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Mean Square Stability and Stabilization for Linear Parabolic Stochastic Partial Difference Systems

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Abstract: This paper studies the mean square stability and stabilization of linear parabolic stochastic partial difference systems, which contain space–time characteristics and stochastic noise. A definition of the mean square stability for this system is proposed. Using stochastic analysis and some mathematical analysis methods, a strict decreasing sequence is constructed to represent the expectation of the sum of squares of the state variable along the spatial dimension. The sufficient conditions of the mean square stability are established based on system parameters, and then the convergence along the time axis is rigorously proven by the Squeeze criterion. Moreover, some stabilization criteria are derived by designing a linear feedback controller. Finally, two examples are given to illustrate the effectiveness of the results.

Keywords: partial difference systems; stochastic noise; mean square stability; stabilization



Academic Editors: José Mario Araújo and Carlos Eduardo Trabuco Dórea

Received: 5 March 2025

Revised: 21 March 2025

Accepted: 10 April 2025

Published: 13 April 2025

Citation: Yong, S.; Dai, X.; Luo, T.; Wang, Z. Mean Square Stability and Stabilization for Linear Parabolic Stochastic Partial Difference Systems. *Processes* **2025**, *13*, 1175. <https://doi.org/10.3390/pr13041175>

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1. Introduction

Partial differential equations (PDEs) are extensively applied in the fields of electromagnetism systems [1], phase transitions [2], plasma physics [3], and heat conduction [4]. The space–time characteristics of partial differential equations usually involve the complex interplay between spatial and temporal dependencies, and the behavior of the solution is determined by the continuous variables that define its dynamics across both spatial and temporal dimensions, resulting in its infinite-dimensional nature. So far, the finite difference method (FDM) and finite element method (FEM) have been widely used for the numerical analysis of partial differential equations (PDEs) [5–7]. Typically, the numerical treatment of PDEs involves their discretization into partial difference equations and significant progress has been achieved in both theoretical and applied aspects. Ref. [8] analyzed the stability and convergence of the solution of partial difference systems by using the finite difference method. It plays a vital role for the partial difference system in many application fields, such as image processing [9], population models, neural networks, molecular orbital models [10], and numerical control systems. For the present time, several results have been achieved in the research related to partial difference systems, such as solution of partial difference systems [11–15], the stability of the partial difference systems [16], the control of partial difference systems [17–22]. However, the traditional partial difference system usually assumes that the system is a completely deterministic discrete model, it is often necessary to consider the stochastic factors; therefore, studying stochastic partial difference systems is crucial.

Compared with stochastic partial difference systems, stochastic difference systems only focus on the time dimension, and stability has become an important research topic in the field of control theory and applied mathematics in recent years. In the classical research framework, based on the Lyapunov method, matrix norm theory and stochastic analysis tools, scholars have established the stability criterion for stochastic difference systems with constant delay and Markov jump systems, etc., and achieved remarkable results; see references [23–32]. Ref. [32] propose a novel approach to the exponential stability in mean square of stochastic difference systems with time-varying delays without Lyapunov functions. Apparently, none of these studies have considered the spatial dimension of the stochastic system, and traditional stability theories focusing on purely temporal dimensions or localized spatial structures face significant challenges in generalizing to stochastic partial difference system, in particular. So relatively little has been done to study stochastic partial difference systems.

Stochastic partial difference systems have an additional spatial characteristic that distinguish them from stochastic difference systems and can effectively describe complex systems that have distributional properties in space and are subject to stochastic perturbations. However, at present, to the best of the authors' knowledge, the research on stochastic partial difference systems has been inconclusive. Therefore, the stability should be considered in system analysis and synthesis. So it is necessary to study the mean square stability of stochastic partial difference systems.

We are researching a new field and providing a simpler method to prove the sufficient conditions of the mean square stability and stabilization of the stochastic partial difference system based on system parameters. In this paper, the linear parabolic stochastic partial difference system with constant coefficients uses Cauchy–Schwarz inequality and discrete Green's formula to depict the relationship of the expectation along the time axis and prove the convergence of the expectation by Squeeze criterion. And the effectiveness of the sufficient conditions is shown through two simulation examples.

The rest of the paper is structured as follows. The preliminaries and problem statement are set out in Sections 2 and 3, respectively. In Section 4, the condition of the mean square stability is demonstrated. There are two illustrations to verify the effectiveness of the main result in Section 5. Finally, some conclusions are mentioned in Section 6.

2. Preliminaries

Let $\mathbb{R}, \mathbb{Z}, \mathbb{Z}_+$ represent a set of real numbers, integral numbers and non-negative integral numbers, respectively. For given $k_1, k_2 \in \mathbb{Z}$, denote $\mathbb{Z}_{[k_1, k_2]} := [k_1, k_2] \cap \mathbb{Z}$. $X_{it} \in \mathbb{R}$ means $X(i, t)$, where $i \in \mathbb{Z}_{[1, N]}$, $t \in \mathbb{Z}_{[0, T]}$ and $N, T \in \mathbb{Z}$. Set $\mathbb{E}$ as the mathematical expectations. If $X_{it} = (X_{1t}, X_{2t}, \dots, X_{Nt})$, then define the squared Euclidean norm as $\|X_{it}\|^2 := \sum_{i=1}^N X_{it}^2$.

To investigate the stability and stabilization of system (2), the following lemma is needed.

Lemma 1. (Discrete Green's formula) [19] introduces the following equality under boundary condition (4) for the system (2)

$$\sum_{i=1}^N X_{it} \Delta_1^2 X_{i-1,t} = - \sum_{i=0}^N (\Delta_1 X_{it})^2. \quad (1)$$

3. Problem Statement

3.1. Model Motivation and Application Scenarios

In practical physical and ecological systems, nonlinear coupled disturbances often manifest as state-dependent stochastic fluctuations, whose intensity is closely tied to the sys-

tem state. In early theoretical studies of stochastic systems, additive noise—characterized by its independence from the system state—was predominantly adopted due to its mathematical tractability. However, in practical scenarios requiring precise characterization of state-dependent disturbances, such as thermal diffusion in nanomaterials and ecological dynamics, multiplicative noise models have emerged as a critical research focus (see References [33,34]). To address these challenges, this work proposes a novel parabolic stochastic partial difference system, whose core innovation lies in the incorporation of multiplicative noise—a formulation where the noise intensity directly couples with the system state.

3.2. Mathematical Model of the System

The proposed parabolic stochastic partial difference system is formulated to rigorously characterize state-dependent multiplicative noise. Let us consider the discrete parabolic stochastic partial difference systems with constant coefficients as follows

$$\Delta_2 X_{it} = a\Delta_1^2 X_{i-1,t} + bX_{it} + cX_{it}W_t + U_{it}, \quad (2)$$

where i, t are discrete spatial and temporal variables, respectively. a, b, c are constants, and $a > 0$. $X_{it} \in \mathbb{R}$, which indicates that X_{it} is a one-dimensional variable, taking values from the set of real numbers. U_{it} is the control input. W_t is a one-dimensional stochastic process defined in the complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with natural filtration $\mathcal{F}_t \subseteq \mathcal{F}_{t+1} \subseteq \mathcal{F}$, $t \in \mathbb{Z}_+$, which is a sequence of σ -algebra of subsets of Ω . And the mathematical expectations satisfy

$$\mathbb{E}W_t = 0; \mathbb{E}W_t W_s = \delta_{ts} = \begin{cases} 1 & \text{if } t = s, \\ 0 & \text{if } t \neq s. \end{cases} \quad (3)$$

Here, the partial difference signs of system (2) are defined, respectively:

$$\begin{cases} \Delta_1 X_{it} = X_{i+1,t} - X_{it}, \\ \Delta_2 X_{it} = X_{i,t+1} - X_{it}, \\ \Delta_1^2 X_{i-1,t} = \Delta_1(\Delta_1 X_{i-1,t}) = X_{i+1,t} - 2X_{it} + X_{i-1,t}. \end{cases} \quad (4)$$

The initial and boundary conditions of the system (2) are as follows

$$X_{i0} = \varphi_i; X_{0t} = X_{N+1,t} = 0, \quad (5)$$

and $\|\varphi_i\|^2 = \sum_{i=1}^N \varphi_i^2 \leq \varphi_0$, and φ_0 is a constant.

Definition 1 ([35]). When t intend to infinity, the expectation of the sum of square of the state variable across the spatial dimension limit converges to 0; that is,

$$\lim_{t \rightarrow \infty} \mathbb{E}\|X_{it}\|^2 = \lim_{t \rightarrow \infty} \mathbb{E} \sum_{i=1}^N X_{it}^2 = 0, \quad (6)$$

the trivial solution of system (2) is said to be mean square stable.

4. The Mean Square Stability

Theorem 1. Assume that the coefficient of the system (2) without control input ($U_{it} = 0$) satisfies

$$|4[(b - 2a)^2 + 2a^2 + c^2] + (1 + 2b)| < 1, a > 0, \quad (7)$$

then the trivial solution of system (2) is mean-square stable.

Proof. By the definition of $\Delta_2 X_{it}$ in (4), we have

$$\begin{aligned}\mathbb{E}\|\Delta_2 X_{it}\|^2 &= \mathbb{E}\|X_{i,t+1} - X_{it}\|^2 \\ &= \mathbb{E}\|X_{i,t+1}\|^2 - 2\mathbb{E}\sum_{i=1}^N (X_{it}X_{i,t+1}) + \mathbb{E}\|X_{it}\|^2.\end{aligned}\quad (8)$$

Then, (8) can be rewritten as

$$\mathbb{E}\|X_{i,t+1}\|^2 = 2\mathbb{E}\sum_{i=1}^N (X_{it}X_{i,t+1}) + \mathbb{E}\|(\Delta_2 X_{it})\|^2 - \mathbb{E}\|X_{it}\|^2. \quad (9)$$

□

On the other hand, according to (2) and (4), we have

$$\begin{aligned}X_{i,t+1} &= \Delta_2 X_{it} + X_{it} \\ &= a\Delta_1^2 X_{i-1,t} + bX_{it} + cX_{it}W_t + X_{it} \\ &= a\Delta_1^2 X_{i-1,t} + (1+b)X_{it} + cX_{it}W_t.\end{aligned}\quad (10)$$

Multiplying by X_{it} on both sides of (10), we can get

$$X_{it}X_{i,t+1} = aX_{it}\Delta_1^2 X_{i-1,t} + (1+b)X_{it}^2 + cX_{it}^2 W_t. \quad (11)$$

Summing both sides of (11) on the spatial discrete indicator i from 1 to N , we obtain

$$\sum_{i=1}^N X_{it}X_{i,t+1} = a\sum_{i=1}^N X_{it}\Delta_1^2 X_{i-1,t} + (1+b)\sum_{i=1}^N X_{it}^2 + c\sum_{i=1}^N X_{it}^2 W_t. \quad (12)$$

Taking the mathematical expectation for (12), we can obtain

$$\mathbb{E}\sum_{i=1}^N (X_{it}X_{i,t+1}) = a\mathbb{E}\sum_{i=1}^N (X_{it}\Delta_1^2 X_{i-1,t}) + (1+b)\mathbb{E}\|X_{it}\|^2 + c\mathbb{E}\sum_{i=1}^N (X_{it}^2 W_t). \quad (13)$$

According to condition (3), (13) yields the following result

$$\mathbb{E}\sum_{i=1}^N (X_{it}X_{i,t+1}) = a\mathbb{E}\sum_{i=1}^N (X_{it}\Delta_1^2 X_{i-1,t}) + (1+b)\mathbb{E}\|X_{it}\|^2. \quad (14)$$

Substituting (14) into (9), we can obtain

$$\begin{aligned}\mathbb{E}\|X_{i,t+1}\|^2 &= \mathbb{E}\|(\Delta_2 X_{it})\|^2 + 2a\mathbb{E}\sum_{i=1}^N (X_{it}\Delta_1^2 X_{i-1,t}) \\ &\quad + 2(1+b)\mathbb{E}\|X_{it}\|^2 - \mathbb{E}\|X_{it}\|^2 \\ &= \mathbb{E}\|(\Delta_2 X_{it})\|^2 + 2a\mathbb{E}\sum_{i=1}^N (X_{it}\Delta_1^2 X_{i-1,t}) + (1+2b)\mathbb{E}\|X_{it}\|^2.\end{aligned}\quad (15)$$

Next, we will proceed with the estimation of $\mathbb{E}\|(\Delta_2 X_{it})\|^2$ and $\mathbb{E}\|(X_{it}\Delta_1^2 X_{i-1,t})\|$.

Again, by system expression (2) and partial difference sign (4), we can obtain

$$\begin{aligned}
\Delta_2 X_{it} &= a\Delta_1^2 X_{i-1,t} + bX_{it} + cX_{it}W_t \\
&= a(X_{i+1,t} - 2X_{it} + X_{i-1,t}) + bX_{it} + cX_{it}W_t \\
&= (b - 2a)X_{it} + aX_{i+1,t} + aX_{i-1,t} + cX_{it}W_t.
\end{aligned} \tag{16}$$

Using the Cauchy–Schwarz inequality for (16), we have

$$(\Delta_2 X_{it})^2 \leq 4(b - 2a)^2 X_{it}^2 + 4a^2 X_{i+1,t}^2 + 4a^2 X_{i-1,t}^2 + 4c^2 X_{it}^2 W_t^2. \tag{17}$$

Summing both sides of (17) on the spatial discrete indicator i from 1 to N

$$\begin{aligned}
\sum_{i=1}^N (\Delta_2 X_{it})^2 &\leq 4(b - 2a)^2 \sum_{i=1}^N X_{it}^2 + 4a^2 \sum_{i=1}^N X_{i+1,t}^2 + 4a^2 \sum_{i=1}^N X_{i-1,t}^2 \\
&+ 4c^2 \sum_{i=1}^N X_{it}^2 W_t^2.
\end{aligned} \tag{18}$$

Taking the mathematical expectation on both sides of (18), we can obtain

$$\begin{aligned}
\mathbb{E}\|(\Delta_2 X_{it})\|^2 &\leq \mathbb{E}\{4(b - 2a)^2 \|X_{it}\|^2 + 4a^2 \|X_{i+1,t}\|^2 + 4a^2 \|X_{i-1,t}\|^2 \\
&+ 4c^2 \|X_{it}W_t\|^2\}.
\end{aligned} \tag{19}$$

Since $\|X_{i+1,t}\|^2 = \sum_{i=2}^N X_{it}^2 + X_{N+1,t}^2$, and $\|X_{i-1,t}\|^2 = \sum_{i=1}^{N-1} X_{it}^2 + X_{0t}^2$ from (19), we can deduce that

$$\begin{aligned}
\mathbb{E}\|(\Delta_2 X_{it})\|^2 &\leq \mathbb{E}\{4(b - 2a)^2 \|X_{it}\|^2 + 4a^2 (\sum_{i=2}^N X_{it}^2 + X_{N+1,t}^2) \\
&+ 4a^2 (\sum_{i=1}^{N-1} X_{it}^2 + X_{0t}^2) + 4c^2 \|X_{it}W_t\|^2\}.
\end{aligned} \tag{20}$$

Substituting (3) and (6) into (20), we can deduce that

$$\mathbb{E}\|(\Delta_2 X_{it})\|^2 \leq 4[(b - 2a)^2 + 2a^2 + c^2]\mathbb{E}\|X_{it}\|^2. \tag{21}$$

According to Lemma 1, taking the mathematical expectation on both sides of (1) and the boundary condition (5), we have

$$\mathbb{E} \sum_{i=1}^N X_{it} \Delta_1^2 X_{i-1,t} = -\mathbb{E} \sum_{i=0}^N (\Delta_1 X_{it})^2 \leq -\mathbb{E}\|(\Delta_1 X_{it})\|^2. \tag{22}$$

Then, substituting (21), (22) into (15), we have

$$\begin{aligned}
\mathbb{E}\|X_{i,t+1}\|^2 &\leq 4[(b - 2a)^2 + 2a^2 + c^2]\mathbb{E}\|X_{it}\|^2 - 2a\mathbb{E}\|(\Delta_1 X_{it})\|^2 \\
&+ (1 + 2b)\mathbb{E}\|X_{it}\|^2.
\end{aligned}$$

Therefore,

$$\mathbb{E}\|X_{i,t+1}\|^2 \leq \{4[(b - 2a)^2 + 2a^2 + c^2] + (1 + 2b)\}\mathbb{E}\|X_{it}\|^2. \tag{23}$$

Let $A_t = \mathbb{E}\|X_{it}\|^2$, $\rho = |4[(b - 2a)^2 + 2a^2 + c^2] + (1 + 2b)|$, then the above equation can be rewritten as follows

$$A_{t+1} \leq \rho A_t. \tag{24}$$

We can obtain

$$A_{t+1} \leq \rho A_t \leq \dots \leq \rho^t A_1 \leq \rho^{t+1} A_0. \quad (25)$$

Based on (5) and (7), we know A_0 is a finite constant and $\rho < 1$. According to Squeeze criterion, we can deduce

$$\lim_{t \rightarrow \infty} A_t = 0. \quad (26)$$

That means

$$\lim_{t \rightarrow \infty} \mathbb{E} \sum_{i=1}^N X_{it}^2 = \lim_{t \rightarrow \infty} \mathbb{E} \|X_{it}\|^2 = 0. \quad (27)$$

Therefore, we conclude that the trivial solution of system is mean square stable in (7).

Theorem 2. We let a state feedback controller $U_{it} = kX_{it}$ in system (2). If the following inequality holds

$$|4[(b+k-2a)^2 + 2a^2 + c^2] + (1+2b+2k)| < 1, a > 0, \quad (28)$$

then the system (3) is mean square stable.

Proof. The proof of Theorem 2 follows the same reasoning as the proof of Theorem 1. The detailed steps are omitted. $\square$

5. Numerical Simulations

In this section, to solve the stochastic system (2), we adopt the Euler-Maruyama method (explicit finite difference scheme) for temporal and spatial discretization. And there are some simulation examples to demonstrate the validity of sufficient conditions for the mean square stability and stabilization.

5.1. The Stability of the Systems Without Controller

Considering the parabolic stochastic partial difference system (3) without control input ($U_{it} = 0$) as follows

$$\Delta_2 X_{it} = a \Delta_1^2 X_{i-1,t} + b X_{it} + c X_{it} W_t, \quad (29)$$

where $X_{it} \in \mathbb{R}$, $i \in \mathbb{Z}_{[1,100]}$, $t \in \mathbb{Z}_{[0,100]}$. $a = 0.033$, $b = -0.067$, $c = 0.1$, let $\rho = |4[(b-2a)^2 + 2a^2 + c^2] + (1+2b)|$, then we can get $\rho = 0.9855 < 1$. Meanwhile, we take the initial value as $X_{i0} = \varphi_i = \sin(\frac{i\pi}{101})$. The simulation results are shown in Figure 1. Figure 1a depicts the state surfaces of the first sample path. And the state trajectories $\|X_{it}\|^2$ of 100 sample paths for the system (29) are shown as the light line in Figure 1b.

We can see that the limit of the expression $\mathbb{E}\|X_{it}\|^2$ converges to 0 as time t tends to infinity in Figure 1b, so that the systems is obviously mean square stable, which validates the effectiveness of Theorem 1.

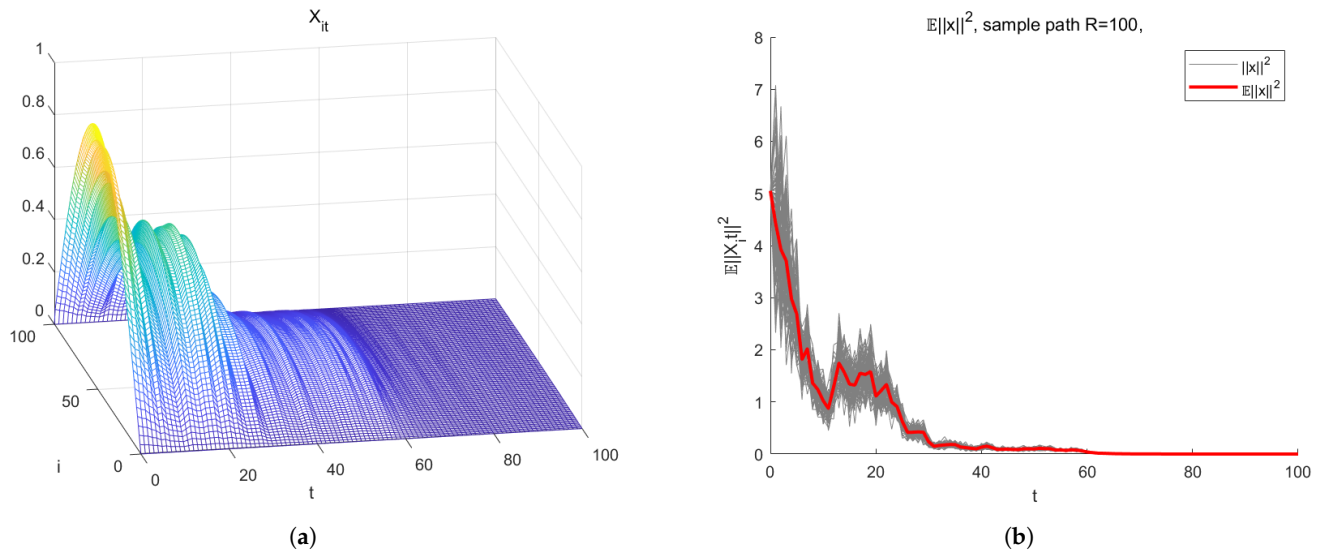


Figure 1. The system with $\rho < 1$. (a) the state variable X_{it} ; (b) the state expectation $\mathbb{E}\|X_{it}\|^2$.

5.2. The Stabilization of the Systems with Controller

In this subsection, we present some illustrative examples to demonstrate the stabilization of the systems with a state feedback controller.

There exists a unstable system, which is closely related to system (29) but with adjusted system parameters $a = 0.04, b = 0.224, c = 0.08$ ($\rho = 1.5693 > 1$). Figure 2 shows the simulation results without control input. The results in instability within the system.

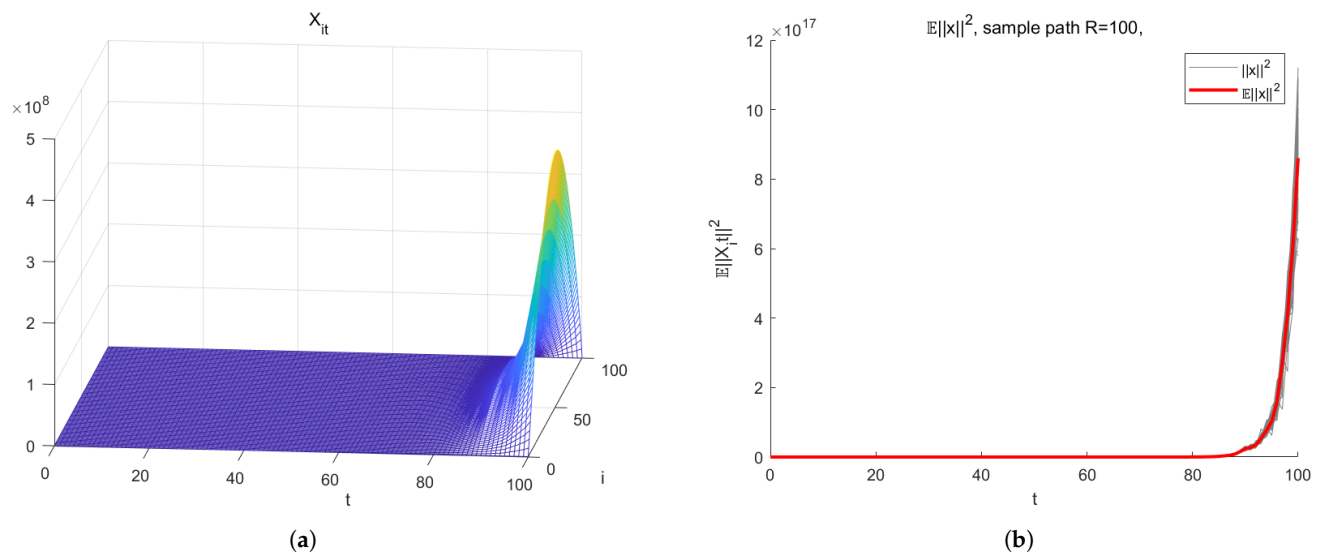


Figure 2. The unstable system. (a) The state variable X_{it} ; (b) the state expectation $\mathbb{E}\|X_{it}\|^2$.

To show the relationship between the feedback gain k and the convergence condition $\rho < 1$ in Theorem 2, five sets of feedback gain for the same control system are selected: $k = 0.379, -0.284, -0.073, -0.379, -0.054$, with corresponding ρ values of 1.0681, 0.9968, 0.9498, 0.9493, and 0.9484, respectively.

As shown in Figures 3 and 4, the state variable X_{it} and the state expectation $\mathbb{E}\|X_{it}\|^2$ are convergent, validating the effectiveness of Theorem 2.

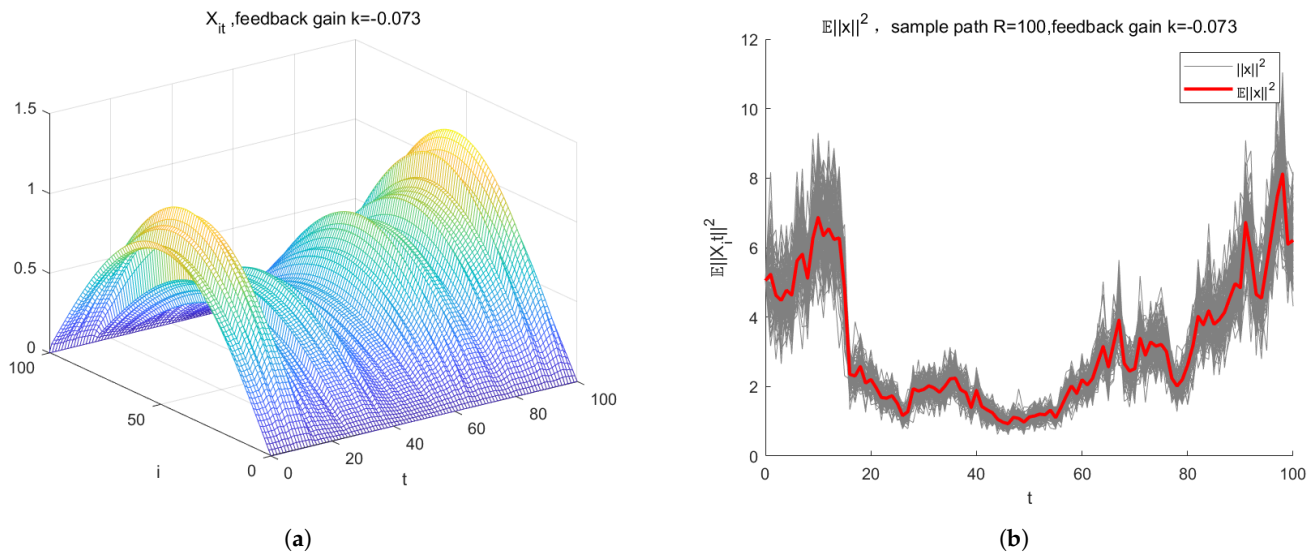


Figure 3. The Non-convergent oscillatory system. (a) The state variable X_{it} ; (b) the state expectation $E||X_{it}||^2$.

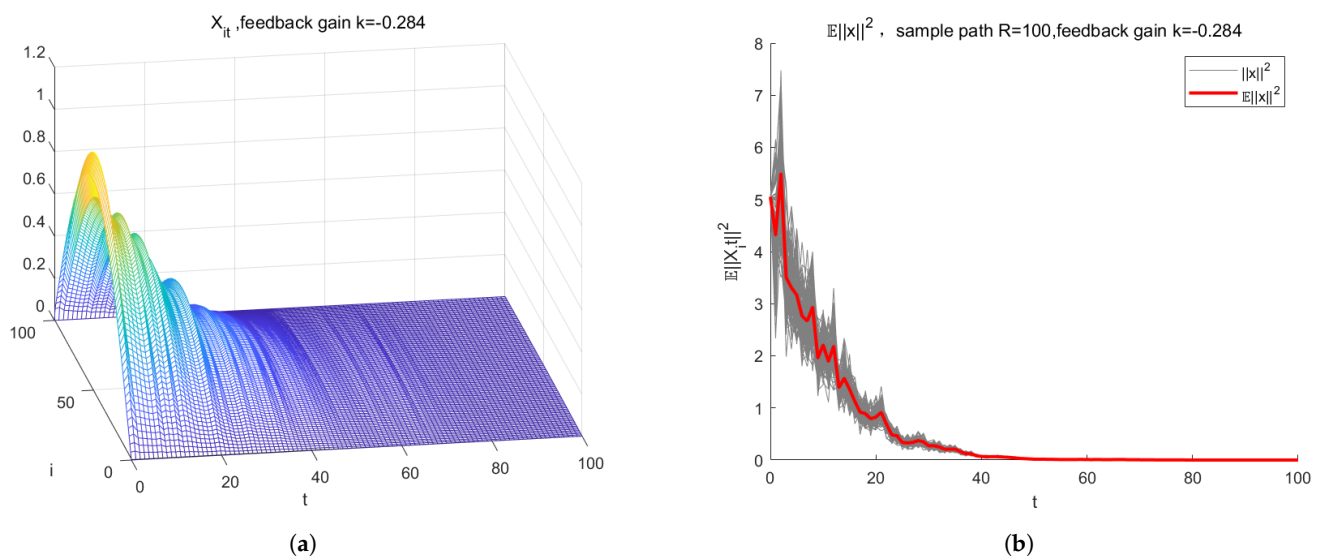


Figure 4. The stable system with feedback controller. (a) The state variable X_{it} ; (b) the state expectation $E||X_{it}||^2$.

To further analyze the convergence performance of different feedback gains, Figure 5 illustrates the state expectations $E||X_{it}||^2$ of the system under different feedback gains, and Table 1 summarizes the key metrics.

From Figure 5 and Table 1, we can learn that the smaller the ρ value is, the faster the convergence speed and the higher the convergence accuracy.

This subsection numerically verifies the core assertion of Theorem 2: when the feedback gain k satisfies $\rho < 1$, the system state converges, and its convergence performance strongly tied to ρ .

Remark: Implicit finite difference schemes have unconditional stability advantages in the numerical analysis of parabolic equations, but their implementation requires an iterative solver, which introduces significant computational overhead. In contrast, the Euler–Maruyama method running under explicit stability constraints is more convenient and suitable for this numerical simulation.

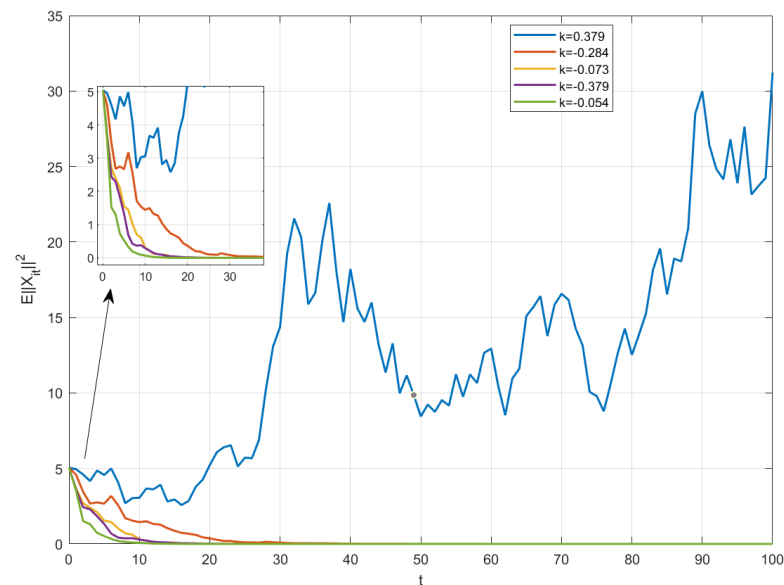


Figure 5. The $\mathbb{E}\|X_{it}\|^2$ with different feedback gains k .

Table 1. The convergence accuracy of the state expectations under different feedback gains k .

k	−0.284	−0.073	− 0.379	−0.054
ρ	0.9968	0.9498	0.9493	0.9484
Accuracy *	2.1483×10^{-5}	6.4620×10^{-13}	3.3759×10^{-15}	9.2385×10^{-17}

* Accuracy refers to convergence accuracy.

6. Conclusions

In this paper, we studied a class of linear parabolic stochastic partial difference systems and developed a simpler method to establish two sufficient conditions on the mean square stability and stabilization of the system (Theorems 1 and 2). The theoretical results are validated through numerical simulations. And in Section 5.2, further experiments by adjusting the feedback gains demonstrate that smaller values of ρ result in faster convergence rates and higher convergence accuracy, as detailed in Figure 5 and Table 1. Through rigorous mathematical proofs and systematic numerical experiments, this work verifies the validity of the proposed theorems. Nonlinear parabolic stochastic partial difference systems with delays will be discussed in the future.

Author Contributions: Conceptualization, X.D.; Data curation, S.Y.; Methodology, S.Y.; Software, S.Y.; Supervision, X.D.; Validation, S.Y.; Writing—original draft, S.Y.; Writing—review & editing, X.D., T.L. and Z.W. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China grant number [62363002].

Data Availability Statement: The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

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