$

$$J_2 = \frac{P_{rod} + P_{ball}}{\dot{m}_{s,fresh}}, \quad (7)$$

$$J_3 = \frac{\dot{m}_{s,UF}}{\dot{m}_{s,OF}}, \quad (8)$$

In this study, the useful recovery R_{util} was defined as a size-based recovery index rather than a mineral-grade recovery. It represents the fraction of the fresh solid feed recovered in the hydrocyclone overflow as useful fine material:

$$R_{\text{util}} = \frac{\dot{m}_{s,OF} F_{\text{OF}}(d_{\text{util}})}{\dot{m}_{s,fresh}},$$

where $\dot{m}_{s,OF}$ is the solid mass flow rate in the hydrocyclone overflow, $\dot{m}_{s,fresh}$ is the fresh solid feed rate, and $F_{\text{OF}}(d_{\text{util}})$ is the mass fraction of the overflow finer than the useful product size d_{util} . This variable was used as a granulometric performance indicator and not as a mineral-grade recovery. Accordingly, the fourth objective was formulated as:

$$J_4 = -R_{\text{util}}, \quad (9)$$

Finally, the specific operating cost was defined in this study as:

$$J_5 = \frac{C_E(P_{rod} + P_{ball}) + C_W \dot{m}_{w,add}}{\dot{m}_{s,fresh}}, \quad (10)$$

where (C_E) and (C_W) represent the unit costs of electrical energy and water, respectively. This implementation used ($C_E = 0.11$ USD/kWh) and ($C_W = 1.25$ USD/m³) [28,29].

3.2. Feasibility Constraints and MOPSO Implementation

The simulator evaluated each particle by running the circuit model to completion. The feasible region was defined by: (i) the bounds of the seven decision variables reported in Table 2; (ii) numerical convergence and physically valid mass and water balances; (iii) admissible diameter ranges of 1.74–2.04 m for the rod mill, 2.89–3.39 m for the ball mill, and 0.54–0.62 m for the hydrocyclone; and (iv) a relative error $\varepsilon_{d_{80,of}} \leq 0.05$ for the

hydrocyclone overflow characteristic size. Solutions violating any of these conditions were considered infeasible.

The relative error associated with the hydrocyclone overflow characteristic size d_{80} was defined by Equation (11):

$$\varepsilon_{d_{80,of}} = \left| \frac{d_{80,of,calc} - d_{80,of,target}}{d_{80,of,target}} \right|, \quad (11)$$

where $d_{80,of,calc}$ and $d_{80,of,target}$ represent the calculated and target d_{80} values in the hydrocyclone *overflow*. The algorithm treated solutions with $\varepsilon_{d_{80,of}} > 0.05$ as infeasible and penalized them during the optimization.

The particle velocity and position were updated using the standard MOPSO formulation [17,18]. The update equations are given by Equations (12) and (13), respectively:

$$\mathbf{v}_i^{t+1} = w \mathbf{v}_i^t + c_1 r_1 (\mathbf{p}_i - \mathbf{x}_i^t) + c_2 r_2 (\mathbf{g}_i - \mathbf{x}_i^t), \quad (12)$$

Here, w is the inertia weight. c_1 and c_2 are the cognitive and social coefficients. r_1 and r_2 are uniformly distributed random numbers in $([0, 1])$. $\mathbf{p}_i$ is the historical best position of the particle. $\mathbf{g}_i$ is the leader chosen from the external archive of non-dominated solutions.

The corresponding particle position was updated as follows:

$$\mathbf{x}_i^{t+1} = \mathbf{x}_i^t + \mathbf{v}_i^{t+1}, \quad (13)$$

while keeping the variables within the bounds defined in Table 2.

To avoid excessive displacements within the search space, the particle velocity was limited according to Equation (14), following the standard MOPSO implementation [17].

$$v_{j,max} = 0.20(x_{j,max} - x_{j,min}), \quad (14)$$

where $x_{j,max}$ and $x_{j,min}$ represent the upper and lower bounds of variable j , respectively. The algorithm constrained the velocities to the interval $[-v_{j,max}, v_{j,max}]$.

The external archive of non-dominated solutions was updated by removing dominated entries and preserving diversity with crowding distance. To encourage exploration, a Gaussian mutation was applied randomly to a fraction of the particles.

The exploration–exploitation balance was controlled by the inertia, cognitive, and social terms, the velocity limit, crowding-distance-based leader selection, and Gaussian mutation. Random initialization, particle inertia, selection of leaders from less populated archive regions, and mutation promoted exploration, whereas attraction to the personal best, guidance by non-dominated leaders, and elitist archive preservation promoted exploitation.

A static penalty of 10^6 was assigned to all objective functions whenever a physical, numerical, granulometric, or geometric constraint was violated.

The DEAP library was used only for parallel particle evaluation via `toolbox.map`.

The optimization started with a swarm of 40 particles. Considering the evaluation of the initial population and the following 4000 iterations, the procedure required a nominal total of 160,040 simulator evaluations. Table 3 summarizes the numerical configuration used during the optimization.

Table 3. Numerical configuration used during the MOPSO optimization.

Parameter	Value
Algorithm	MOPSO implemented in Python
Decision variables	7 continuous variables
Objectives	5
Swarm size	40 particles
Iterations	4000
Nominal evaluations	160,040
External archive	200 non-dominated solutions
Parallelization	24 cores
Inertia weight	$w = 0.72$
Cognitive coefficient	$c_1 = 1.49$
Social coefficient	$c_2 = 1.49$
Maximum velocity	$0.20(x_{\max} - x_{\min})$
Mutation probability	$p_{\text{mut}} = 0.10$
Mutation standard deviation	$\sigma = 0.05(x_{\max} - x_{\min})$
Penalty	10^6

4. Comparison with USIM PAC[®]

This study compared the Python model systematically with the reference simulation built in USIM PAC[®]. The comparison included four aspects: mass-balance closure, consistency of the particle-size distributions, behavior of the hydrocyclone partition curve, and main equipment sizing. These criteria reflect the main physical and design responses of a closed grinding–classification circuit and help assess the model’s reliability before integrating it with the multiobjective optimization algorithm.

4.1. Mass-Balance Comparison

Table 4 compares the solid mass flow rates obtained with USIM PAC[®] and with the Python implementation for the main streams of the rod mill–hydrocyclone–ball mill circuit. The relative percentage error was calculated using Equation (15):

$$\varepsilon_i = \left| \frac{Q_{i,\text{Python}} - Q_{i,\text{USIM}}}{Q_{i,\text{USIM}}} \right| \times 100, \quad (15)$$

Here, Q_i represents the solid mass flow rate of stream i . Most streams showed zero relative error. The fine product showed errors below $10^{-3}\%$. Therefore, numerical rounding explains the observed differences.

Table 4. Solid mass-balance comparison between USIM PAC[®] and the Python model.

Stream	USIM PAC [®] (tpd)	Python (tpd)	Relative Error (%)
Fresh feed	550.0000	550.0000	0.000
Rod mill feed	550.0000	550.0000	0.000
Rod mill discharge	550.0000	550.0000	0.000
Hydrocyclone feed	1181.7006	1181.7006	0.000
Hydrocyclone overflow	550.0008	550.0000	$<10^{-3}$
Hydrocyclone underflow	631.6998	631.6998	0.000
Ball mill feed	631.6998	631.6998	0.000
Ball mill discharge	631.6998	631.6998	0.000

In the reference case, the fresh ore feed is set at 550 tpd. Following primary grinding, the hydrocyclone receives 1181.7006 tpd, as the rod mill product is combined with the

recirculating stream from the ball mill. The hydrocyclone then divides this stream into approximately 550 tpd of overflow and 631.6998 tpd of underflow, thereby reproducing the characteristic structure of the closed circuit.

The circulating load percentage was calculated using Equation (16):

$$CL = \frac{631.6998}{550.0000} \times 100 \approx 114.9\%, \quad (16)$$

which confirms that the Python model conserves solid mass and reproduces the internal recirculation predicted by USIM PAC®.

4.2. Particle-Size Distribution Comparison

The second comparison criterion examined the cumulative particle-size distributions obtained with USIM PAC® and the Python model. Figure 5 presents the cumulative passing curves for the main circuit streams. The Python model reproduces the overall particle-size evolution, including the progressive size reduction from the fresh feed to the hydrocyclone overflow. The closest agreement occurs for the fresh feed, rod mill discharge, hydrocyclone overflow, and ball mill discharge, whereas the larger deviations in the hydrocyclone feed and underflow result from the amplification of small partition differences through the closed ball mill–hydrocyclone recirculating loop.

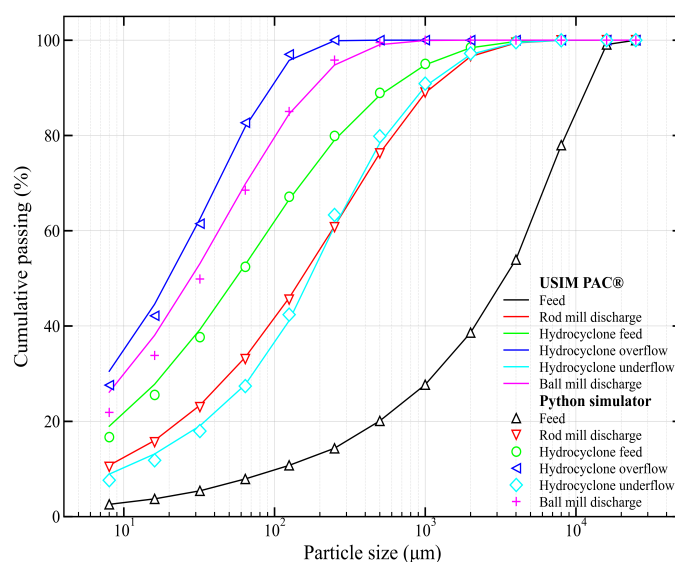


Figure 5. Comparison of cumulative particle-size distributions for the main circuit streams obtained with USIM PAC® and the Python model.

The close overlap between the two representations shows that the open simulator adequately reproduces the circuit’s particle-size evolution. This includes the fresh feed, rod mill discharge, hydrocyclone feed, and classification products.

Table 5 summarizes the characteristic d_{80} values for the main streams. These values verify the progressive size reduction and the separation created by the hydrocyclone.

The d_{80} values confirm size reduction from fresh feed to the circuit’s fine product. In the Python model, d_{80} dropped from 8772.7 μm in the fresh feed to 647.3 μm after the rod mill, and to 60.0 μm in the hydrocyclone overflow. The fresh feed, rod mill discharge, overflow, and ball mill discharge showed the lowest discrepancies, with errors below 2%.

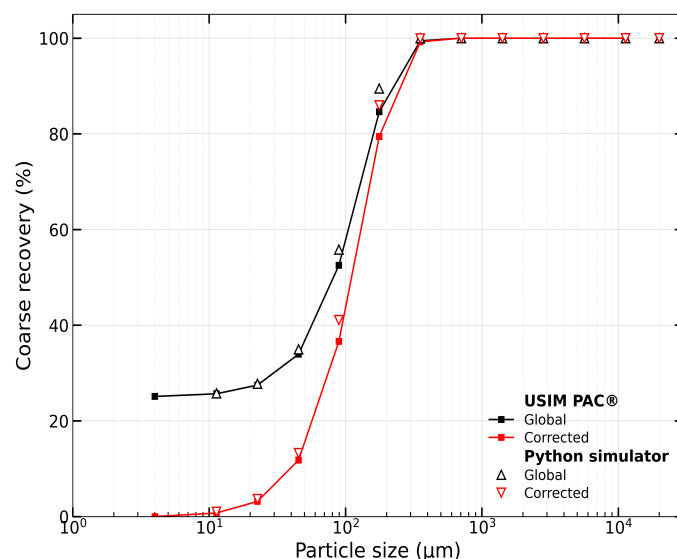
Table 5. Comparison of characteristic d_{80} values between USIM PAC[®] and the Python model.

Stream	Stream Number	d_{80} (μm)		Relative Error (%)
		USIM PAC [®]	Python	
Fresh feed	1	8772.7	8772.7	0.0003
Rod mill discharge	5	646.8	647.3	0.0828
Hydrocyclone feed	6	275.8	252.1	8.5982
Hydrocyclone overflow	8	61.0	60.0	1.6260
Hydrocyclone underflow	9	566.9	508.1	10.3676
Ball mill discharge	12	106.3	106.4	0.1399

The hydrocyclone feed and underflow showed the largest differences, with errors of 8.60% and 10.37%, respectively. The sensitivity of the internal recirculating streams to small variations in the hydrocyclone particle-size partition explains these discrepancies, since the iterative nature of the secondary grinding closed loop tends to amplify them. Nevertheless, both implementations preserved the particle-size hierarchy of the circuit and the overall trend of size reduction.

4.3. Hydrocyclone Partition Curve Comparison

Figure 6 compares the global and corrected hydrocyclone partition curves obtained with USIM PAC[®] and with the Python model. The results show close overlap between the two approaches, especially in the transition region around the cut size.

**Figure 6.** Comparison of the global and corrected hydrocyclone partition curves obtained with USIM PAC[®] and the Python model.

The global curve shows a nonzero recovery of fine particles to the underflow, reflecting the effect of hydraulic bypass. As particle size increases, recovery to the underflow progressively increases until it approaches 100%, which indicates that most of the coarse material returns to the secondary grinding stage. In both implementations, this transition occurs around the cut size and converges to values close to 100% for particles larger than approximately 300 μm .

The corrected curve removes the effect of hydraulic bypass, allowing the hydrocyclone's intrinsic classification behavior to be evaluated. The observed agreement confirms

that the implemented model reproduces both the global separation behavior and the classifier's corrected efficiency.

For the corrected partition curves, the probable error was evaluated as $E_p = (d_{75c} - d_{25c})/2$, and the imperfection as $I = E_p/d_{50c}$. The USIM PAC® Models reference configuration yielded $E_p = 41.18 \mu\text{m}$ and $I = 0.400$, whereas the Python Plitt model yielded $E_p = 41.04 \mu\text{m}$ and $I = 0.397$. The relative difference in imperfection was only 0.72%, confirming that both approaches predict comparable classification sharpness.

4.4. Main Equipment Sizing Comparison

The comparison also checked main equipment sizing to ensure the open model not only matches circuit balances and particle-size distributions but also provides consistent geometric estimates. Figure 7 shows the Python scanning procedure's results for the rod mill, ball mill, and hydrocyclone.

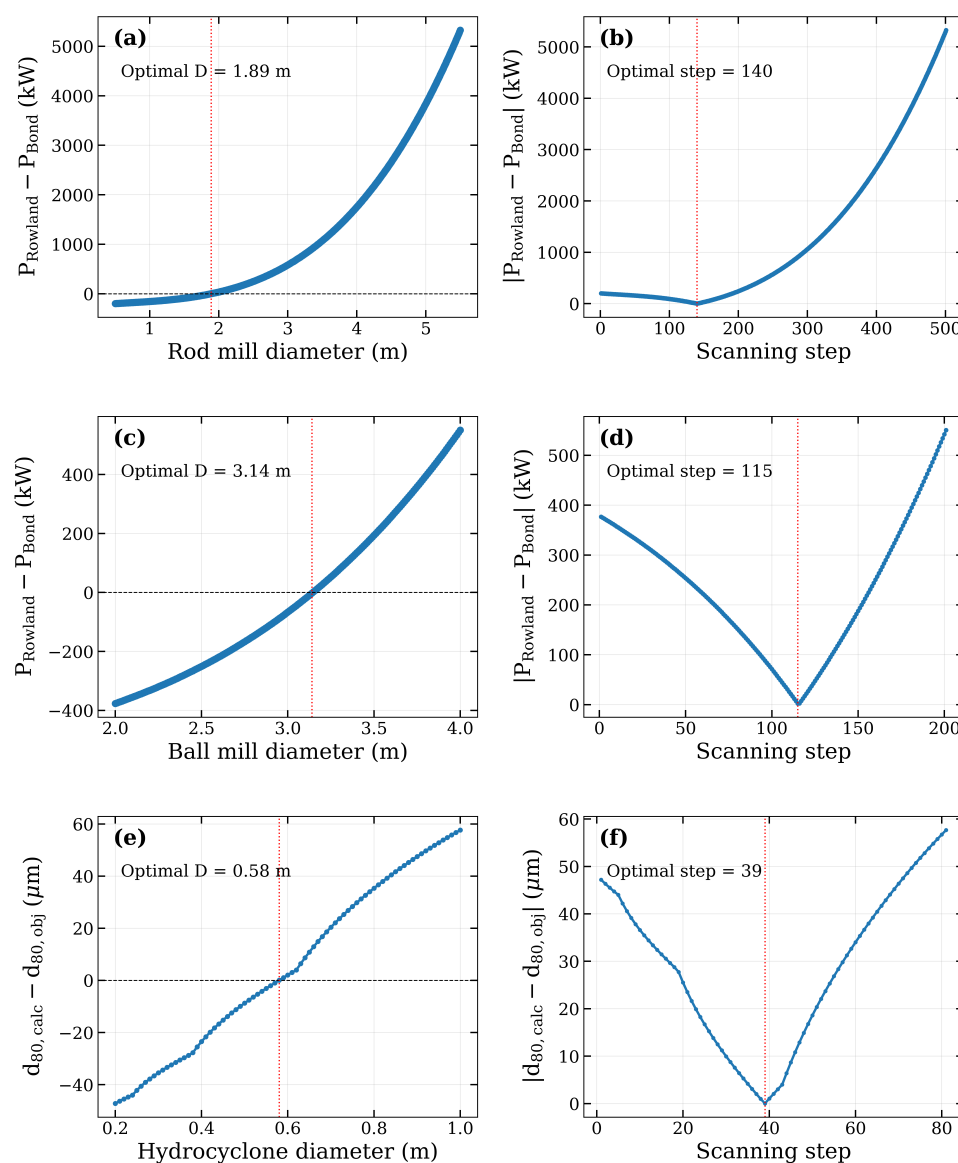


Figure 7. Parametric scanning for model-based sizing of the main equipment using power and d_{80} criteria.

Figure 7a,c,e show the diameter scanning for each equipment item, whereas Figure 7b,d,f present the error response used as the selection criterion. For the mills, the scanning routine identifies the diameter that minimizes the difference between the

power required by the Bond equation and the available power estimated through sizing correlations. The curves show well-defined minima, with optimal diameters close to 1.89 m for the rod mill and 3.03 m for the ball mill. For the hydrocyclone, the selection criterion minimized the difference between the calculated d_{80} in the overflow and the target value of the circuit, giving a diameter close to 0.58 m. These results indicate that the selected solutions correspond to internal minima within the search domain and not to values imposed by the scanning limits.

Table 6 summarizes the direct comparison with USIM PAC[®]. The rod mill diameter reached 1.87 m in USIM PAC[®] and 1.89 m in Python, whereas the hydrocyclone diameter reached 0.58 m in both approaches. For the ball mill, the Python model estimated a diameter of 3.03 m, compared with 2.80 m in USIM PAC[®].

The discretization of the scanning procedure and differences in the implementation of sizing correlations and available-power calculations may explain this difference. Nevertheless, both values lie within the same design range and lead to equivalent operating conditions from a preliminary sizing perspective.

Table 6. Model-based sizing comparison between USIM PAC[®] and the Python model.

Equipment	USIM PAC [®]	Python	Relative Error (%)
Rod mill	1.87 m	1.89 m	1.07
Ball mill	2.80 m	3.03 m	8.21
Hydrocyclone	0.58 m	0.58 m	0.00

These results support using the sizing model as a feasibility criterion in multiobjective optimization, thereby avoiding solutions that improve the numerical objectives at the expense of geometric infeasibility.

Overall, the comparison through mass balances, particle-size distributions, partition curves, and main equipment dimensions demonstrates that the Python model consistently reproduces the steady-state behavior of the reference circuit. The solid balances showed zero or below $10^{-3}\%$ relative errors, whereas the particle-size distributions preserved the expected hierarchy among feed, grinding products, and hydrocyclone streams. Likewise, the agreement observed in the partition curves and in the sizing of the main equipment provides evidence that the simulator consistently reproduces the circuit's hydraulic, particle-size, and geometric responses.

5. Results and Discussion

After comparison establishes the physical and numerical consistency of the open model with USIM PAC[®], this section analyzes the results from the MOPSO-based multiobjective optimization. The analysis focuses on the distribution of feasible solutions in the objective space and on the operating trade-offs among specific water consumption, specific grinding energy, mass recirculation ratio, useful recovery, and operating cost.

5.1. MOPSO Search Behavior and Convergence

Figure 8 shows the evolution of normalized swarm diversity, particle movement, feasible solutions, and external-archive occupancy during the MOPSO run.

The centroid-based diversity decreased from 0.284968 in the initial population to 0.151724 at the end of the run, corresponding to a reduction of 46.76%. Most of this reduction occurred during the initial stage, after which the diversity fluctuated within a nonzero range. Particle displacement and velocity magnitude also remained above zero throughout the run, indicating sustained swarm movement without complete collapse to a single operating condition.

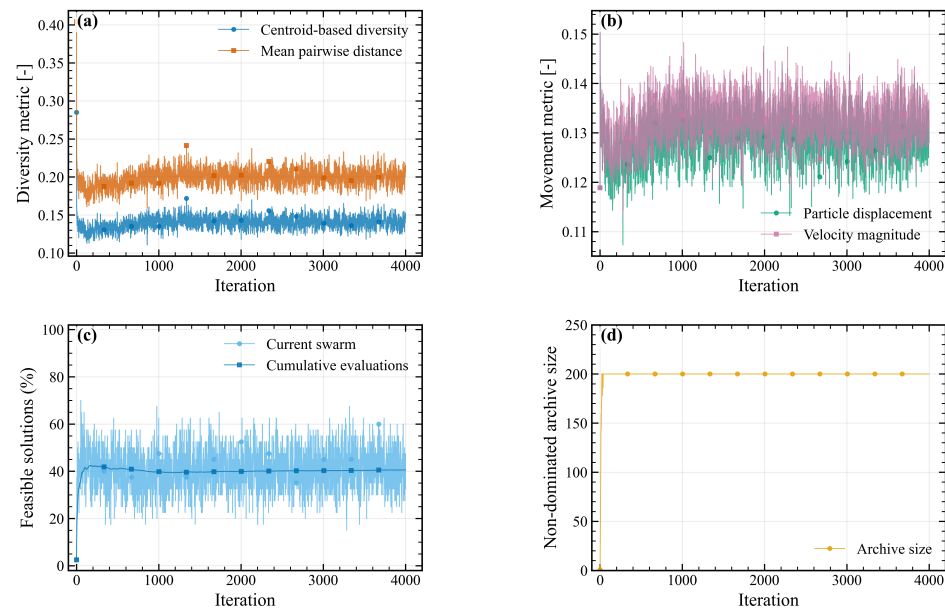


Figure 8. Exploration–exploitation diagnostics of the MOPSO run: (a) normalized swarm diversity; (b) normalized particle displacement and velocity magnitude; (c) current and cumulative feasible-solution ratios; and (d) external non-dominated archive size.

The current-swarm feasible-solution ratio was 2.5% in the initial population and 32.5% in the final iteration, although it fluctuated throughout the optimization. The cumulative feasible-evaluation ratio reached 40.54%, while the external archive first reached its maximum capacity of 200 non-dominated solutions at iteration 22 and ended the run at this capacity. These trends indicate that MOPSO continued identifying feasible solutions while preserving a populated non-dominated archive, without clear evidence of premature convergence.

Convergence was also evaluated through the best feasible value obtained for each objective during the optimization. Figure 9 shows the evolution of these values over the 4000 iterations.

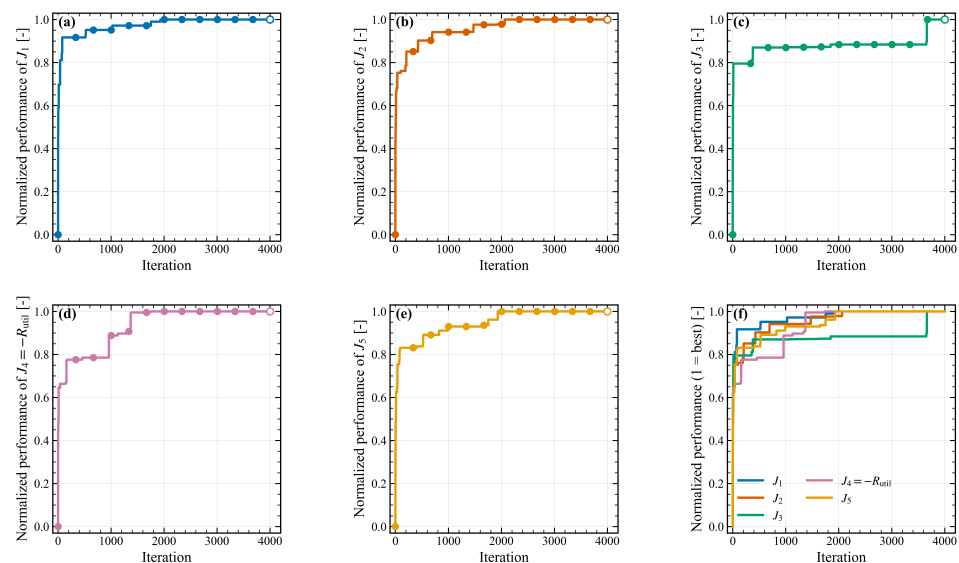


Figure 9. Best feasible objective values obtained during the 4000-iteration MOPSO run: (a) specific water consumption J_1 ; (b) specific grinding energy J_2 ; (c) mass recirculation ratio J_3 ; (d) useful recovery R_{util} ; (e) specific operating cost J_5 ; and (f) combined normalized best-so-far trajectories of the five objectives. Each panel represents the best-so-far value of the corresponding objective and does not imply that all extrema belong to the same solution.

The largest improvements occurred during the initial iterations, followed by progressively longer plateaus and smaller refinements. The objectives J_1 , J_2 , $J_4 = -R_{\text{util}}$, and J_5 reached their final best-so-far levels before approximately iteration 2400, whereas J_3 showed its last improvement near iteration 3700. No additional improvement was observed during the final 300 iterations. The final best values were $J_1 = 1.8746$ tW/tS, $J_2 = 24.3939$ kWh/t, $J_3 = 1.0839$, $R_{\text{util}} = 0.8445$, and $J_5 = 4.4668$ USD/t. These values correspond to the individual extrema of each objective and do not necessarily belong to the same solution. The progressively longer plateaus and the absence of further improvement during the final iterations indicate practical stabilization of the best feasible solutions.

5.2. Pareto Front Analysis

This study integrated the Python model with MOPSO to explore the operating space of the circuit under five simultaneous criteria: specific water consumption J_1 , specific grinding energy J_2 , mass recirculation ratio J_3 , the useful-recovery objective $J_4 = -R_{\text{util}}$, and specific operating cost J_5 . Figure 10 shows the normalized scatter matrix and marginal histograms of the objective functions for the feasible solutions generated during the optimization.

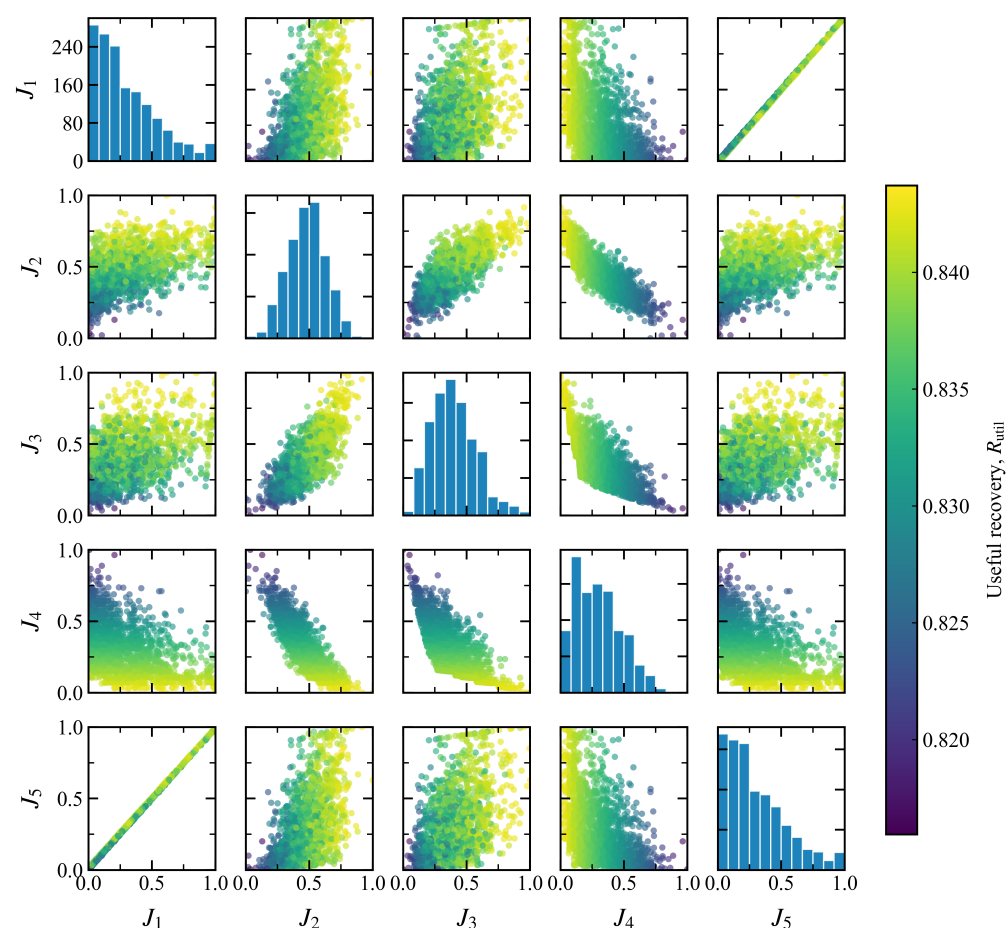


Figure 10. Scatter matrix and marginal histograms of objectives J_1 – J_5 for the feasible solutions generated by MOPSO, where $J_4 = -R_{\text{util}}$. Color indicates R_{util} .

Figure 10 shows that the solution space presents complex relationships among the objectives and confirms the existence of operating trade-offs that characterize grinding-classification circuits. Specific water consumption J_1 and specific operating cost J_5 show a strong positive and nearly linear relationship because the cost function explicitly includes water and energy consumption. In contrast, specific grinding energy J_2 and

mass recirculation ratio J_3 exhibit greater dispersion, reflecting the interaction among size reduction, classification, and recirculation within the circuit.

The mass recirculation ratio J_3 acts as a structural indicator of the closed loop. Low J_3 values reduce recirculating material and may reduce the internal solids load and favor more stable hydraulic conditions; however, they do not necessarily maximize useful recovery. Moderate recirculation can increase the probability of regrinding coarse particles and help a larger fraction of the product reach the required fine size.

Useful recovery involves trade-offs in terms of specific energy, water consumption, and operating costs. Solutions with higher R_{util} tend to occupy regions associated with greater grinding intensity or more demanding classification conditions, which increase resource consumption and, consequently, the specific cost. Conversely, lower-cost configurations concentrate in regions with moderate water and energy consumption.

The marginal histograms also indicate that most solutions concentrate in regions of low operating cost and moderate water and energy consumption, whereas the solutions associated with higher useful recovery occupy a smaller fraction of the search space. This behavior suggests that improvements in useful fine-product recovery require more demanding operating conditions and, therefore, appear less frequently within the feasible solution set identified by MOPSO.

Overall, the patterns in Figure 10 indicate that the feasible solution set generated by MOPSO contains multiple technically feasible operating configurations, each associated with distinct trade-offs among useful fine-product recovery, hydraulic conditions, and resource consumption. Therefore, the selection of an operating condition depends on the adopted decision criteria and the plant's specific priorities.

5.3. Representative Optimization Scenarios

Because the problem involves five conflicting objective functions, MOPSO does not yield a unique optimum but a set of non-dominated solutions. Therefore, the term optimal parameters refers to the decision-variable values associated with representative Pareto-optimal solutions, each reflecting a different operating trade-off.

To quantify these trade-offs, Table 7 summarizes representative scenarios generated during the multiobjective optimization. The comparison includes the base case and configurations aimed at minimizing specific water consumption and operating cost, minimizing specific grinding energy, minimizing the mass recirculation ratio, and maximizing useful recovery.

In addition to the objective values and equipment dimensions, Table 7 reports the corresponding fresh solid feed rate and fine-product d_{80} error. The fresh solid feed rate was calculated as $\dot{m}_{s,fresh} = \alpha \times 550$ t/d. Table 8 reports the fresh-feed scaling factor α and the remaining decision-variable values associated with the same representative scenarios. Together, both tables allow the hydraulic, energetic, granulometric, economic, and geometric performance of each solution to be evaluated simultaneously.

Table 7. Representative solutions from the multiobjective optimization of the rod mill–hydrocyclone–ball mill circuit.

Scenario	Fresh Feed [t/d]	J_1 [tW/tS]	J_2 [kWh/t]	J_3 [-]	R_{util} [-]	J_5 [USD/t]	D_{rod} [m]	D_{ball} [m]	D_{cyc} [m]	ϵ_{d80}
Base case	550.0	2.4830	25.4751	1.2071	0.8267	4.9738	1.890	3.140	0.580	—
Min J_1 + Min J_2 + Min J_5	715.0	1.8746	24.3939	1.1009	0.8178	4.4668	2.040	3.340	0.540	0.0070
Min J_3	582.3	2.0932	24.7635	1.0839	0.8147	4.6464	1.950	3.140	0.560	0.0008
Max R_{util}	616.2	8.5394	26.5632	1.3545	0.8445	8.6865	1.980	3.290	0.550	0.0042

Table 8 reports the operating variables that generated the performance values in Table 7. The solids contents define the hydraulic conditions of the three main equipment

feeds, α scales the fresh solid feed relative to the base case, and the target d_{80} values define the granulometric conditions selected by MOPSO.

Table 8. Decision-variable values for the base case and representative Pareto-optimal solutions obtained with MOPSO.

Scenario	reg_{rod} [wt.%]	reg_{hc} [wt.%]	reg_{ball} [wt.%]	α [-]	$d_{80,rod}$ [mm]	$d_{80,ball}$ [mm]	$d_{80,OF}$ [mm]
Base case	70.0000	40.0000	55.0000	1.0000	0.600000	0.100000	0.060000
Min J_1 + Min J_2 + Min J_5	44.6780	46.4648	46.9984	1.3000	0.561483	0.090000	0.061150
Min J_3	55.4983	42.7476	35.2838	1.0587	0.540000	0.090000	0.061654
Max R_{util}	12.4137	41.0094	36.7564	1.1203	0.540000	0.110000	0.058000

Compared with the base case, the Min J_1 + Min J_2 + Min J_5 solution reduced the specific water consumption from 2.4830 to 1.8746 tW/tS, which represents a decrease of 24.50%. In the same configuration, the specific grinding energy decreased from 25.4751 to 24.3939 kWh/t, equivalent to a 4.24% reduction, and the specific operating cost decreased from 4.9738 to 4.4668 USD/t, equivalent to a 10.19% reduction. The mass recirculation ratio also decreased from 1.2071 to 1.1009, equivalent to an 8.80% reduction, whereas useful recovery decreased only slightly from 82.67% to 81.78%, corresponding to 0.89 percentage points or a relative decrease of 1.08%.

The corresponding decision vector combines a higher fresh-feed scaling factor with increased solids contents in the hydrocyclone and ball mill feeds. These conditions reduce downstream water addition and distribute the water and energy requirements over a higher fresh-solid throughput. The slightly coarser hydrocyclone overflow target is also consistent with the small decrease in useful recovery.

The equipment-sizing results remained within the feasible ranges imposed by the scanning routine. Across the representative scenarios, the selected diameters varied within narrow preliminary-design ranges: 1.89–2.04 m for the rod mill, 3.14–3.34 m for the ball mill, and 0.54–0.58 m for the hydrocyclone. Thus, equipment geometry was used mainly as a feasibility criterion rather than as a primary comparison metric among the optimized scenarios.

The reported α values indicate that the optimized scenarios modify both the internal operating conditions and the fresh-feed throughput, thereby affecting the water, energy, and cost indicators normalized per tonne of fresh solids. Therefore, these specific indicators must be interpreted together with the corresponding feed rate and solids distribution.

The Min J_1 + Min J_2 + Min J_5 solution achieved the lowest specific grinding energy, with 24.3939 kWh/t, while also reducing specific water consumption and operating cost relative to the base case. This solution combines an increase in fresh-feed throughput with lower rod mill feed solids and finer grinding targets. In turn, the Min J_3 solution yielded the lowest mass recirculation ratio, $J_3 = 1.0839$, while also reducing specific water consumption and operating cost relative to the base case. Its finer rod mill discharge is consistent with a smaller coarse fraction reporting to the hydrocyclone underflow. These results show that minimizing a single objective does not necessarily lead to simultaneous improvements in the remaining performance indicators.

The Max R_{util} solution increased useful recovery from 82.67% to 84.45%, corresponding to an absolute gain of 1.78 percentage points. This scenario achieved the highest granulometric performance among the analyzed alternatives. This configuration combines the lowest hydrocyclone overflow target d_{80} , a finer rod mill discharge, and the lowest rod mill feed solids content. These conditions favor fine-particle recovery but require greater water and energy use. However, this improvement occurred with the highest specific water consumption, the highest specific grinding energy, and the highest operating cost.

This behavior highlights the inherent trade-off between useful fine-product recovery and resource consumption within the circuit.

The differences among scenarios show that each region of the Pareto front represents a distinct operating strategy. Some configurations prioritize reductions in water consumption and operating cost, whereas others favor useful recovery or lower internal recirculation. Therefore, the Pareto front provides a quantitative basis for selecting operating conditions according to specific technical, economic, or environmental criteria.

From an operating perspective, the $\text{Min } J_1 + \text{Min } J_2 + \text{Min } J_5$ solution represents the most balanced alternative in terms of hydraulic, energetic, and economic efficiency. Although this solution slightly reduces useful recovery relative to the base case, from 82.67% to 81.78%, it maintains comparable granulometric performance while simultaneously reducing water consumption, specific grinding energy, mass recirculation ratio, and operating cost. This behavior makes the solution particularly attractive for operations located in regions with water restrictions or high energy costs, provided that the 0.89 percentage points decrease in useful recovery is operationally acceptable.

Overall, the results show that the open simulator integrated with MOPSO can identify technically feasible operating configurations and quantify the trade-offs among water consumption, specific grinding energy, mass recirculation ratio, useful recovery, and operating cost. These findings support the proposed approach as a reproducible tool for the multiobjective analysis and optimization of grinding–classification circuits.

6. Conclusions

This study developed an open Python model for steady-state simulation, model-based sizing, and multiobjective optimization of a rod mill–hydrocyclone–ball mill circuit applied to a gold ore. The model integrates mass and water balances, particle-size distribution reconstruction, simplified comminution models, hydraulic classification, and sizing criteria for the main equipment.

The simulator-to-simulator benchmark against USIM PAC[®] showed strong agreement in mass balances, particle-size distributions, partition curves, and equipment dimensions. The solid balances showed zero or below $10^{-3}\%$ relative errors, whereas the sizing routine reproduced the reference dimensions with errors of 1.07% for the rod mill, 8.21% for the ball mill, and 0.00% for the hydrocyclone.

The integration of the simulator with MOPSO enabled the exploration of feasible solutions that consider specific water consumption, specific grinding energy, mass recirculation ratio, useful recovery, and operating cost simultaneously. Compared with the base case, the $\text{Min } J_1 + \text{Min } J_2 + \text{Min } J_5$ solution reduced water consumption from 2.4830 to 1.8746 tW/tS, specific grinding energy from 25.4751 to 24.3939 kWh/t, and specific operating cost from 4.9738 to 4.4668 USD/t, while useful recovery decreased only slightly from 82.67% to 81.78%. These changes correspond to reductions of 24.50%, 4.24%, and 10.19% in specific water consumption, specific grinding energy, and specific operating cost, respectively, together with a reduction of 8.80% in the mass recirculation ratio. Useful recovery decreased by 0.89 percentage points. Conversely, the maximum-recovery solution increased useful recovery from 82.67% to 84.45%, corresponding to an absolute gain of 1.78 percentage points, although with higher water and energy consumption and operating cost.

The results showed that no single operating condition can simultaneously optimize all objectives. Consequently, the Pareto front provided a set of operating alternatives from which configurations can be selected according to specific industrial priorities, such as water savings, energy consumption reduction, recirculation reduction, cost minimization, or useful recovery maximization. The convergence analysis conducted for the 4000-iteration run showed that no additional improvement was observed during the final 300 iterations,

indicating that this computational budget was empirically sufficient for the present case, although global optimality cannot be guaranteed for a stochastic metaheuristic.

This study is limited to steady-state simulator-to-simulator benchmarking for a single reference ore and does not constitute experimental validation of the Pareto-optimal operating conditions. Future work will evaluate selected solutions through pilot- or plant-scale tests and will extend the framework to dynamic operation, parameter uncertainty, and ore variability.

Overall, the proposed approach provides an open, reproducible framework that can integrate optimization techniques for the analysis, sizing, and evaluation of operating strategies in grinding–classification circuits.

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