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Estimating Stochastic Volatility under the Assumption of Stochastic Volatility of Volatility

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Received: 12 March 2020; Accepted: 9 April 2020; Published: 11 April 2020

Abstract: We propose novel nonparametric estimators for stochastic volatility and the volatility of volatility. In doing so, we relax the assumption of a constant volatility of volatility and therefore, we allow the volatility of volatility to vary over time. Our methods are exceedingly simple and far simpler than the existing ones. Using intraday prices for the Standard & Poor's 500 equity index, the estimates revealed strong evidence that both volatility and the volatility of volatility are stochastic. We also proceeded in a Monte Carlo simulation analysis and found that the estimates were reasonably accurate. Such evidence implies that the stochastic volatility models proposed in the literature with constant volatility of volatility may fail to approximate the discrete-time short rate dynamics.

Keywords: nonparametric estimators; stochastic volatility; stochastic volatility of volatility

JEL Classification: C0; G0

1. Introduction

Financial market volatility is a key factor for many issues in finance, ranging from asset management to risk management (Poon and Granger 2003). In light of this, market participants are preoccupied with both the nature of volatility and its level. Since volatility can be used in investment decisions, derivative pricing and financial market regulation, several approaches have been proposed in the existing literature with regard to its estimation. A key assumption is that that volatility can change over time, thus many changes in volatility can be modeled stochastically. In the existing literature, there is evidence that stochastic volatility models outperform constant volatility models (see Hull and White 1987; Ghysels et al. 1996; Andersen and Lund 1997; among many others). Indeed, there is strong evidence of non-stationarity in the variance (see e.g., Cohen et al. 1972). Stochastic volatility models resolve the shortcoming of the Black and Scholes model that the volatility is constant over time and is unaffected by the changes in the price level of the underlying asset. Although volatility can be estimated by parametric, semi-parametric and nonparametric estimators (see Asai et al. 2006; Maasoumi and McAleer 2008; Asai and McAleer 2011; Caporin and McAleer 2012 for a detailed discussion on this topic), statistical inferences for stochastic volatility models are mainly parametric. Cox et al. (1985) and Heston (1993) offered some indicative examples of parametric estimation.[1] However, Alghalith (2012)

[1] Stochastic volatility is an essential component in an asset pricing model, in option pricing. See (Wong and Lo 2009; Mrázek et al. 2016; Cui et al. 2017; among others).

highlighted some important limitations of these procedures. Few studies have used nonparametric approaches employing integrated volatility. Examples include Vetter (2015), Comte et al. (2010) and Renò (2006). Stochastic volatility stands for one of the main concepts in financial literature used in finance, that is, the endemic time-varying behavior of volatility and the co-dependence observed in markets (see Mandelbrot 1963; Officer 1973; Shephard 2005; among others).

The volatility of volatility is also a separate key risk factor which affects, for example, option returns, beyond volatility itself (Huang et al. 2019). Park (2015) showed that the volatility of volatility implied by VIX options has predictability for tail risk hedging returns. However, the recent literature on the volatility of volatility has mainly used parametric methods. For example, Barndorff-Nielsen and Veraart (2013) considered a non-Gaussian Ornstein–Uhlenbeck process, while Corsi et al. (2008) considered the realized volatility. More recently, Alghalith (2016) considered a stochastic volatility approach. In particular, Alghalith (2016) estimated the means of volatility and its volatility, not the volatility at each time. Furthermore, although the latest asset price models considered the stochastic volatility of the asset price to vary over time, the existing literature still considers the volatility of volatility as constant over time. Additionally, the existing literature does not provide statistical inferences for stochastic volatility models under the assumption of the stochastic volatility of volatility.

In this paper, we propose novel nonparametric estimators for stochastic volatility and the volatility of volatility. The contribution to the existing literature is that, our approach is far simpler than the previous methods (e.g., GARCH type models, Corsi et al. 2008; Alghalith 2016). In addition, we relax the assumption of a constant volatility which allows the volatility of volatility to vary over time, giving a new insight in option pricing. Furthermore, we provided statistical inferences for the stochastic volatility models under the assumption of stochastic volatility of volatility testing if both volatility and the volatility of volatility are stochastic. To this end, we introduced a separate time-varying volatility of volatility model which drives the conditional variance of the variance.

In order to illustrate the usefulness of our approach, we presented an empirical application to the equity market. We focused on the Standard & Poor's 500 (S&P 500) equity index using intraday (1-min, 5-min and 30-min data) and daily data. Our estimates revealed strong evidence that both volatility and the volatility of volatility are stochastic. Then, we proceeded in a Monte Carlo simulation analysis and found that our estimates are reasonably accurate. This suggests that the stochastic volatility models proposed in the literature with constant volatility of volatility may fail to approximate the discrete-time short rate dynamics. Using predictive regressions, we also showed that both volatility measures are significant predictors of the future market's volatility expectations and sentiment. By including both measures at the same time, we indicated that the time-varying nature of the volatility of volatility stands as an additional source of risk and such evidence highlights the importance of there being be two different types of risk premia: one for the uncertainty induced by the volatility per se and one induced by the uncertainty of the volatility of volatility. The volatility of volatility is often ignored in most asset price models and can still be associated with deep uncertainty in the market.

This paper is organized as follows: Section 2 details the model specification proposed in this study. Section 3 deals with the estimation technique. Section 4 presents an application with financial data, alone with a Monte Carlo simulation study. Section 5 provides financial implications for volatility and its volatility risk premia. Section 6 concludes by emphasizing the importance of our estimators.

2. Model Specification

We begin by assuming that the asset at time t follows the diffusion:

$$dS_t = S_t[\mu dt + v_t dW_{1t}], \tag{1}$$

where S_t is the price of the asset at time t, μ is the expected rate of return and v_t is the volatility; W_{1t} is a Brownian motion defined in the probability space $(\Omega,\ F,\ F_t,\ P)$, where $\{F_t\}_{s\leq t\leq T}$ is the augmentation of filtration which drives equity prices. Models such as the model in (1) are routinely adopted in finance to model equity prices, currency prices or interest rates (see e.g., Heston 1993; Alghalith 2016).

The purpose of this paper is to propose an exact solution for stochastic volatility and its volatility and nonparametric suitable estimators. More specifically, we introduced a separate time-varying volatility of volatility model which drives the conditional variance of the variance. The idea is the following, using Ito's differentiation rule in (1), we have:

$$\begin{aligned}(dS_t)^2 &= (S_t[\mu d_t + v_t dW_{1t}])^2 = {S_t}^2[\mu d_t + v_t dW_{1t}]^2 \\ &= {S_t}^2\left[\mu^2 \underbrace{d_t^2}_{0} + 2\mu v_t \underbrace{d_t dW_{1t}}_{0} + {v_t}^2 \underbrace{{dW_{1t}}^2}_{dt}\right] = v_t^2 S_t^2 dt.\end{aligned} \tag{2}$$

Thus, solving (2) for v_t, we have:

$$v_t = \sqrt{\frac{(dS_t)^2}{S_t^2 dt}}. \tag{3}$$

If the volatility is stochastic, it may be given by (see Heston 1993):

$$dv_t^2 = \left(\alpha_1 - b_1 v_t^2\right)dt + \gamma_t v_t dW_{2t}, \tag{4}$$

where α_1 and b_1 are constants, γ_t is the volatility of volatility (the volatility of the variance v_t^2) and W_{2t} is also a Brownian motion which now drives the variance. Note that the drift of v_t only depends on itself and thus, not on the S_t or γ_t. Hence, using Ito's differentiation rule in (4), we have:

$$\begin{aligned}\left(dv_t^2\right)^2 &= \left[\left(\alpha_1 - b_1 v_t^2\right)dt + \gamma_t v_t dW_{2t}\right]^2 \\ &= \left[\left(\alpha_1 - b_1 v_t^2\right)^2 \underbrace{d_t^2}_{0} + 2\left(\alpha_1 - b_1 v_t^2\right)\underbrace{d_t dW_{2t}}_{0} + \gamma_t^2 v_t^2 \underbrace{{dW_{2t}}^2}_{dt}\right] \\ &= \gamma_t^2 v_t^2 dt.\end{aligned} \tag{5}$$

Thus, solving (5) for γ_t, we have:

$$\gamma_t = \sqrt{\frac{\left(dv_t^2\right)^2}{v_t^2 dt}}. \tag{6}$$

Similarly, if the volatility of volatility γ_t is stochastic, it may be given by:

$$d\gamma_t^2 = \left(\alpha_2 - b_2 \gamma_t^2\right)dt + \alpha_2 \gamma_t dW_{3t}, \tag{7}$$

where α_2 and b_2 are constants, γ_t is the volatility of volatility (the volatility of the variance γ_t^2) and W_{3t} is a Brownian motion which in turn drives the volatility of volatility. In the same way as in (4), the drift of the γ_t is a function of the γ_t. Consequently, again using Ito's differentiation rule in (7), we have:

$$\begin{aligned}\left(d\gamma_t^2\right)^2 &= \left[\left(\alpha_2 - b_2 \gamma_t^2\right)dt + \alpha_2 \gamma_t dW_{3t}\right]^2 \\ &= \left[\left(\alpha_2 - b_2 \gamma_t^2\right)^2 \underbrace{d_t^2}_{0} + 2\left(\alpha_2 - b_2 \gamma_t^2\right)\underbrace{d_t dW_{3t}}_{0} + a_2^2 \gamma_t^2 \underbrace{{dW_{3t}}^2}_{dt}\right] = a_2^2 \gamma_t^2 dt.\end{aligned} \tag{8}$$

Finally, it should be noted that the Brownian components W_{1t}, W_{2t} and W_{3t} in (1), (4) and (7), respectively, can be correlated, that is $dW_{it}dW_{jt} = \rho_{i,j}dt$ for all $i \neq j$.

3. Estimation Technique

Using (3), the volatility per unit of time is estimated nonparametrically as the ratio (square root) of the squared first difference of the prices and the squared prices, which is given as follows:

$$v_t = \sqrt{\frac{(\Delta S_t)^2}{S_t^2}}, \tag{9}$$

where Δ denotes the first difference of the data and S_t is the observed data. Consequently, the annual volatility is given by

$$v_t = \sqrt{\frac{n(\Delta S_t)^2}{S_t^2}}, \tag{10}$$

where n measures the data frequency.

Building on Equation (2), we test whether the volatility is stochastic. In doing so, we estimate the following linear regression equation, while in the case where the true model is non-linear, Taylor expansion can be used to linearize the relationship:

$$\left(\Delta v_t^2\right)^2 = \delta_1 v_t^2 + \varepsilon_{1t}, \tag{11}$$

where δ_1 is the parameter to be estimated and ε_{1t} is the error term. If δ_1 is statistically significantly different from zero, then we assume that the underlying volatility is stochastic and not constant over time affected by the changes of the underlying asset. In particular, considering the following equation, $\left(dv_t^2\right)^2 = \gamma_t^2 v_t^2 dt$, if volatility is deterministic, then dv_t^2 is equal to zero requiring that γ_t^2 is also equal to zero. At a discrete time, we can test this hypothesis by (11). In the case when δ_1 is equal to zero, $\left[\left(\Delta v_t^2\right)\right]$ must be equal to zero too, which in turn suggests that the volatility is deterministic (i.e., cannot change over time).[2]

Similarly, using (6), the volatility of volatility per unit of time is estimated nonparametrically as the ratio (square root) of the squared first difference of the variance and the variance:

$$\gamma_t = \sqrt{\frac{\left(\Delta v_t^2\right)^2}{v_t^2}}, \tag{12}$$

where Δ denotes the first difference of the variance v_t^2. Furthermore, the annual volatility of volatility is given by

$$\gamma_t = \sqrt{\frac{n\left(\Delta v_t^2\right)^2}{v_t^2}}. \tag{13}$$

where again n measures the frequency of the data.

Finally, we use (4) to test whether the volatility of volatility is stochastic. In doing so, we estimate the following regression equation:

$$\left(\Delta \gamma_t^2\right)^2 = \delta_2 \gamma_t^2 + \varepsilon_{2t}, \tag{14}$$

where δ_2 is the parameter to be estimated and ε_{2t} is the error term. In the case where the true model is non-linear, Taylor expansion can be also used to linearize the relationship. In line with the concept

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2 Heston (1993) assumed that the variance v_t follows a square root process, $dv_t = \kappa^*[\theta^* - v_t]dt + \sigma\sqrt{v_t}dW_t$, where σ controls the volatility of volatility. When σ is equal to zero, the volatility is deterministic. Following Heston (1993) and applying Ito's rule (4), we are able to test whether volatility is stochastic via linear regression equations. We refer to the study implemented by Heston (1993) for more information regarding this issue.

of Heston (1993) for stochastic volatility, if δ_2 is statistically significantly different from zero, then we assume that the volatility of volatility is stochastic, thus it is not constant over time. More specifically, let us assume the following equation, $\left(d\gamma_t^2\right)^2 = a_2^2\gamma_t^2 dt$, in order for the volatility of volatility to be deterministic, the $d\gamma_t^2$ would have to be zero requiring that a_2 is equal to zero, too. At a discrete time, we can test this hypothesis by (14). In the case when δ_1 is equal to zero, $\left[\left(\Delta\gamma_t^2\right)\right]$ must be equal to zero too, which in turn suggests that the volatility of volatility is deterministic (i.e., cannot change over time).

4. Application

We used intraday prices for the S&P 500 equity index; the 1-min, 5-min and 30-min data cover the period from 11 August 2016 to 9 November 2016. This sample is strictly restricted by the availability of the intraday data available. We also used daily data for the same index ranging from 31 March 2015 to 31 November 2020. Data was retrieved from the Thomson Reuters database. The S&P 500 is the most common benchmark for the broader US equity markets, while it is one of the most heavily traded and liquid equity indexes. We employed intraday data due to the fact that they reveal important information not easily seen at lower sampling frequencies (e.g., daily data), such as the intraday changes and the market microstructures. Hence, volatility is more accurately estimated by employing high frequency data (see Hansen and Huang 2016; among others). We constructed the corresponding returns of each frequency, as the differences between prices at consecutive time points. Even though each frequency may be just enough to use for our estimators, there is an adequate balance between high and low sampling frequencies. For example, Liu et al. (2015) found that 5 min is an adequate sampling frequency for liquid assets, while 1 min was more appropriate for the diversified set of asset classes. Consequently, our empirical results of volatility and the volatility of volatility were compatible among different intraday frequencies. For our datasets, we applied the data-adjustment (cleaning) procedure suggested by Barndorff-Nielsen et al. (2009).[3] Furthermore, we ignored trading days with recorded prices for less than 60% of the operating time's expected observations and short trading days around major holidays.

We present the empirical results of volatility and the volatility of volatility for the S&P 500 equity index in Tables 1–3. An illustrated representation of our estimators is also given in Figures 1–4 for each frequency under consideration. We present these figures in an Appendix. As we can see from the figures, there is clear evidence that not only the volatility varies over time, but also its volatility varies over time (mainly when considering lower frequencies i.e., 30-min returns or daily frequency). This also suggests that the stochastic volatility models proposed in the literature with constant volatility of volatility may fail to approximate the discrete-time short rate dynamics. Furthermore, we also see that the volatility of volatility behaves rather differently to the volatility itself. Such results are consistent with the views of Huang et al. (2019). Consistent with a setup of our model which separated volatility from its volatility, the correlation between volatility and its volatility was almost zero for the 1-min data, −0.0228 for the 5-min data and for the 30-min data was equal to −0.0671, while for the daily frequency it was equal to −0.0081. Although both measures have some common peaks, the overall picture is that large increases in volatility are relatively independent of the increases in the volatility of volatility.

[3] Following Barndorff-Nielsen et al. (2009), (1) we deleted the entries with a timestamp outside the 9:30 a.m.–4 p.m. window when the market is open and (2) we deleted the entries with a bid, ask or a transaction price equal to zero.

Table 1. Descriptive statistics for volatility and the volatility of volatility of the S&P 500.

Panel A: 1-Min Sample				
	v_t	γ_t	v_t **(annual)**	γ_t **(annual)**
Mean	1.59E-04	1.05E-03	2.4883E-02	1.65E-01
Median	1.03E-04	1.73E-04	1.6142E-02	2.71E-02
Maximum	1.15E-02	2.66E+00	1.8062E+00	4.16E+02
Minimum	0.00E+00	4.16E-11	0.0000E+00	6.52E-09
Std. Dev.	2.32E-04	1.91E-02	3.6384E-02	3.00E+00
Skewness	1.37E+01	1.16E+02	1.3678E+01	1.16E+02
Kurtosis	4.21E+02	1.55E+04	4.2115E+02	1.55E+04
Jarque–Bera	1.80E+08	2.43E+11	1.8000E+08	2.43E+11
p-value	[0.00E+00]	[0.00E+00]	[0.00E+00]	[0.00E+00]
No. of obs.	24,224	24,224	24,224	24,224
Panel B: 5-Min Sample				
	v_t	γ_t	v_t **(annual)**	γ_t **(annual)**
Mean	4.16E-04	2.96E-03	2.96E-02	2.11E-01
Median	2.66E-04	4.43E-04	1.89E-02	3.15E-02
Maximum	1.37E-02	1.33E+00	9.79E-01	9.48E+01
Minimum	4.56E-06	1.06E-09	3.25E-04	7.52E-08
Std. Dev.	5.67E-04	2.47E-02	4.04E-02	1.76E+00
Skewness	6.98E+00	3.69E+01	6.98E+00	3.69E+01
Kurtosis	1.02E+02	1.78E+03	1.02E+02	1.78E+03
Jarque–Bera	2.08E+06	6.63E+08	2.08E+06	6.63E+08
Probability	[0.00E+00]	[0.00E+00]	[0.00E+00]	[0.00E+00]
p-value	5042	5042	5042	5042
Panel C: 30-Min Sample				
	v_t	γ_t	v_t **(annual)**	γ_t **(annual)**
Mean	1.14E-03	3.31E-02	9.56E-03	2.78E-01
Median	7.18E-04	2.09E-02	1.32E-03	3.83E-02
Maximum	1.53E-02	4.46E-01	5.91E-01	1.72E+01
Minimum	4.66E-06	1.35E-04	2.49E-07	7.25E-06
Std. Dev.	1.38E-03	4.02E-02	3.90E-02	1.13E+00
Skewness	3.63E+00	3.63E+00	9.34E+00	9.34E+00
Kurtosis	2.48E+01	2.48E+01	1.13E+02	1.13E+02
Jarque–Bera	1.84E+04	1.84E+04	4.38E+05	4.38E+05
p-value	[0.00E+00]	[0.00E+00]	[0.00E+00]	[0.00E+00]
No. of obs.	841	841	841	841
Panel D: Daily Sample				
	v_t	γ_t	v_t **(annual)**	γ_t **(annual)**
Mean	6.60E-03	4.84E-02	2.34E-01	1.71E+00
Median	3.77E-03	7.28E-03	1.33E-01	2.58E-01
Maximum	1.36E-01	7.49E+00	4.83E+00	2.66E+02
Minimum	6.87E-06	3.20E-05	2.43E-04	1.13E-03
Std. Dev.	9.54E-03	2.87E-01	3.38E-01	1.02E+01
Skewness	5.45E+00	1.93E+01	5.45E+00	1.93E+01
Kurtosis	5.18E+01	4.48E+02	5.18E+01	4.48E+02
Jarque-Bera	1.31E+05	1.04E+07	1.31E+05	1.04E+07
p-value	[0.00E+00]	[0.00E+00]	[0.00E+00]	[0.00E+00]
No. of obs.	1256	1256	1256	1256

Note: This table reports the descriptive statistics for the following series: v_t (volatility per unit), γ_t (volatility of volatility per unit), v_t (annual volatility), γ_t (annual volatility of volatility) estimated by using 1-min returns, 5-min returns, 30-min returns and daily returns. The hypothesis of normality is studied by a Jarque–Bera test. The p-value of the test is given below in brackets.

Table 2. Estimation results for the stochastic volatility.

Panel A: 1-Min Sample					
Case 1: Returns			**Case 2: Simulation**		
	δ_1	R^2		δ_1	$\overline{R}^2$
$H_0 : \delta_1 = 0$	8.82E-05 *** (7.05E-07)	0.3892	$H_0 : \delta_1 = 0$	8.49E-05 (7.15E-07)	0.3789
Panel B: 5-Min Sample					
Case 1: Returns			**Case 2: Simulation**		
	δ_1	R^2		δ_1	$\overline{R}^2$
$H_0 : \delta_1 = 0$	1.23E-04 *** (2.22E-06)	0.3779	$H_0 : \delta_1 = 0$	1.16E-04 (2.28E-06)	0.3688
Panel C: 30-Min Sample					
Case 1: Returns			**Case 2: Simulation**		
	δ_1	R^2		δ_1	$\overline{R}^2$
$H_0 : \delta_1 = 0$	1.36E-04 *** (6.40E-06)	0.341	$H_0 : \delta_1 = 0$	1.76E-05 (5.46E-06)	0.3184
Panel D: Daily Sample					
Case 1: Returns			**Case 1: Returns**		
	δ_2	R^2		δ_2	$\overline{R}^2$
$H_0 : \delta_1 = 0$	6.54E-03 *** (2.40E-04)	0.3677	$H_0 : \delta_1 = 0$	3.21-E04 (1.37E-05)	0.3810

Note: This table reports the estimates of the tests for the null hypothesis $H_0 : \delta_1 = 0$, that is, if volatility is stochastic, applying linear least squares regression. Two cases are considered. Case 1 is based on real data estimated by using 1-min returns, 5-min returns, 30-min returns and daily returns for the Standard & Poor's 500 (S&P 500) equity index. Case 2 is based on a Monte Carlo analysis using 1000 replications with a sample size equal to the number of observations 24,224, 5042, 841 and 1259 for each panel, respectively. *** refer to significant levels of 1%.

Table 3. Estimation results for the stochastic volatility of volatility.

Panel A: 1-Min Sample					
Case 1: Returns			**Case 2: Simulation**		
	δ_2	R^2		δ_2	$\overline{R}^2$
$H_0 : \delta_2 = 0$	6.95E+00 *** (4.56E-02)	0.4937	$H_0 : \delta_2 = 0$	6.25E+00 (6.01E-02)	0.4889
Panel B: 5-Min Sample					
Case 1: Returns			**Case 2: Simulation**		
	δ_2	R^2		δ_2	$\overline{R}^2$
$H_0 : \delta_2 = 0$	1.68E+00 *** (2.48E-02)	0.4770	$H_0 : \delta_2 = 0$	1.22E+00 (1.85E-02)	0.4462
Panel C: 30-Min Sample					
Case 1: Returns			**Case 2: Simulation**		
	δ_2	R^2		δ_2	$\overline{R}^2$
$H_0 : \delta_2 = 0$	2.87E-01 *** (1.09E-02)	0.4491	$H_0 : \delta_2 = 0$	3.16E-01 (2.40E-02)	0.4214
Panel D: Daily Sample					
Case 1: Returns			**Case 2: Simulation**		
	δ_2	R^2		δ_2	$\overline{R}^2$
$H_0 : \delta_2 = 0$	5.02E+01 *** (1.50E+00)	0.4710	$H_0 : \delta_2 = 0$	7.99E+01 (1.20E-01)	0.3914

Note: This table reports the estimates of the tests for the null hypothesis $H_0 : \delta_2 = 0$, that is, if the volatility of volatility is stochastic, applying linear least squares regression. Two cases are considered. Case 1 is based on our real data estimated by using 1-min returns, 5-min returns, 30-min returns and daily returns for the S&P 500 equity index. Case 2 is based on a Monte Carlo analysis using 1000 replications with a sample size equal to the number of observations 24,224, 5042, 841 and 1259 for each panel, respectively. *** refer to significant levels of 1%.

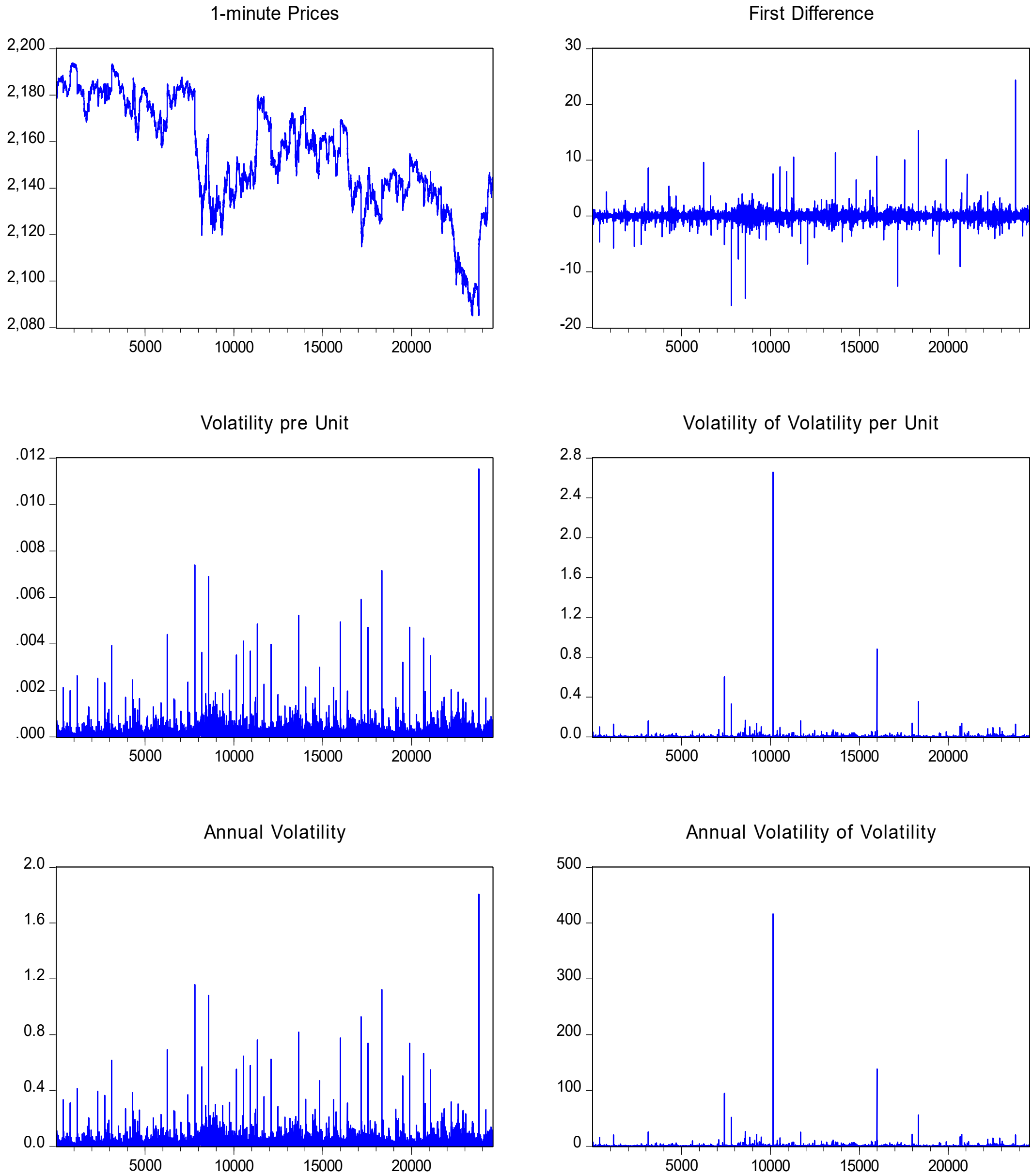

Figure 1. Plot of the S&P 500 1-min sample.

In Table 1, we report the descriptive statistics for the following series: u_t (volatility per unit), γ_t (volatility of volatility per unit), u_t (annual volatility), γ_t (annual volatility of volatility). Furthermore, we consider three panels: Panel A corresponds to the empirical results using the 1-min returns, Panel B to the empirical results using the 5-min returns, Panel C to the empirical results using the 30-min returns and Panel D to the empirical results using the daily returns. It is worth mentioning that the results are consistent with the Jarque–Bera test, in which the null hypothesis of normality is rejected for both volatility and the volatility of volatility series.

In Table 2, we test if volatility is stochastic via (11) for the null hypothesis $H_0 : \delta_1 = 0$. For our study, we applied linear least squares regression and we considered two cases. The first case was based on our real data. The second case was based on a Monte Carlo analysis using 1000 replications with a sample size equal to the number of observations in each sample (24,224, 5042, 841, 1259). We proceeded to a Monte Carlo analysis to check the reliability of the proposed estimators. To this end, we tested them on simulated time series of a diffusion model.[4] As for the former case, as expected we rejected the null hypothesis $H_0 : \delta_1 = 0$ in each sample, which in fact means that volatility is stochastic. As for the latter case, we observed that the standard deviations are very small related to the means and therefore, our regression estimates are reasonably accurate.

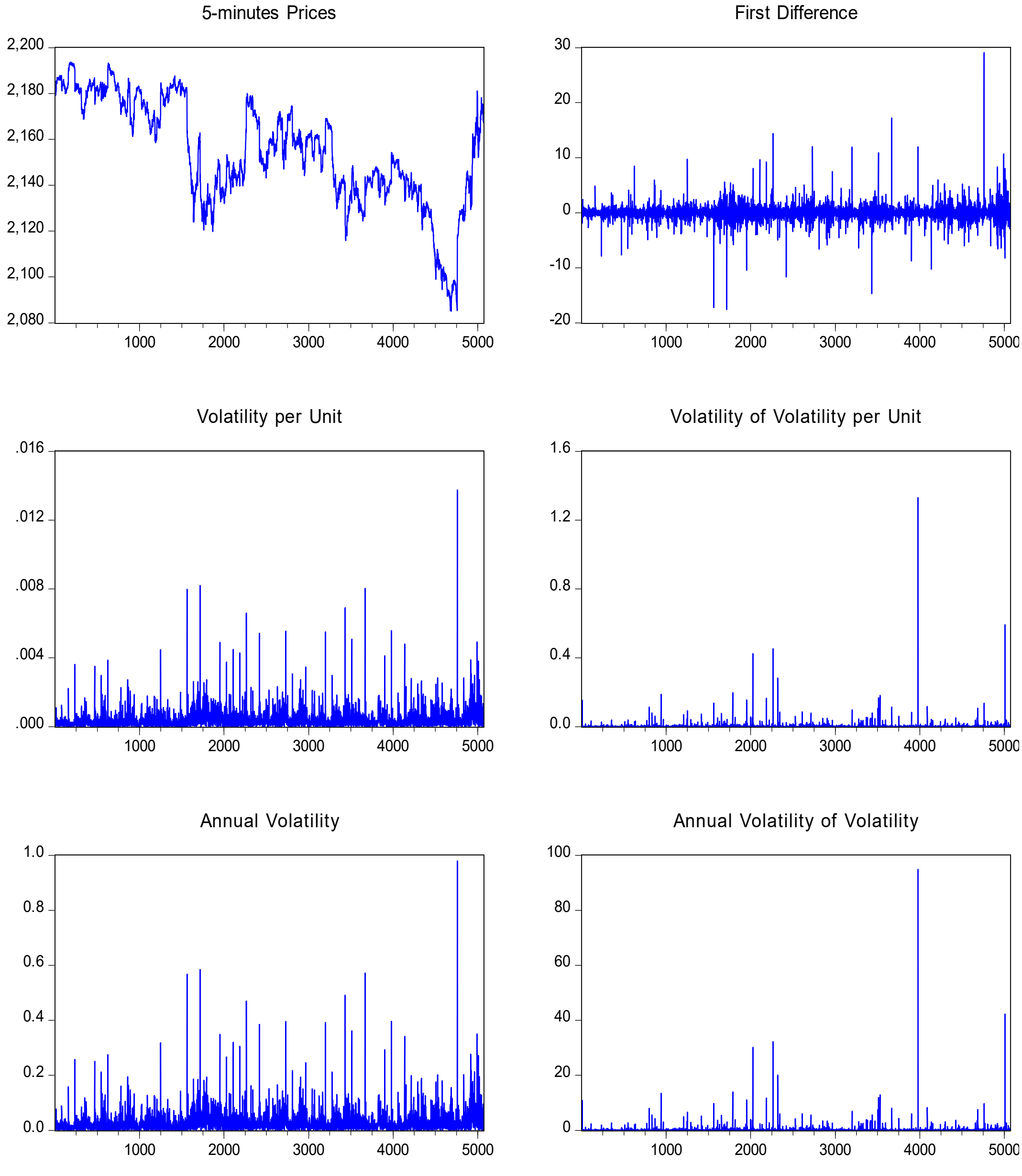

Figure 2. Plot of the S&P 500 5-min sample.

4 For the simulation analysis, we considered a stochastic process of a single random variable based on the S&P 500 index assuming that the variance varies over time (i.e., exhibits serial correlated properties). To this end, we used a stochastic differential equation of the Ito type. The drift and the diffusion parameters are estimated from the original series by the method of maximum pseudo-likelihood (see Boukhetala 1996).

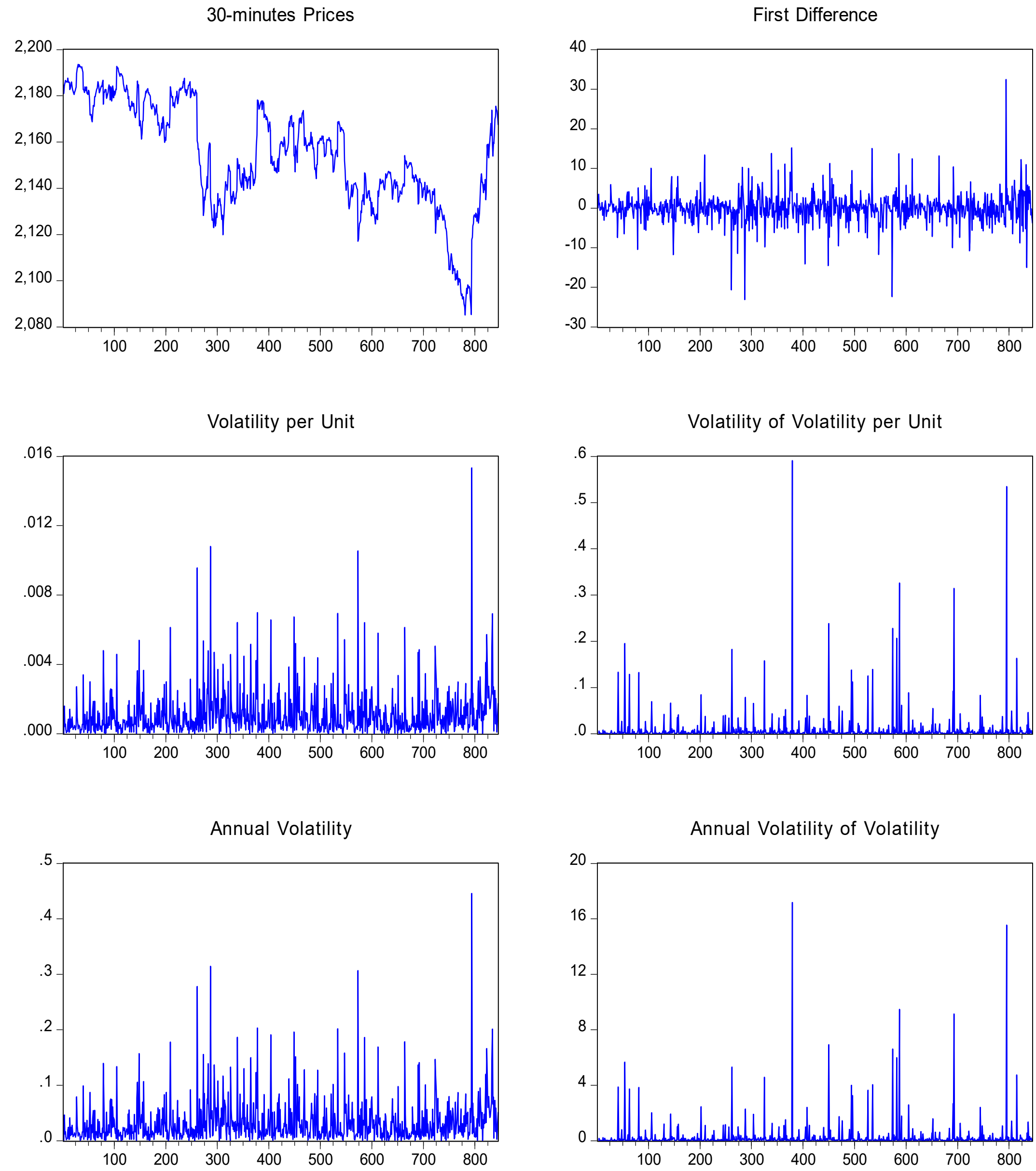

Figure 3. Plot of the S&P 500 30-min sample.

In Table 3, we test if the volatility of volatility is stochastic via (14) for the null hypothesis $H_0 : \delta_2 = 0$. For our study, we applied linear least squares regression and we considered two cases. The first case was based on our real data. The second case was based on a Monte Carlo analysis using 1000 replications with a sample size equal to the number of observations in each sample (24,224, 5042, 841, 1259). As for the former case, we rejected the null hypothesis $H_0 : \delta_2 = 0$ in each sample, which in fact means that the volatility of volatility is also stochastic. As for the Monte Carlo simulation analysis, we can see that the standard deviations are very small related to the means and therefore, our regression estimates are reasonably accurate. From a financial point of view, such evidence in the results is consistent with a no-arbitrage asset pricing model which features that volatility and its volatility vary over time. Below we further evaluate this concept using predictive regressions.

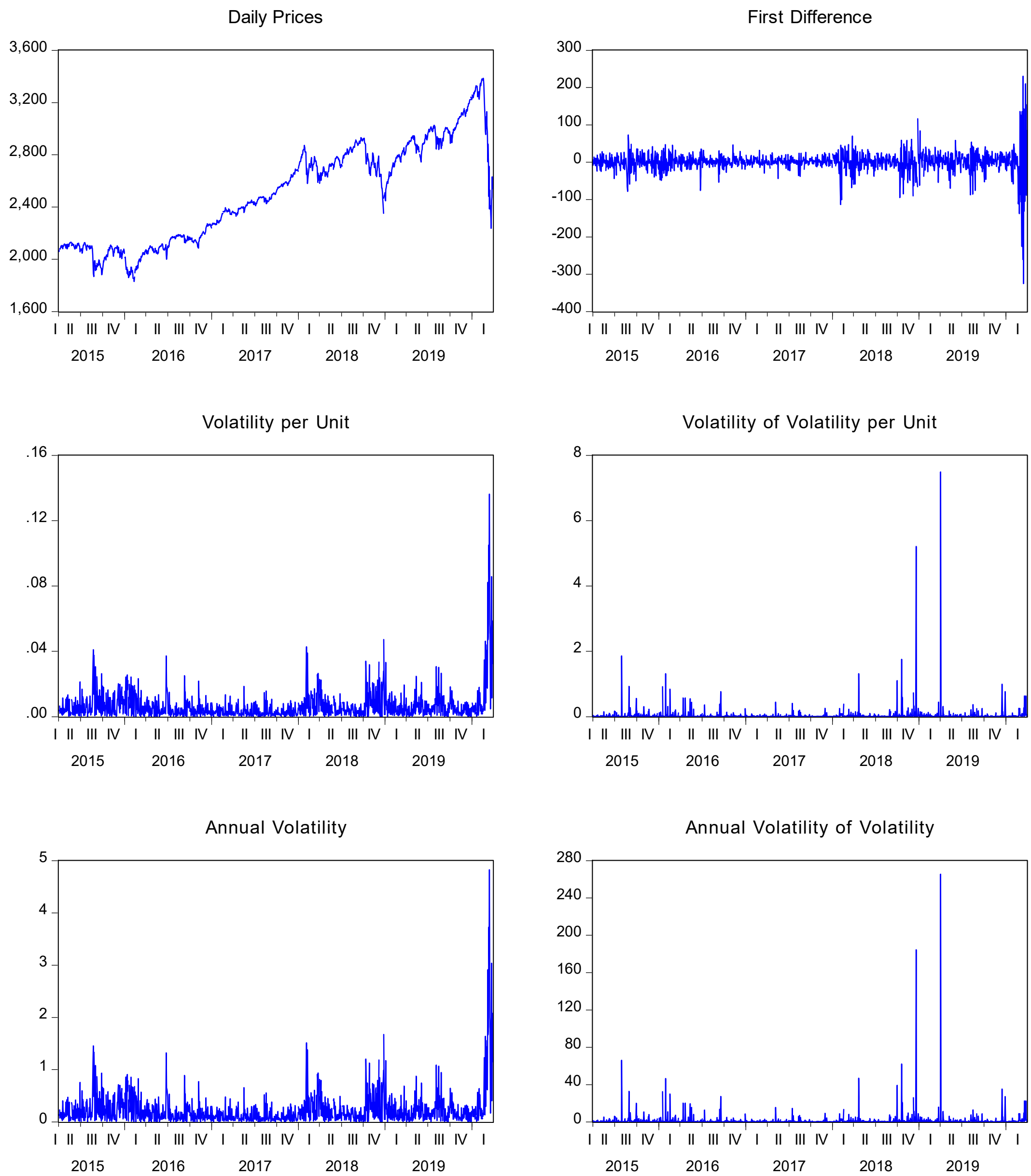

Figure 4. Plot of the S&P 500 daily sample.

5. Financial Implications

To evaluate the implications of the model for volatility and its volatility risks, we studied their predictability of the future market's volatility expectations. To this end, we used the daily sample used in this study for the S&P 500 index (again from 31 March 2015 to 31 November 2020), along with the Volatility Index (*VIX*) and the Volatility of the *VIX* Index (*VIX* of *VIX* or *VVIX*) for the same time period. We focused on daily data frequency due to the lack of availability of intraday data for the *VIX* and *VVIX* indices. These indices were created by the Chicago Board Options Exchange (CBOE). The *VIX* index (also called "*Fear Gauge*" or "*Fear Index*") acts as a real-time market index capturing the market's expectation of 30-day-forward-looking volatility. It is derived from the price inputs of the S&P 500 index options and provides a measure of market risk. As for the *VVIX* index, it can be considered as a measure of the volatility of the *VIX* capturing the short-term volatility of the S&P 500 indices. It indicates how quickly market sentiment can change.

In particular, we estimate the following regression equations for VIX and $VVIX$ indices:

$$VIX_{t+1} = \beta_0^{VIX} + \beta_1^{VIX} v_t + \beta_2^{VIX} \gamma_t + u_{1t+1}, \tag{15}$$

$$VVIX_{t+1} = \beta_0^{VVIX} + \beta_1^{VVIX} v_t + \beta_2^{VVIX} \gamma_t + u_{2t+1}, \tag{16}$$

where v_t is the annualized volatility estimated by (10), γ_t is the annualized volatility of volatility estimated by (13) and u_{t+1} is the error-term (u_{1t} for the VIX index and u_{2t} for the $VVIX$ index).

Table 4 refers to the estimates of (15) and (16) for the VIX and $VVIX$ indices, respectively. For this analysis, we applied robust linear least squares and we considered two specifications. The first specification refers to the case when the VIX and $VVIX$ indices are expressed in units (Panel A), while in the second specification these indices are expected in logarithmic units (Panel B). We also tested the issue of equality between the coefficients β_1 and β_2, i.e., $H_0 : \beta_1^{VIX} - \beta_2^{VIX} = 0$ and $H_0 : \beta_1^{VVIX} - \beta_2^{VVIX} = 0$ by Wald tests. The p-values of the tests are given below in brackets.

Table 4. Estimation results for the stochastic volatility of volatility.

Panel A: Units			
***VIX* Index**			
v_{t-1}	γ_{t-1}	$Adj. R^2$	Wald test $H_0 : \beta_1^{VIX} - \beta_2^{VIX} = 0$
4.23E+02 ***	1.33E+01 ***	0.2381	46.2936
(8.86E+00)	(2.94E-01)		[0.00E+00]
***VVIX* Index**			
v_{t-1}	γ_{t-1}	$Adj. R^2$	Wald test $H_0 : \beta_1^{VIX} - \beta_2^{VIX} = 0$
7.60E+02 ***	7.63E+00 ***	0.1175	25.6088
2.94E+01	9.76E-01		[0.00E+00]
Panel B: Logarithm Units			
***VIX* Index**			
v_{t-1}	γ_{t-1}	$Adj. R^2$	Wald test $H_0 : \beta_1^{VVIX} - \beta_2^{VVIX} = 0$
2.50E+01 ***	6.19E-01 ***	0.2953	39.8077
(6.47E-01)	(2.05E-02)		[0.00E+00]
***VVIX* Index**			
v_{t-1}	γ_{t-1}	$Adj. R^2$	Wald test $H_0 : \beta_1^{VIVX} - \beta_2^{VVIX} = 0$
7.73E+00 ***	8.16E-02 ***	0.1414	23.9296
(8.16E-02)	(1.06E-02)		[0.00E+00]

Note: This table reports estimates of the predictive regressions in (15) and (16) for VIX and $VVIX$ indices, respectively. For the analysis, robust linear least squares are employed. Two specifications are considered. The first specification refers to the case when the VIX and $VVIX$ indices are expressed in units (Panel A), while in the second specification these indices are expected in logarithmic units (Panel B). *** refer to significant levels of 1%. The issue of equality, i.e., $H_0 : \beta_1^{VIX} - \beta_2^{VIX} = 0$ and $H_0 : \beta_1^{VVIX} - \beta_2^{VVIX} = 0$ is tested by a Wald test. The p-value of the test is given below in brackets.

Using predictive regressions, we showed that the v_t and γ_t are significant predictors of future market's volatility expectations and sentiment. Both past volatility and its volatility significantly forecast the future VIX and $VVIX$ indices. By including both measures at the same time, we found that the predictive power is positive and shared between volatility and its volatility. However, the impact of the volatility is always higher. Although this result for the volatility per se was expected, the difference in the coefficient estimates between β_1 and β_2—as indicated by the Wald tests implemented—can provide new insights in pricing volatility risk premia.[5] In particular, the null hypothesis of equality was rejected in all cases. Such evidence shows that the volatility of volatility is a significant additional source of risk which indeed separately can increase financial stress in equity markets. In other words, the time-varying nature of the volatility of volatility stands for an additional source of stress, which in turn implies that there are two different types of risk premia, one for the uncertainty induced by the volatility itself and one induced by the uncertainty of the volatility of volatility. The latter risk factor is often ignored in stochastic volatility models and can still be associated with deep uncertainty in the market.

6. Conclusions

In this paper, we devised new nonparametric estimators for the stochastic volatility and the volatility of volatility. Our method has two main advantages compared to the previous methods in the existing literature. Firstly, they are exceedingly simple, far simpler than the existing methods. Secondly, we relax the assumption of a constant volatility of volatility and thus, we allow the volatility of volatility to vary over time. We also provide statistical inferences for stochastic volatility models under the assumption of stochastic volatility of volatility testing, via simple statistical tests, whether both volatility measures vary over time.

We illustrated the usefulness of our approach giving an empirical example using intraday and daily data for the S&P 500 equity index. Our findings revealed strong evidence that both volatility and the volatility of volatility are stochastic. Then, we proceeded in a Monte Carlo simulation analysis and found that our estimates are reasonably accurate. We also evaluated our estimators by studying their predictability for the future values of the VIX and $VVIX$ indices. As we expected, the volatility is a significant predictor of the future market's volatility expectations, yet surprisingly, we find that the time-varying nature of the volatility of volatility is an additional source of risk. In other words, the volatility of volatility separately increases financial stress in the market. From a financial point of view, both past volatility and its volatility significantly forecast the future VIX and $VVIX$ indices, in a no-arbitrage model which features both time-varying volatility factors—as the model employed in this study—are priced by the investors.

Finally, our approach has important implications in identifying volatility risk premia. According to the findings, there are two different types of risk premia, one for the uncertainty induced by the volatility per se and one induced by the uncertainty of the volatility of volatility. The latter is often ignored in most asset price models but can still definitely be associated with deep uncertainty in the market. Future research may compare our estimators with other relevant estimators, either parametric or nonparametric, for stochastic volatility. Finally, another idea for future research would be to expand our estimators in a multivariate framework (see Amendola et al. 2020) or extend them to allow for jumps in the equation driving the observable variable.

Author Contributions: The authors have equal contributions. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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[5] Following Bekaert and Hoerova (2014), the squared VIX index can be decomposed into the variance of equity returns and the variance risk premium. For robustness purposes, in order to isolate the predictive power of past volatility and the volatility on volatility risk premium, we estimated (15) and (16) using the difference between the VIX indices and daily realized variance (estimated as the sum of 5-min intraday returns of the S&P 500) as dependent variables. In this regard, we also used in (15) and (16) the residuals obtained from $VIX_t = \beta_0^{VIX} + \beta_1^{VIX} v_t + u_{3t}$ and their squared specifications as dependent variables. In all cases, the results remain qualitative and quantitatively similar.

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