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# Reliability Estimation in Multicomponent Stress-Strength Based on Inverse Weibull Distribution

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**Abstract:** The present study focuses on the multi-component stress-strength (MCSS) model based on inverse Weibull distribution (IWD). Both stress and strength are assumed to follow IWD with a common shape parameter. In such a system, reliability is obtained by the maximum likelihood (ML) method. The results are extracted using Monte Carlo simulation for comparing the performance of the reliability component $R_{s,k}$ using different sample sizes and different combinations of the parameters ($s$,$k$). The procedure is further illustrated through a real data set to show how the proposed technique may be employed to study the strength and stress of multicomponent model.

**Keywords:** inverse Weibull; multicomponent; stress-strength model; asymptotic confidence interval; maximum likelihood estimation

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## 1. Introduction

The stress-strength (SS) reliability of a system defines the probability that the system will function properly until the strength exceeds the stress. The basic underlying philosophy in reliability studies is to examine whether a part or a product can sustain a certain amount of stress under some conditions so that it can survive for a longer period. However, with the availability of highly sophisticated simulation techniques, researchers are now studying the stress-strength of more than one component simultaneously, commonly known as multicomponent SS models using varied probability distributions. In designing mechanical components, one comes across the study of SS. If the probability of strength is less than the probability of stress, then we study their behavior in SS studies. Within the reliability environment, the term SS was first used by Church and Harris [1]. Bhattacharyya and Johnson [2] observed that, in several practical scenarios, the performance of a system depends on more than one component and these components have their strengths. Multicomponent stress-strength (MCSS) models have great applications range from communication and industrial systems to logistic and military systems. For examples, an aircraft generally contains more than one engine ($k$) and assume that for takeoff at least $s$ ($1 \leq s \leq k$) engines are needed, see Hanagal [3], Turkkan and Pham-Gia [4], Serkan [5,6], Serkan and Funda [7], Shawky and Al-Gashgari [8], Pak et al. [9,10], and Rao et al. [11]. SS models have been discussed in the literature many a times. In a SS model, a unit operates if its strength exceeds the stress applied on it. Hence, the reliability $R$ is defined as the probability that the strength of the unit exceeds the stress it is subjected to, i.e., $R = P(X > Y)$, where $X$ is the random strength of the unit and $Y$ is the random stress applied to the unit. The reliability and its estimation have been well studied under different distributional assumptions on $X$ and $Y$. Theory and applications of SS has been thoroughly discussed by Kotz et al. [12].

The series and parallel systems are special cases of a general class of systems called $k$-out-of-$n$ systems. A system belonging to this class can be one of two types: (i) a system that fails with the failure of the kth component, denoted by $k$-out-of-$n$: F system; or (ii) a system that functions if at least $k$ components are working; and is denoted by $k$-out-of-$n$:

G system. The present study will focus on the second type of study in which the failure distribution of the components will follow IWD.

Suppose a system with “$k$” identical components functions if $s$ $(1 \leq s \leq k)$ or more of the components operate simultaneously. In its operating environment, the system is subjected to a common stress $Y$, which is a random variable with CDF $G(.)$. Bhattacharya and Roychowdhury [13] gave interesting examples of a multi-component system. Assume that the strengths of the components are independent and identically distributed random variables with CDF $F(.)$ and subjected to the common random stress $Y$ having CDF $G(.)$. Then the system reliability, i.e., the probability that the system does not fail, $R_{s,k}$ given by

$$\begin{aligned} R_{s,k} &= P(\text{at least } s \text{ of the } X_1,\ X_2,\ \ldots,\ X_k \text{ exceed } Y) \\ &= \textstyle\sum_{i=s}^{k} \left( \begin{array}{c} k \\ i \end{array} \right) [P(X_i > Y]^i [P(X_i \leq Y]^{k-i} \\ &= \textstyle\sum_{i=s}^{k} \left( \begin{array}{c} k \\ i \end{array} \right) \int_{-\infty}^{\infty} [1 - F(y)]^i [F(y)]^{(k-i)}\, dG(y), \end{aligned} \tag{1}$$

where $X_1, X_2, \ldots, X_n$ are independently and identically distributed (*iid*) with common distribution function $F(x)$, the system is subject to common random stress $Y$ with distribution function $G(y)$. The probability given in Equation (1) is called reliability in an MCSS model given by Bhattacharyya and Johnson [2], also see Ebrahimi [14], Pandey and BorhanUddin [15], Paul and BorhanUddin [16], Rao and Kantam [17], Rao [18–20], Hassan and Basheikh [21], Dey et al. [22], Kızılaslan [23], Badr et al. [24], and Akgül [25].

Weibull distribution was initially presented by Weibull in 1935; this distribution does not provide a satisfactory parametric fit for lifetime distributions with non-monotone failure rates, such as the unimodal failure rate functions, which are common in reliability and biological studies. In this case, instead, it is recommended to use a special case of the Weibull distribution, the IWD. If “$Z$” denotes a random variable (r.v) from the Weibull model, and if we define $X$ as follows $X = 1/Z$, then r.v $X$ is said to follow IWD. Extensive work has been done on the IWD; see, for example, Keller et al. [26], Calabria, and Pulcini [27–30] provide an interpretation of the IWD in the context of the load strength relationship for a component. Maswadah [31,32] has fitted IWD to the flood data reported in Dumonceaux and Antle [33], for more details see, e.g., Murthy et al. [34] and Bi and Gui [35]. IWD is a very flexible distribution model that approaches different distributions when its shape parameter varies. When $\beta = 1$, the distribution is the same as the inverse exponential distribution; when $\beta < 1$, it follows the inverse gamma distribution and when $\beta = 2$, it is known as inverse Rayleigh distribution. The IWD model can be used in reliability analysis. It can be successful in modeling life for several devices and variables such as electron tubes, automotive radiators, fatigue in textiles, the marketing life expectancy of drugs, etc.

A random variable $X$ is said to have a two-parameter IWD if it has the following probability density function (PDF):

$$f(x : \alpha, \lambda) = \alpha\lambda x^{-(\lambda+1)} e^{-\alpha x^{-\lambda}}, x \geq 0, \alpha, \lambda > 0 \tag{2}$$

where $\alpha > 0$ is called scale parameter and $\lambda > 0$ is called the shape parameter of this family and it will be denoted by IWD $(\alpha; \lambda)$. If $X$~IWD $(\alpha; \lambda)$, then the cumulative distribution function (CDF),

$$F(x : \alpha, \lambda) = e^{-\alpha x^{-\lambda}}, x \geq 0, \alpha, \lambda > 0. \tag{3}$$

IWD takes many different names such as Fréchet distribution (Johnson et al. [36]) and complementary Weibull distribution (Drapella [37]). For the theoretical analysis of the IWD, see Khan et al. [38]. The utility of the IWD for modeling reliability data was further discussed by researching the failure of mechanical components subject to degradation, see Keller et al. [26]. The ability of IWD to model failure rates is quite common in reliability and biological studies, see Bi and Gui [35] and Li and Hao [39]. In the field of engineering sciences, several lifetime distributions are used to study stress-strength reliability models, and these models are frequently used to estimate the system reliability R, see Yadav et al. [40]. The role of dependent evidence in system reliability evaluation and a full Bayesian approach that is applied to various system reliability models was studied by Yang et al. [41]. Graves and Hamada [42] worked on the likelihood for simultaneous failure time data when monitoring is stopped when the system fails and this method is based on the reliability structure of the system, listing all possible events consistent with the simultaneous data and calculating their contributions to the likelihood. Estimation of reliability in multicomponent stress-strength based on two parameter exponentiated Weibull distribution was studied by Rao et al. [43]. However, scrolling through the literature the authors could not come across. Estimation of reliability in multicomponent stress-strength based on IWD. The work of Palumbo and Pallotta [44] also motivates to study IWD. In their work, authors considered real data sets for four generative mechanisms following the principles "deterioration", "stress-strength", "shocks", and "extreme maximum value", and observed that IWD is more reliable as compared to inverse Gaussian and lognormal distributions One main advantage of considering IWD from computational aspect is that the cumulative distribution function (CDF) of the IWD admits a closed form.

The main objective of this paper is to study the reliability in an MCSS based on $X$, $Y$ being two independent random variables, where $X$~IWD ($\alpha$; $\lambda$), and $Y$~IWD ($\beta$; $\lambda$) having a common shape parameter $\lambda$. Going through the relevant literature, one seldom comes across work on the estimation of the survival probability in an MCSS system when stress follows a two-parameter IWD hence, the rationale for the current study emerges. In Section 2, the expression for the $R_{s,k}$ is derived, and a procedure for estimating it is developed. For the said purpose, ML estimators are employed to obtain the asymptotic distribution and confidence intervals for $R_{s,k}$. Comparison based on small samples using Monte Carlo simulation is carried out in Section 3, along with using real-life data for illustrating the estimation process. Section 4 briefly concludes the study.

## 2. Maximum Likelihood Estimation of $R_{s,k}$

Now, we assume $(X_1,\ X_2, \ldots,\ X_n)$ is a random sample of strength variables following IWD ($\alpha$; $\lambda$) with common distribution function $F(x)$ and $(Y_1, Y_2, \ldots, Y_m)$ is a random sample of stress variables following IW ($\beta$; $\lambda$) with common distribution function $G(y)$. The probability given in Equation (1) is called reliability in an MCSS model given by Bhattacharyya and Johnson [2]. Thus, the reliability in multicomponent stress-strength for two-parameter IWD using (1) is

$$\begin{aligned} R_{s,k} &= \sum_{i=s}^{k} \binom{k}{i} \int_0^\infty \left[1-e^{-\alpha y^{-\lambda}}\right]^i \left[e^{-\alpha y^{-\lambda}}\right]^{k-i} \cdot \beta\lambda y^{-(\lambda+1)} e^{-\beta y^{-\lambda}} dy \\ &= \gamma \sum_{i=s}^{k} \binom{k}{i} \int_0^1 (1-Z)^i Z^{k+\gamma-i-1} dz \\ &= \gamma \sum_{i=s}^{k} \binom{k}{i} \beta(k+\gamma-i, i+1), \end{aligned}$$

where $z = e^{-\alpha y^{-\lambda}}$ and $\gamma = \frac{\beta}{\alpha}$.

After simplification, we get

$$R_{s,k} = \gamma \sum_{i=s}^{k} \frac{k!}{(k-i)!} \left[\prod_{j=0}^{i} (k+\gamma-j)\right]^{-1} \tag{4}$$

The reliability can also be written as

$$R_{s,k} = \frac{\Gamma(k+1)\Gamma(k+\gamma-s+1)}{\Gamma(k-s+1)\Gamma(k+\gamma+1)}. \tag{5}$$

The probability in (4) or (5) is called reliability in an MCSS. If $\alpha$ and $\beta$ are unknown, it is necessary to estimate $\alpha$ and $\beta$ to estimate $R_{s,k}$ using the ML method. Once the unknown parameters are obtained, then $R_{s,k}$ can be computed using Equation (4) or (5).

For a random sample of size n from strength component and a random sample of size m is available from stress component, the likelihood function of these observed samples for $\theta = (\alpha, \beta, \lambda)$ can be written as:

$$\begin{aligned} \mathrm{L}(X, Y; \alpha, \beta, \lambda) &= \prod_{i=1}^{n} f(x_i, \alpha, \lambda) \prod_{j=1}^{m} g(y_j, \beta, \lambda) \\ &= \alpha^n \beta^m \lambda^{n+m} \prod_{i=1}^{n} x_i^{-(\lambda+1)} e^{-\alpha \sum x_i^{-\lambda}} \prod_{j=1}^{m} y_j^{-(\lambda+1)} e^{-\beta \sum y_j^{-\lambda}}. \end{aligned}$$

Thus, the log-likelihood function is

$$\begin{aligned} \ell = lnL = n\, ln\alpha + m\, ln\beta + (n+m)\, ln\lambda - (\lambda+1) \sum_{i=1}^{n} lnx_i - \alpha \sum_{i=1}^{n} x_i^{-\lambda} \\ -(\lambda+1) \sum_{j=1}^{m} lny_j - \beta \sum_{j=1}^{m} y_j^{-\lambda}. \end{aligned} \tag{6}$$

The MLEs of $\alpha, \beta,$ and $\lambda$ say $\hat{\alpha}, \hat{\beta}$, and $\hat{\lambda}$ respectively can be obtained by the solution of

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} x_i^{-\hat{\lambda}}}, \tag{7}$$

$$\hat{\beta} = \frac{m}{\sum_{j=1}^{m} y_j^{-\hat{\lambda}}}. \tag{8}$$

Using (7) and (8), the MLE for $\lambda$ is the solution of the nonlinear equation

$$g(\hat{\lambda}) = \frac{n+m}{\hat{\lambda}} - \left( \sum_{i=1}^{n} lnx_i + \sum_{j=1}^{m} lny_j \right) + \frac{n}{\sum_{i=1}^{n} x_i^{-\hat{\lambda}}} \sum_{i=1}^{n} x_i^{-\hat{\lambda}} lnx_i + \frac{m}{\sum_{j=1}^{m} y_j^{-\hat{\lambda}}} \sum_{j=1}^{m} y_j^{-\hat{\lambda}} lny_j = 0. \tag{9}$$

Once we obtain $\hat{\lambda}$, thus, $\hat{\alpha}$ and $\hat{\beta}$ can be calculated from (7) and (8), respectively. Therefore, the MLE of $R_{s,k}$ becomes:

$$\hat{R}_{s,k} = \hat{\gamma} \sum_{i=s}^{k} \frac{k!}{(k-i)!} \left[ \prod_{j=0}^{i} (k + \hat{\gamma} - j) \right]^{-1}, \tag{10}$$

or

$$\hat{R}_{s,k} = \frac{\Gamma(k+1)\Gamma(k+\hat{\gamma}-s+1)}{\Gamma(k-s+1)\Gamma(k+\hat{\gamma}+1)}, \hat{\gamma} = \frac{\hat{\beta}}{\hat{\alpha}}. \tag{11}$$

*Asymptotic Confidence Intervals*

To construct the asymptotic confidence intervals (CI, *s*), we need to compute the asymptotic variance (*AV*) of the reliability coefficient $\hat{R}_{s,k}$. The asymptotic variance $\hat{R}_{s,k}$, given by Rao [40], is:

$$AV(\hat{R}_{s,k}) = Var(\hat{\alpha}) \left( \frac{\partial \hat{R}_{s,k}}{\partial \hat{\alpha}} \right)^2 + Var(\hat{\beta}) \left( \frac{\partial \hat{R}_{s,k}}{\partial \hat{\beta}} \right)^2. \tag{12}$$

To obtain the asymptotic confidence interval for $\hat{R}_{s,k}$, we need asymptotic variances of the MLE, and they are given by:

$$Var(\hat{\alpha}) = \left[ E(-\frac{\partial^2 lnL}{\partial \hat{\alpha}^2}) \right]^{-1} = \frac{\hat{\alpha}^2}{n} \text{ and } Var(\hat{\beta}) = \left[ E(-\frac{\partial^2 lnL}{\partial \hat{\beta}^2}) \right]^{-1} = \frac{\hat{\beta}^2}{m}. \tag{13}$$

In addition, from (4) or (5), we have

$$\frac{\partial R_{s,k}}{\partial \alpha} = -\frac{\gamma\Gamma(k+1)\Gamma(k+\gamma-s)}{\alpha\Gamma(k-s+1)\Gamma(k+\gamma+1)}\left[1-(k+\gamma-s)\left(H_{k+\gamma}-H_{k+\gamma-s-1}\right)\right], \quad (14)$$

and

$$\frac{\partial R_{s,\,k}}{\partial \beta} = \frac{\Gamma(k+1)\Gamma(k+\gamma-s)}{\alpha\Gamma(k-s+1)\Gamma(k+\gamma+1)}\left[1-(k+\gamma-s)\left(H_{k+\gamma}-H_{k+\gamma-s-1}\right)\right], \quad (15)$$

where $H_r = \sum_{t=1}^{r} t^{-1}$ is the Harmonic series. Using the values from (13), (15), and (16) at $\alpha = \hat{\alpha}$ and $\beta = \hat{\beta}$ in (12), the asymptotic variance of $\hat{R}_{s,k}$ as

$$AV\left(\hat{R}_{s,k}\right) = \left(\frac{\hat{\gamma}\Gamma(k+1)\Gamma(k+\hat{\gamma}-s)}{\Gamma(k-s+1)\Gamma(k+\hat{\gamma}+1)}\right)^2\left[1-(k+\hat{\gamma}-s)\sum_{t=k+\hat{\gamma}-s}^{k+\hat{\gamma}} t^{-1}\right]^2\left(\frac{1}{n}+\frac{1}{m}\right). \quad (16)$$

To avoid extra effort for the derivation of the $\hat{R}_{s,k}$, derivatives of $\hat{R}_{s,k}$ are worked for ($s$, $k$) = (1, 3) and (3, 5) separately, and after simplification they are

$$\hat{R}_{1,3} = \frac{3}{3+\hat{\gamma}}, \frac{\partial \hat{R}_{1,3}}{\partial \hat{\alpha}} = \frac{3\hat{\gamma}}{\hat{\alpha}(3+\hat{\gamma})^2}, \frac{\partial \hat{R}_{1,3}}{\partial \hat{\beta}} = \frac{-3}{\hat{\alpha}(3+\hat{\gamma})^2}, \hat{R}_{3,5} = \frac{60}{(5+\hat{\gamma})(4+\hat{\gamma})(3+\hat{\gamma})},$$

$$\frac{\partial \hat{R}_{3,5}}{\partial \hat{\alpha}} = \frac{60\left(3\hat{\gamma}^2+24\hat{\gamma}+47\right)\hat{\gamma}}{\hat{\alpha}\left[(5+\hat{\gamma})(4+\hat{\gamma})(3+\hat{\gamma})\right]^2}, \ \frac{\partial \hat{R}_{3,5}}{\partial \hat{\beta}} = \frac{-60\left(3\hat{\gamma}^2+24\hat{\gamma}+47\right)}{\hat{\alpha}\left[(5+\hat{\gamma})(4+\hat{\gamma})(3+\hat{\gamma})\right]^2}.$$

Therefore,

$$AV\left(\hat{R}_{1,3}\right) = \frac{9\hat{\gamma}^2}{(3+\hat{\gamma})^4}\left(\frac{1}{n}+\frac{1}{m}\right), AV\left(\hat{R}_{3,5}\right) = \left(\frac{60\hat{\gamma}\left(3\hat{\gamma}^2+24\hat{\gamma}+47\right)^2}{\left[(5+\hat{\gamma})\ (4+\hat{\gamma})(3+\hat{\gamma})\right]^2}\right)^2\left(\frac{1}{n}+\frac{1}{m}\right).$$

As $n \to \infty$, $m \to \infty$ $\frac{\hat{R}_{s,k}-R_{s,k}}{\sqrt{AV\left(\hat{R}_{s,k}\right)}} \xrightarrow{d} N(0,1)$. Then, the asymptotic 95% confidence interval (CI) of the system reliability $R_{s,k}$ is given by

$$\hat{R}_{s,k} \mp 1.96\sqrt{AV\left(\hat{R}_{s,k}\right)}$$

and hence the asymptotic confidence 95% confidence interval for $R_{1,3}$ is given by

$$\hat{R}_{1,3} \mp 1.96\frac{3\hat{\gamma}}{(3+\hat{\gamma})^2}\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)}, \text{where } \hat{\gamma} = \frac{\hat{\beta}}{\hat{\alpha}}$$

Similarly, the asymptotic confidence 95% confidence interval for $R_{3,5}$ is given by

$$\hat{R}_{3,5} \mp 1.96\frac{60\hat{\gamma}\left(3\hat{\gamma}^2+24\hat{\gamma}+47\right)}{\left[(5+\hat{\gamma})(4+\hat{\gamma})(3+\hat{\gamma})\right]^2}\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)}, \qquad \text{where}\hat{\gamma} = \frac{\hat{\beta}}{\hat{\alpha}}.$$

## 3. Simulation Study and Data Analysis

In this section, a simulation study is carried out coupled with the application of the proposed technique on a real-life data set to estimate reliability coefficient when samples are available from IWD.

### 3.1. Simulation Study

Based on the Monte Carlo simulation performance of the $R_{s,k}$ is compared using different sample sizes. Ten thousand random samples of size 10 (5) 30 each from the stress population and strength population are generated. This section investigates some empirical results based on Monte Carlo simulations to study the behavior of the proposed methods using different sample sizes. Sample sizes $(n, m)$ = (10, 10), (15, 15), (20, 20), (25, 25), and (30, 30) with 1000 iterations are generated for $(\alpha, \beta)$ = (3.0, 1.5), (2.5, 1.5), (2.0, 1.5), (1.5, 1.5), (1.5, 2.0), (1.5, 2.5), and (1.5, 3.0) and in all cases we take $\lambda = 2$. The same procedure is repeated for both combinations of $(s, k)$ = (1, 3) and (3, 5). The true value of reliability in multicomponent stress-strength with the given combination for given parameter values are shown in Table 1.

**Table 1.** True Values of $R_{s,k}$ for various combination of parameters.

| **(s, k)** | **(α, β)** | | | | | | |
|---|---|---|---|---|---|---|---|
| | **(3, 1.5)** | **(2.5, 1.5)** | **(2, 1.5)** | **(1.5, 1.5)** | **(1.5, 2)** | **(1.5, 2.5)** | **(1.5, 3)** |
| (1, 3) | 0.857143 | 0.833333 | 0.800000 | 0.750000 | 0.692308 | 0.642857 | 0.600000 |
| (3, 5) | 0.692641 | 0.646998 | 0.585812 | 0.500000 | 0.409919 | 0.340330 | 0.285714 |

Thus, the true values for reliability in the MCSS model for the IW decrease as $\beta$ increases for fixed values of $\alpha$, the same pattern is witnessed when $\beta$ is fixed and $\alpha$ decreases, and the same pattern emerges for both combinations of $(s, k)$. The MLE of $\alpha$ and $\beta$ are obtained from (6) and (7), while the MLE of $\lambda$ is obtained as the solution of the non-linear equation in (8). The MLE thus obtained is used to calculate the reliability of a multi-component system for $(s, k)$ = (1, 3) and (3, 5).

The average bias (ABias), the average mean square errors (AMSE), the average standard errors (ASE), the average length of the simulated 95% confidence intervals (ALCI), and the average coverage probabilities of the simulated 95% confidence intervals (ACP) of the reliability estimate $\hat{R}_{s,k}$ over the 10,000 samples are presented in Table 2.

**Table 2.** The AveBias, AMSE, ASE, ALCI, and ACP of $\hat{R}_{s,k}$.

| $n$ | $m$ | $(\alpha, \beta)$ | **ABias** | | **AMSE** | | **ASE** | | **ALCI** | | **ACP** | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $R_{1,3}$ | $R_{3,5}$ | $R_{1,3}$ | $R_{3,5}$ | $R_{1,3}$ | $R_{3,5}$ | $R_{1,3}$ | $R_{3,5}$ | $R_{1,3}$ | $R_{3,5}$ |
| 10 | 10 | (3, 1.5) | −0.0564 | −0.1044 | 0.0045 | 0.0150 | 0.0708 | 0.1258 | 0.2774 | 0.4931 | 0.9989 | 0.9951 |
| 15 | 15 | | −0.0561 | −0.1032 | 0.0040 | 0.0134 | 0.0579 | 0.1030 | 0.2268 | 0.4039 | 0.9966 | 0.9863 |
| 20 | 20 | | −0.0550 | −0.1021 | 0.0037 | 0.0125 | 0.0500 | 0.0893 | 0.1959 | 0.3501 | 0.9905 | 0.9720 |
| 25 | 25 | | −0.0548 | −0.1010 | 0.0045 | 0.0120 | 0.0447 | 0.0799 | 0.1752 | 0.3131 | 0.9753 | 0.9458 |
| 30 | 30 | | −0.0544 | −0.1011 | 0.0034 | 0.0116 | 0.0408 | 0.0730 | 0.1598 | 0.2861 | 0.9526 | 0.9102 |
| 10 | 10 | (2.5, 1.5) | −0.0473 | −0.0832 | 0.0036 | 0.0111 | 0.0746 | 0.1293 | 0.2924 | 0.5068 | 0.9986 | 0.9956 |
| 15 | 15 | | −0.0468 | −0.0826 | 0.0031 | 0.0097 | 0.0610 | 0.1060 | 0.2390 | 0.4154 | 0.9984 | 0.9957 |
| 20 | 20 | | −0.0452 | −0.0815 | 0.0027 | 0.0088 | 0.0526 | 0.0919 | 0.2062 | 0.3602 | 0.9977 | 0.9900 |
| 25 | 25 | | −0.0450 | −0.0810 | 0.0026 | 0.0083 | 0.0470 | 0.0823 | 0.1844 | 0.3225 | 0.9960 | 0.9851 |
| 30 | 30 | | −0.0447 | −0.0808 | 0.0025 | 0.0079 | 0.0429 | 0.0752 | 0.1682 | 0.2946 | 0.9899 | 0.9763 |
| 10 | 10 | (2, 1.5) | −0.0303 | −0.0509 | 0.0024 | 0.0068 | 0.0786 | 0.1327 | 0.3082 | 0.5203 | 0.9992 | 0.9983 |
| 15 | 15 | | −0.0293 | −0.0501 | 0.0018 | 0.0053 | 0.0642 | 0.1088 | 0.2516 | 0.4266 | 0.9994 | 0.9984 |
| 20 | 20 | | −0.0291 | −0.0498 | 0.0016 | 0.0047 | 0.0556 | 0.0944 | 0.2180 | 0.3701 | 0.9992 | 0.9981 |
| 25 | 25 | | −0.0288 | −0.0497 | 0.0014 | 0.0041 | 0.0497 | 0.0846 | 0.1950 | 0.3316 | 0.9994 | 0.9981 |
| 30 | 30 | | −0.0289 | −0.0490 | 0.0013 | 0.0038 | 0.0454 | 0.0772 | 0.1781 | 0.3028 | 0.9989 | 0.9974 |
| 10 | 10 | (1.5, 1.5) | −0.0023 | −0.0004 | 0.0016 | 0.0041 | 0.0837 | 0.1360 | 0.3280 | 0.5331 | 0.9989 | 0.9989 |
| 15 | 15 | | −0.0095 | −0.0019 | 0.0010 | 0.0028 | 0.0682 | 0.1116 | 0.2673 | 0.4375 | 0.9993 | 0.9993 |
| 20 | 20 | | −0.0014 | −0.0002 | 0.0008 | 0.0022 | 0.0593 | 0.0968 | 0.2324 | 0.3794 | 0.9995 | 0.9997 |
| 25 | 25 | | −0.0010 | −0.0003 | 0.0006 | 0.0017 | 0.0530 | 0.0867 | 0.2077 | 0.3399 | 0.9995 | 0.9997 |
| 30 | 30 | | −0.0004 | −0.0007 | 0.0005 | 0.0014 | 0.0483 | 0.0793 | 0.1895 | 0.3107 | 0.9995 | 0.9998 |
| 10 | 10 | (1.5, 2) | 0.0321 | 0.0525 | 0.0030 | 0.0073 | 0.0884 | 0.1378 | 0.3465 | 0.5402 | 0.9928 | 0.9959 |
| 15 | 15 | | 0.0332 | 0.0517 | 0.0024 | 0.0058 | 0.0723 | 0.1132 | 0.2833 | 0.4436 | 0.9913 | 0.9958 |
| 20 | 20 | | 0.0328 | 0.0522 | 0.0020 | 0.0050 | 0.0627 | 0.0983 | 0.2459 | 0.3852 | 0.9932 | 0.9963 |
| 25 | 25 | | 0.0336 | 0.0521 | 0.0019 | 0.0045 | 0.0561 | 0.0880 | 0.2198 | 0.3451 | 0.9888 | 0.9954 |
| 30 | 30 | | 0.0332 | 0.0524 | 0.0017 | 0.0043 | 0.0513 | 0.0805 | 0.2009 | 0.3154 | 0.9887 | 0.9928 |

**Table 2.** *Cont.*

| $n$ | $m$ | $(\alpha, \beta)$ | ABias | | AMSE | | ASE | | ALCI | | ACP | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $R_{1,3}$ | $R_{3,5}$ | $R_{1,3}$ | $R_{3,5}$ | $R_{1,3}$ | $R_{3,5}$ | $R_{1,3}$ | $R_{3,5}$ | $R_{1,3}$ | $R_{3,5}$ |
| 10 | 10 | (1.5, 2.5) | 0.0623 | 0.0907 | 0.0061 | 0.0131 | 0.0920 | 0.1383 | 0.3607 | 0.5422 | 0.9769 | 0.9877 |
| 15 | 15 | | 0.0622 | 0.0906 | 0.0053 | 0.0115 | 0.0754 | 0.1136 | 0.2956 | 0.4454 | 0.9645 | 0.9855 |
| 20 | 20 | | 0.0615 | 0.0902 | 0.0049 | 0.0106 | 0.0655 | 0.0987 | 0.2568 | 0.3870 | 0.9536 | 0.9777 |
| 25 | 25 | | 0.0622 | 0.0902 | 0.0048 | 0.0101 | 0.0586 | 0.0885 | 0.2296 | 0.3468 | 0.9324 | 0.9677 |
| 30 | 30 | | 0.0621 | 0.0901 | 0.0046 | 0.0098 | 0.0535 | 0.0808 | 0.2098 | 0.3169 | 0.9121 | 0.9491 |
| 10 | 10 | (1.5, 3) | 0.0872 | 0.1194 | 0.0101 | 0.0194 | 0.0950 | 0.1380 | 0.3725 | 0.5409 | 0.9443 | 0.9787 |
| 15 | 15 | | 0.0867 | 0.1186 | 0.0091 | 0.0175 | 0.0780 | 0.1134 | 0.3056 | 0.4444 | 0.9136 | 0.9605 |
| 20 | 20 | | 0.0868 | 0.1179 | 0.0088 | 0.0165 | 0.0676 | 0.0985 | 0.2651 | 0.3860 | 0.8618 | 0.9325 |
| 25 | 25 | | 0.0871 | 0.1176 | 0.0086 | 0.0159 | 0.0605 | 0.0883 | 0.2373 | 0.3461 | 0.8008 | 0.8984 |
| 30 | 30 | | 0.0867 | 0.1174 | 0.0083 | 0.0155 | 0.0553 | 0.0807 | 0.2169 | 0.3163 | 0.7391 | 0.8356 |

The AMSE decreases as the sample size increases for both combinations of the parameters. In addition, it is witnessed that the direction of bias is both positive and negative in both situations of $(s, k)$. The values of ABias are positive for fixed values of "$\alpha$" and increasing values of "$\beta$" and vice versa the values of ABias are negative. The values of the ASE decrease as the sample size increases and the same pattern follows for the same combination of the parameters $(s, k)$. The ALCI also decreases as the sample size increases. The coverage probability is close to the nominal value in all cases but slightly less than 0.95 in some cases e.g., when $\alpha = 1.5$ and $\beta = 3.0$ and sample sizes 25 and 30. Overall, the performance of the confidence interval (CI) is quite good for all combinations of parameters.

### 3.2. Real Data Application

For an illustration of how the proposed methods will work two real data sets are analyzed. The first and second data sets, see Surles and Padgett [45], Kundu and Gupta [46], and Bi and Gui [35], are:

Strength data ($X$):

$X$ = {0.762, 0.761, 0.676, 0.644, 0.588, 0.555, 0.537, 0.536, 0.514, 0.509, 0.501, 0.499, 0.495, 0.493, 0.487, 0.485, 0.477, 0.467, 0.459, 0.450, 0.446, 0.444, 0.441, 0.440, 0.440, 0.435, 0.435, 0.424, 0.420, 0.420, 0.412, 0.411, 0.411, 0.404, 0.402, 0.398, 0.398, 0.394, 0.392, 0.390, 0.389, 0.387, 0.380, 0.380, 0.379, 0.378, 0.373, 0.371, 0.367, 0.361, 0.361, 0.357, 0.356, 0.355, 0.354, 0.351, 0.347, 0.339, 0.332, 0.326, 0.324, 0.324, 0.323, 0.320, 0.309, 0.291, 0.279, 0.279}

Stress data ($Y$):

$Y$ = {0.526, 0.469, 0.454, 0.449, 0.443, 0.426, 0.424, 0.417, 0.409, 0.407, 0.404, 0.397, 0.397, 0.396, 0.395, 0.388, 0.383, 0.382, 0.382, 0.381, 0.376, 0.374, 0.365, 0.365, 0.350, 0.343, 0.342, 0.340, 0.340, 0.336, 0.334, 0.330, 0.320, 0.319, 0.318, 0.311, 0.310, 0.309, 0.308, 0.306, 0.306, 0.304, 0.300, 0.299, 0.296, 0.293, 0.291, 0.286, 0.286, 0.283, 0.281, 0.281, 0.276, 0.260, 0.258, 0.257, 0.252, 0.249, 0.248, 0.237, 0.228, 0.199}

Initially, we have fitted IW distribution to both the data sets to see whether the data follows IW Model or not. The summary measures for the above data sets are shown in Table 3.

From Table 3, both data sets are positively skewed, and the variables follow IW distribution as the $p$-value of the KS test for both variables exceeds 0.05.

For the MCSS model when data follows the IW distribution, using an iterative process, the MLE of $\lambda$ using (8), and the MLEs of $\alpha$ and $\beta$ are obtained by substituting MLE of $\lambda$ in (6) and (7). The final estimates are $\hat{\lambda} = 5.2605$, $\hat{\alpha} = 0.0060$, and $\hat{\beta} = 0.00179$. Based on the estimates of $\alpha$ and $\beta$, the MLE of $R_{s,k}$ becomes $\hat{R}_{1,3} = 0.910125$ and $\hat{R}_{3,5} = 0.799968$. The 95% confidence interval for $R_{1,3}$ become (0.882187, 0.938062) and for $R_{3,5}$ become (0.741289, 0.858648).

**Table 3.** Summary Measures for Both Data Sets.

| | Strength ($X$) | Stress ($Y$) |
|---|---|---|
| Mean | 0.42688 | 0.33987 |
| Median | 0.40400 | 0.33400 |
| Standard Deviation | 0.09936 | 0.06703 |
| Standard Error | 0.01196 | 0.00844 |
| Skewness | 1.38091 | 0.31303 |
| Kurtoses | 5.38230 | 2.67273 |
| $\hat{\alpha}$, $\hat{\beta}$ | 0.00469 | 0.00235 |
| $\hat{\lambda}$ | 5.50125 | 5.04997 |
| Log-Likelihood | 71.12399 | 76.47376 |
| KS Test Statistic | 0.05329 | 0.08753 |
| KS Test $p$-value | 0.98371 | 0.68698 |

Figures 1 and 2 depict the goodness of fit of IWD for the Strength ($X$) and Stress ($Y$) along with the histograms fitted by the estimated pdfs, and also P-P plot, Q-Q plot, and comparison of Empirical and theoretical CDFs.

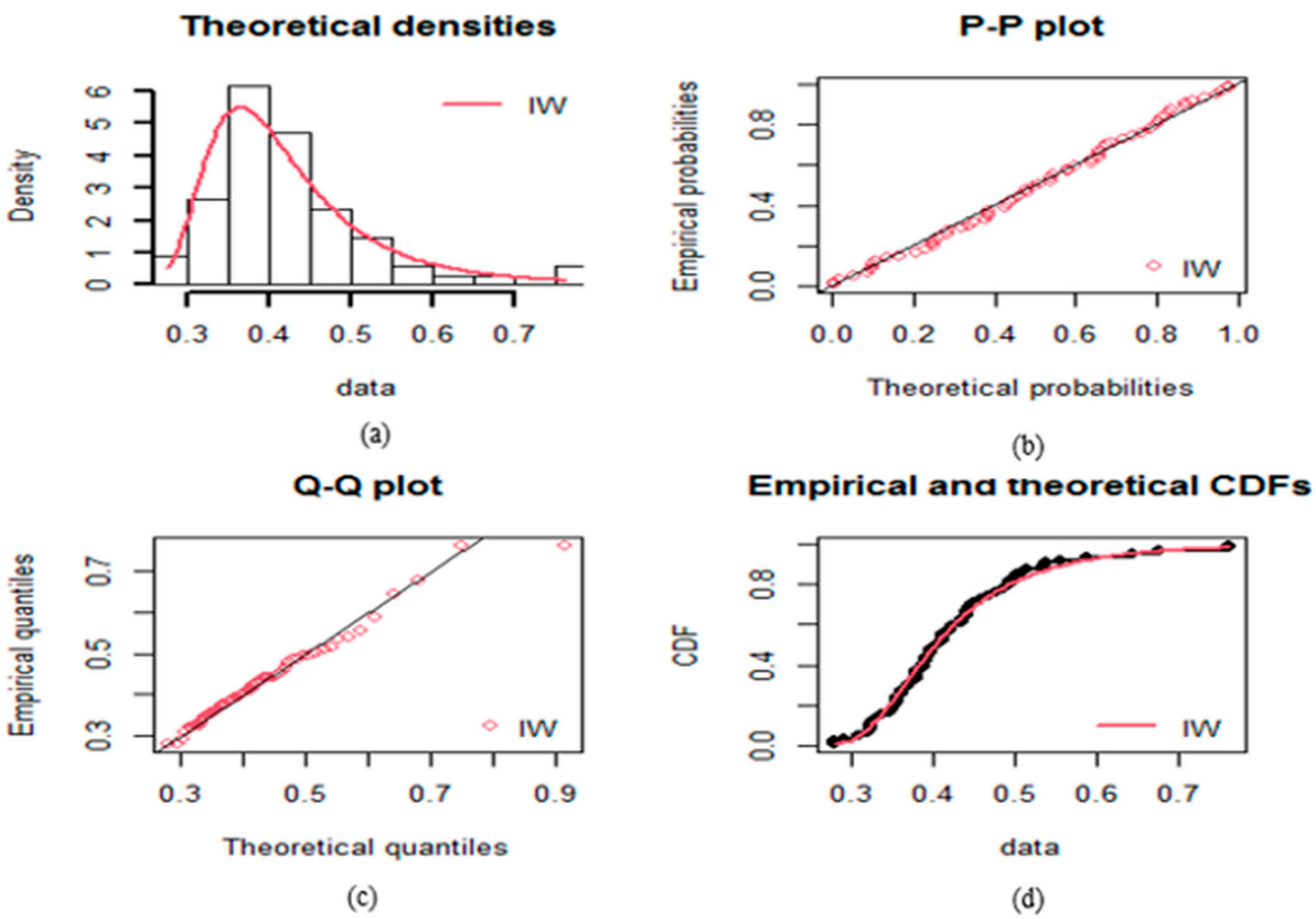


**Figure 1.** Examples of fits of IWD for the Strength ($X$) dataset: (**a**) histogram fitted by the estimated pdf, (**b**) empirical cdf fitted by the estimated cdf, (**c**) P-P plot, and (**d**) Q-Q plot.

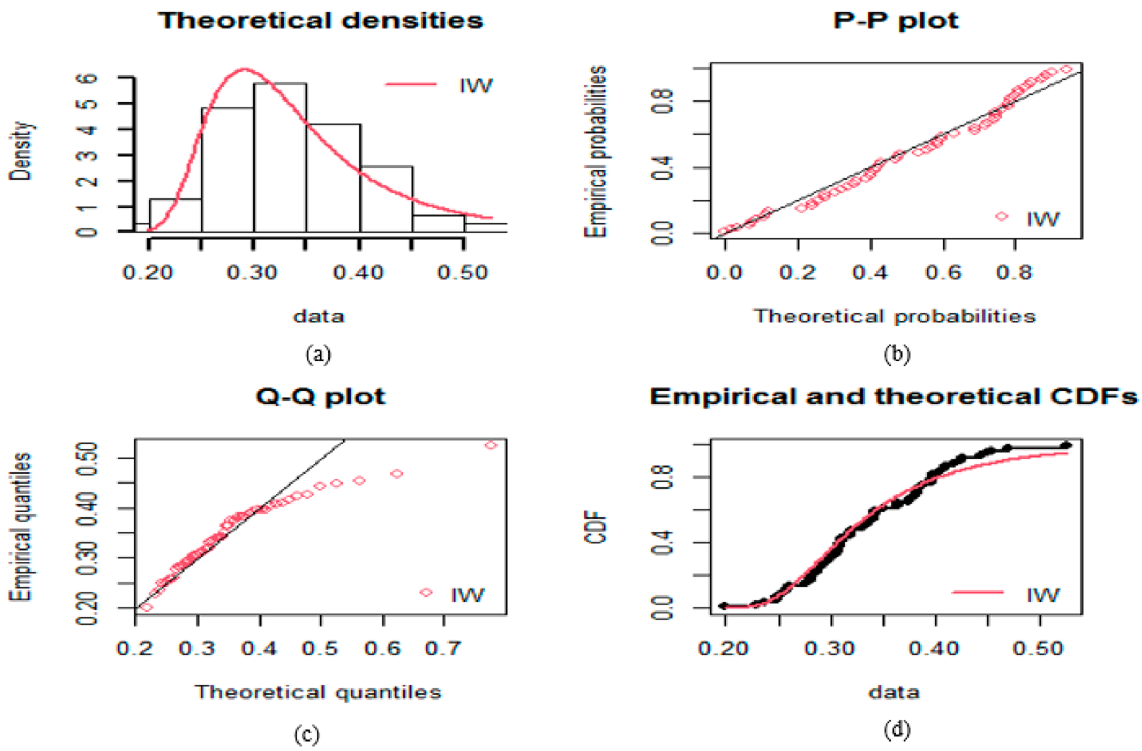


**Figure 2.** Examples of fits of IWD for the Stress ($Y$) dataset: (**a**) histogram fitted by the estimated pdf, (**b**) empirical cdf fitted by the estimated cdf, (**c**) P-P plot, and (**d**) Q-Q plot.

## 4. Conclusions

In this article, we studied the MCSS for two parameters IWD when both the stress and strength follow IWD. Asymptotic confidence intervals were computed using the ML method. From the simulation study, it is observed that, the average bias and AMSE decrease as the sample size increases for both values of ($s$, $k$). It verifies the consistency property of the MLE of $R_{s,k}$. In addition, the lengths of asymptotic confidence intervals of $R_{s,k}$ decrease as the sample size increases. Overall, the performance of the confidence interval is quite good for all combinations of parameters. Regarding the sizes of record value samples for the strengths and stress variables ($n$, $m$), it is observed that the MSEs and the lengths of asymptotic confidence intervals tend to decrease as ($n$, $m$) increases. The ACP, it becomes closer to the nominal value for all sets of parameters considered in the study except when $\alpha = 1.5$ and $\beta = 3.0$ with sample sizes 25 and 30. The whole procedure has been illustrated through a real-life data set. The results of the current study support the outcome of the earlier studies, e.g., Rao [18–20]. In future reliability of Bayes approach for IWD under MCSS and comparison with the present study be studied.

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## References


1. Church, J.D.; Harris, B. The estimation of reliability from stress-strength Relationships. *Technometrics* **1970**, *12*, 49–54. [CrossRef]
2. Bhattacharyya, G.K.; Johnson, R.A. Estimation of reliability in the multi-component stress-strength model. *J. Am. Stat. Assoc.* **1974**, *69*, 966–970. [CrossRef]
3. Hanagal, D.D. Estimation of system reliability in multicomponent series stress–strength models. *J. Indian Stat. Assoc.* **2003**, *41*, 1–7.
4. Turkkan, N.; Pham-Gia, T. System stress-strength reliability: The multivariate Case. *IEEE Trans. Reliab.* **2007**, *56*, 115–124. [CrossRef]
5. Serkan, E. Consecutive k-out-of n: G system in stress-strength setup. *Commun. Stat. Simul. Comput.* **2008**, *37*, 579–589.
6. Serkan, E. Multivariate stress-strength reliability model and its evaluation for coherent structures. *J. Multivar. Anal.* **2008**, *99*, 1878–1887.
7. Serkan, E.; Funda, I. Reliability evaluation for a multi-state system under stress-strength setup. *Commun. Stat. Theory Methods* **2011**, *40*, 547–558.
8. Shawky, A.I.; Al-Gashgari, F.H. Bayesian and non-bayesian estimation of stress-strength model for Pareto type I Distribution. *Iran. J. Sci. Technol. Trans. A Sci.* **2013**, *37*, 335–342.
9. Pak, A.; Gupta, A.K.; Khoolenjani, N.B. On Reliability in a Multicomponent Stress-Strength Model with Power Lindley Distribution. *Rev. Colomb. Estadística* **2018**, *41*, 251–267. [CrossRef]
10. Pak, A.; Khoolenjani, N.B.; Rastogi, M.K. Bayesian inference on reliability in multicomponent stress-strength bathtub shaped model based on record values. *Pak. J. Stat. Oper. Res.* **2019**, *15*, 431–444. [CrossRef]
11. Rao, G.S.; Bhatti, F.A.; Aslam, M.; Albassam, M. Estimation of reliability in a multicomponent stress–strength system for the exponentiated moment-based exponential distribution. *Algorithms* **2019**, *12*, 246. [CrossRef]
12. Kotz, S.; Lumelskii, Y.; Pensky, M. *The Stress-Strength Model and Its Generalizations*; World Scientific: Singapore, 2003.
13. Bhattacharya, D.; Roychowdhury, S. Reliability of a coherent system in a multicomponent stress strength Model. *Am. J. Math. Manag. Sci.* **2013**, *32*, 40–52.
14. Ebrahimi, N. Estimation of reliability for a series stress-strength system. *IEEE Trans. Reliab.* **1982**, *31*, 202–205. [CrossRef]
15. Pandey, M.; Uddin, M.B.; Ferdous, J. Reliability estimation of an s-out-of-k system with nonidentical component strengths: The Weibull case. *Reliab. Eng. Syst. Saf.* **1992**, *36*, 109–116. [CrossRef]
16. Paul, R.K.; Uddin, M.B. Estimation of the reliability of the stress-strength model with nonidentical component strengths. *Microelectron. Reliab.* **1997**, *37*, 923–927. [CrossRef]
17. Rao, G.S.; Kantam, R.R.L. Estimation of reliability in the multicomponent stress-strength model: Log-logistic Distribution. *J. Appl. Stat. Sci.* **2010**, *3*, 75–84.
18. Rao, G.S. Estimation of reliability in multicomponent stress-strength model based on Rayleigh distribution. *ProbStat Forum* **2012**, *5*, 150–161.
19. Rao, G.S. Estimation of reliability in multicomponent stress-strength model based on generalized inverted Exponential distribution. *Int. J. Curr. Res. Rev.* **2012**, *4*, 48–56.
20. Rao, G.S. Estimation of reliability in multicomponent stress-strength model based on generalized exponential Distribution. *Rev. Colomb. Estadística* **2012**, *35*, 67–76.
21. Hassan, A.S.; Basheikh, H.M. Estimation of reliability in multicomponent stress-strength model following exponential Pareto distribution. *Egypt. Stat. J.* **2012**, *56*, 82–95.
22. Dey, S.; Mazucheli, J.; Anis, M.Z. Estimation of the reliability of multi-component stress–strength for a Kumaraswamy distribution. *Commun. Stat. Theory Methods* **2017**, *46*, 1560–1572. [CrossRef]
23. Kizilaslan, F. Classical and Bayesian estimation of reliability in a multicomponent stress-strength model based on the proportional reversed hazard rate model. *Math. Comput. Simul.* **2017**, *136*, 36–62. [CrossRef]
24. Badr, M.M.; Shawky, A.I.; Alharby, A.H. The Estimation of Reliability from Stress–Strength for Exponentiated Fréchet Distribution. *Iran. J. Sci. Technol. Trans. A Sci.* **2019**, *43*, 863–874. [CrossRef]
25. Akgül, F.G. Reliability estimation in multicomponent stress–strength model for Topp- Leone distribution. *J. Stat. Comput. Simul.* **2019**, *89*, 2914–2929. [CrossRef]
26. Keller, A.Z.; Kamath, A.R.R.; Perera, U.D. Reliability analysis of CNC Machine Tools. *Reliab. Eng.* **1982**, *3*, 449–473. [CrossRef]
27. Calabria, R.R.; Pulcini, G. Confidence limits for reliability and tolerance limits in the inverse Weibull Distribution. *Reliab. Eng. Syst. Saf.* **1989**, *24*, 77–85. [CrossRef]
28. Calabria, R.; Pulcini, G. On the maximum likelihood and least-squares estimation in the inverse Weibull Distributions. *Ital. J. Appl. Stat.* **1990**, *2*, 53–66.
29. Calabria, R.; Pulcini, G. Bayes probability intervals in a load-strength model. *Commun. Stat.-Theory Methods* **1992**, *21*, 3393–3405. [CrossRef]
30. Calabria, R.; Pulcini, G. Bayes 2-sample prediction for the inverse Weibull Distribution. *Commun. Stat.-Theory Methods* **1994**, *23*, 1811–1824. [CrossRef]
31. Maswadah, M. Conditional confidence interval estimation for the inverse Weibull distribution based on Censored generalized order statistics. *J. Stat. Comput. Simul.* **2003**, *73*, 887–898. [CrossRef]
32. Maswadah, M. Kernel Inference on the Inverse Weibull Distribution. *Korean Commun. Stat.* **2006**, *13*, 503–512. [CrossRef]

33. Dumonceaux, R.; Antle, C. Discrimination between the Lognormal and Weibull Distribution. *Technometrics* **1973**, *15*, 923–926. [CrossRef]
34. Prabhakar Murthy, D.N.; Xie, M.; Jiang, R. *Weibull Models*; Wiley: New York, NY, USA, 2004.
35. Bi, Q.; Gui, W. Bayesian and classical estimation of stress-strength reliability for inverse Weibull lifetime models. *Algorithms* **2017**, *10*, 71. [CrossRef]
36. Johnson, N.L.; Kotz, S.; Balakrishnan, N. Distributions. In *Continuous Univariate Distributions*, 2nd ed.; Wiley: New York, NY, USA, 1995; Volume 2.
37. Drapella, A. The complementary Weibull distribution: Unknown or just forgotten? *Qual. Reliab. Eng. Int.* **1993**, *9*, 383–385. [CrossRef]
38. Khan, M.S.; Pasha, G.R.; Pasha, A.H. Theoretical analysis of inverse Weibull Distribution. *WEAS Trans. Math.* **2008**, *7*, 30–38.
39. Li, C.; Hao, H. Reliability of a Stress–Strength Model with Inverse Weibull Distribution. *IAENG Int. J. Appl. Math.* **2017**, *47*, 302–306.
40. Yadav, A.S.; Singh, S.K.; Singh, U. Estimation of stress–strength reliability for inverse Weibull Distribution under the progressive type-II censoring scheme. *J. Ind. Prod. Eng.* **2018**, *35*, 48–55. [CrossRef]
41. Yang, L.; Wang, P.; Wang, Q.; Bi, S.; Peng, R.; Behrensdorf, J.; Beer, M. Reliability analysis of a complex system with hybrid structures and multi-level dependent life metrics. *Reliab. Eng. Syst. Saf.* **2021**, *209*, 107469. [CrossRef]
42. Graves, L.; Hamada, M.S. A Note on Incorporating Simultaneous Multi-level Failure Time Data in System Reliability Assessments. *Qual. Reliab. Eng. Int.* **2016**, *32*, 1127–1135. [CrossRef]
43. Rao, G.S.; Aslam, M.; Arif, O.H. Estimation of reliability in multicomponent stress–strength based on two parameter exponentiated Weibull Distribution. *Commun. Stat.-Theory Methods* **2017**, *46*, 7495–7502.
44. Palumbo, B.; Pallotta, G. New approach to the identification of the inverse Weibull model. In Proceedings of the 46th Scientific Meeting of the Italian Statistical Society, Rome, Italy, 20–22 June 2012.
45. Surles, J.G.; Padgett, W.J. Inference for reliability and stress-strength for a scaled Burr type X distribution. *Lifetime Data Anal.* **2001**, *7*, 187–200. [CrossRef] [PubMed]
46. Kundu, D.; Gupta, R.D. Estimation of P[Y < X] for Weibull distributions. *IEEE Trans. Reliab.* **2016**, *55*, 270–280.