

Article

Effects of the Age Process on Aggregate Discounted Claims

Ghislain Léveillé * , Ilie-Radu Mitric and Victor Côté 

École D'actuariat, Université Laval, Québec, QC G1V 0A6, Canada; Ilie-Radu.Mitric@act.ulaval.ca (I.-R.M.); Victor.Cote.1@act.ulaval.ca (V.C.)

* Correspondence: Ghislain.Leveille@act.ulaval.ca

Received: 31 August 2018; Accepted: 21 September 2018; Published: 24 September 2018



Abstract: In this document, we examine the effects of the age process on aggregate discounted claims by studying the conditional raw and joint moments, the moment generating function and the distribution function of the increments of compound renewal sums with discounted claims, taking into account the past experience of an insurance portfolio.

Keywords: age process; aggregate discounted claims; increments; Lundberg-type bounds; moments; net force of interest; stochastic ordering; risk measures

1. Introduction

Compound renewal sums with discounted claims have been extensively studied under many aspects for the last few decades. The interested reader will find a relevant overview of this subject in [Garrido and Léveillé \(2014\)](#). Whether it is their raw moments as discussed in [Léveillé and Garrido \(2001a, 2001b\)](#), their joint moments derived in [Léveillé and Adékambi \(2011\)](#) or their moment generating functions (mgf) and related distributions such as examined by [Léveillé et al. \(2010\)](#), many challenging problems still remain within this context.

One of these challenges is to take into account all the information recorded by the insurer up to time t , i.e., number of claims, severities and occurrence times of the claims, in order to examine their incidence on the increment of the risk process on a subsequent time interval open at left $]t, t + h]$. This question has been largely studied in credibility theory (see [Bühlmann and Gisler \(2005\)](#)), but not within this context, compound renewal sums with discounted claims, where very detailed attention will be given to the age of our counting process, an aspect not covered by the credibility theory, but which is however very important.

Thus, in order to carry out this analysis, we have to make first some assumptions on the interest rate, on the counting process, on the severity of claims and on their possible dependence with the inter-occurrence times. Then, it will be also important to recall some results related to non-conditional moments of the increments of this risk process in order to compare them to their corresponding conditional moments. Finally, as a function of the age process, we will show some orderings of our risk process in terms of its distribution functions, moments and Lundberg-type bounds of ruin probabilities.

Hence, consider the following risk process:

$$Z(t) = \sum_{k=1}^{N(t)} e^{-\delta T_k} X_k,$$

where,

- $\{\tau_k; k \in N^* = \{1, 2, \dots\}\}$ is a sequence of continuous positive independent and identically distributed (i.i.d.) random variables, with common distribution function $F_\tau(x)$, such that τ_k represents the inter-occurrence time between the $(k - 1)$ -th and the k -th claims.
- $\{T_k; k \in N^*\}$ is a sequence of random variables such that $T_k = \sum_{i=1}^k \tau_i$, $T_0 = 0$, and then, T_k represents the occurrence time of the k -th claim.
- $\{X_k; k \in N^*\}$ is a sequence of positive i.i.d. random variables, with common distribution function $F_X(x)$, independent of $\{\tau_k; k \in N^*\}$, such that X_k represents the deflated amount of the claim.
- $\{N(t); t \geq 0\}$ is a counting process, which is generated by the inter-occurrence times τ_k and represents the number of corresponding claims occurring in $[0, t]$.
- $\delta > 0$ is a constant force of net interest.

Assuming only that the first two raw moments of the claim exist, a particular case of heavy tail distributions relatively frequent in actuarial problems, the following results were obtained by [Léveillé and Garrido \(2001a\)](#) and [Léveillé and Adékambi \(2011\)](#):

Proposition 1. Consider the risk process $\{Z(t); t \geq 0\}$, such as defined previously, and assume that $E[X] < \infty$, $E[X^2] < \infty$ and $h \geq 0$. Then,

$$\begin{aligned} (i). \quad E[Z(t)] &= E[X] \int_0^t e^{-\delta v} dm(v), \\ (ii). \quad E[Z^2(t)] &= E[X^2] \int_0^t e^{-2\delta v} dm(v) + 2E^2[X] \int_0^t \int_0^{t-v} e^{-\delta(2v+u)} dm(u) dm(v), \\ (iii). \quad E[Z(t)Z(t+h)] &= E[Z^2(t)] + E^2[X] \int_0^t \int_{t-v}^{t+h-v} e^{-\delta(2v+u)} dm(u) dm(v), \end{aligned}$$

where $m(t) = E[N(t)] = \sum_{k=1}^{\infty} F_\tau^{*k}(t)$ is the renewal function.

From these results, we are able to calculate a premium, based on the expectation and the standard deviation of our risk process, and we can also examine the autocorrelation function of our risk process. However, for an insurance business, the insurer must rather evaluate his/her risk from one period to another by also taking into account his/her past experience. Thus, our purpose is to extend the preceding results to this dynamical context and then analyze the incidence of the age process on the moments, distribution functions, risk measures and even on the ruin probability. This work must be seen as a useful complement to credibility theory (see [Bühlmann and Gisler \(2005\)](#)) and not just a pure incremental study on aggregate discounted claims.

2. First Raw and Joint Moments of the Conditional Increments

Now, assume that we are currently at the time t and that the past experience is given by the following set,

$$\Sigma_{n,t} = \{N(t) = n, T_i = t_i, X_i = x_i; i = 1, \dots, n\},$$

where $\Sigma_{n,0} = \emptyset$, $\Sigma_{0,t} = \{N(t) = 0; t > 0\}$.

Then, the conditional increment of the aggregate discounted claims on $]t, t+h]$, at time t , will be denoted by:

$$Z_{n,t}(h) = e^{\delta t} [Z(t+h) - Z(t)] | \Sigma_{n,t}.$$

Hence, by observing that our conditional counting process on $]t, t+h]$ is generated by a delayed renewal process $\{N_d(h); h \geq 0\}$, i.e., by the sequence of times $\{\tau_{n+1}^*, \tau_k; k > n+1\}$, where:

$$\tau_{n+1}^* = T_{n+1} - t | \Sigma_{n,t} = \tau_{n+1} - (t - t_n) | \tau_{n+1} > t - t_n, F_{\tau^*}(v) = F_{\tau_{n+1}^*}(v),$$

one can start by adapting the results obtained by Léveillé and Garrido (2001a) and Léveillé and Adékambi (2011) to our risk $Z_{n,t}(h)$ when the counting process is a delayed renewal process. We then have the following theorem.

Theorem 1. Consider the risk process $\{Z_{n,t}(h); h \geq 0\}$ such as defined previously. Then, by assuming that $E[X] < \infty$ and $E[X^2] < \infty$, we get the following moments:

$$\begin{aligned}
 (i). \quad E[Z_{n,t}(h)] &= E[X] \int_0^h e^{-\delta v} \left[1 + \int_0^{h-v} e^{-\delta u} dm(u) \right] dF_{\tau^*}(v). \\
 (ii). \quad E[Z_{n,t}^2(h)] &= E[X^2] \int_0^h e^{-2\delta v} \left[1 + \int_0^{h-v} e^{-2\delta u} dm(u) \right] dF_{\tau^*}(v) \\
 &\quad + 2E^2[X] \int_0^h e^{-2\delta v} \left[\int_0^{h-v} e^{-\delta u} dm(u) + \int_0^{h-v} \int_0^{h-v-w} e^{-\delta(u+2w)} dm(u) dm(w) \right] dF_{\tau^*}(v). \\
 (iii). \quad E[Z_{n,t}(h)Z_{n,t}(h+h')] &= E[Z_{n,t}^2(h)] \\
 &\quad + E^2[X] \int_0^h e^{-2\delta v} \left\{ \int_{h-v}^{h+h'-v} e^{-\delta u} dm(u) + \int_0^{h-v} \int_{h-v-w}^{h+h'-v-w} e^{-\delta(u+2w)} dm(u) dm(w) \right\} dF_{\tau^*}(v).
 \end{aligned}$$

Proof. (i) By conditioning on τ^* and applying Proposition 1 (i), we get the first moment, i.e.,

$$\begin{aligned}
 E[Z_{n,t}(h)] &= E[E[Z_{n,t}(h)|\tau^*]] \\
 &= E \left[e^{-\delta\tau^*} X_{n+1} + e^{-\delta\tau^*} E \left[\sum_{k=n+2}^{N(t+h)} e^{-\delta(\tau_{n+2}+\dots+\tau_k)} X_k | \tau^* \right] \right] \\
 &= E[X] E \left[e^{-\delta\tau^*} \left\{ 1 + E \left[\sum_{k=1}^{N(h-\tau^*)} e^{-\delta T_k} \right] \right\} \right] \\
 &= E[X] \left\{ \int_0^h e^{-\delta v} \left[1 + \int_0^{h-v} e^{-\delta u} dm(u) \right] dF_{\tau^*}(v) \right\},
 \end{aligned}$$

where $F_{\tau^*}(v) = \frac{F_{\tau}(v+t-t_n)}{F_{\tau}(t-t_n)}$.

(ii) By conditioning on τ^* and applying Propositions 1 (i) and (ii), we get the second moment, i.e.,

$$\begin{aligned}
 E[Z_{n,t}^2(h)] &= E[E[Z_{n,t}^2(h)|\tau^*]] \\
 &= E \left[E \left[\left(e^{-\delta\tau^*} X_{n+1} + e^{-\delta\tau^*} \sum_{k=n+2}^{N(t+h)} e^{-\delta(\tau_{n+2}+\dots+\tau_k)} X_k \right)^2 | \tau^* \right] \right] \\
 &= E \left[E \left[e^{-2\delta\tau^*} X_{n+1}^2 + 2e^{-2\delta\tau^*} X_{n+1} \sum_{k=n+2}^{N(t+h)} e^{-\delta(\tau_{n+2}+\dots+\tau_k)} X_k + e^{-2\delta\tau^*} \left(\sum_{k=n+2}^{N(t+h)} e^{-\delta(\tau_{n+2}+\dots+\tau_k)} X_k \right)^2 | \tau^* \right] \right]
 \end{aligned}$$

Hence,

$$\begin{aligned}
 E[Z_{n,t}^2(h)] &= E[X^2] E[e^{-2\delta\tau^*}] + 2E^2[X] E \left[e^{-2\delta\tau^*} E \left[\sum_{k=n+2}^{N(t+h)} e^{-\delta(\tau_{n+2}+\dots+\tau_k)} | \tau^* \right] \right] \\
 &\quad + E \left[e^{-2\delta\tau^*} \left(\sum_{k=n+2}^{N(t+h)} e^{-\delta(\tau_{n+2}+\dots+\tau_k)} X_k \right)^2 | \tau^* \right] \\
 &= E[X^2] \int_0^h e^{-2\delta v} dF_{\tau^*}(v) + 2E^2[X] \int_0^h e^{-2\delta v} E \left[\sum_{k=1}^{N(h-v)} e^{-\delta T_k} \right] dF_{\tau^*}(v) \\
 &\quad + \int_0^h e^{-2\delta v} E \left[\left(\sum_{k=1}^{N(h-v)} e^{-\delta T_k} X_k \right)^2 \right] dF_{\tau^*}(v) \\
 &= E[X^2] \int_0^h e^{-2\delta v} dF_{\tau^*}(v) + 2E^2[X] \int_0^h e^{-2\delta v} \int_0^{h-v} e^{-2\delta u} dm(u) dF_{\tau^*}(v) \\
 &\quad + \int_0^h e^{-2\delta v} \left\{ E[X^2] \int_0^{h-v} e^{-2\delta u} dm(u) + 2E^2[X] \int_0^{h-v} \int_0^{h-v-w} e^{-\delta(u+2w)} dm(u) dm(w) \right\} dF_{\tau^*}(v),
 \end{aligned}$$

which is equivalent to the following expression:

$$E[Z_{n,t}^2(h)] = E[X^2] \int_0^h e^{-2\delta v} \left[1 + \int_0^{h-v} e^{-2\delta u} dm(u) \right] dF_{\tau^*}(v) \\ + 2E^2[X] \int_0^h e^{-2\delta v} \left[\int_0^{h-v} e^{-\delta u} dm(u) + \int_0^{h-v} \int_0^{h-v-w} e^{-\delta(u+2w)} dm(u) dm(w) \right] dF_{\tau^*}(v).$$

(iii) By conditioning on τ^* and applying Propositions 1 (i)–(iii), we get the first joint moment, i.e.,

$$E[Z_{n,t}(h)Z_{n,t}(h+h')] = E[Z_{n,t}^2(h)] \\ + E \left[E \left[\left(\sum_{k=n+1}^{N(t+h)} e^{-\delta(\tau_{n+1}^* + \tau_{n+2}^* + \dots + \tau_k^*)} X_k \right) \left(\sum_{j=N(t+h)+1}^{N(t+h+h')} e^{-\delta(\tau_{n+1}^* + \tau_{n+2}^* + \dots + \tau_k^*)} X_j \right) \middle| \tau_{n+1}^* \right] \right] \\ = E[Z_{n,t}^2(h)] + E^2[X] E \left[E \left[\left(e^{-\delta\tau^*} + e^{-\delta\tau^*} \sum_{k=n+2}^{N(t+h)} e^{-\delta(\tau_{n+2}^* + \dots + \tau_k^*)} \right) \left(e^{-\delta\tau^*} \sum_{j=N(t+h)+1}^{N(t+h+h')} e^{-\delta(\tau_{n+2}^* + \dots + \tau_k^*)} \right) \middle| \tau^* \right] \right] \\ = E[Z_{n,t}^2(h)] \\ + E^2[X] E \left[e^{-2\delta\tau^*} \left\{ E \left[\sum_{j=N(t+h)+1}^{N(t+h+h')} e^{-\delta(\tau_{n+2}^* + \dots + \tau_k^*)} \middle| \tau^* \right] + E \left[\sum_{k=n+2}^{N(t+h)} e^{-\delta(\tau_{n+2}^* + \dots + \tau_k^*)} \sum_{j=N(t+h)+1}^{N(t+h+h')} e^{-\delta(\tau_{n+2}^* + \dots + \tau_k^*)} \middle| \tau^* \right] \right\} \right]$$

Hence,

$$E[Z_{n,t}(h)Z_{n,t}(h+h')] = E[Z_{n,t}^2(h)] \\ + E^2[X] \int_0^h e^{-2\delta v} \left\{ E \left[\sum_{j=N(h-v)+1}^{N(h+h'-v)} e^{-\delta T_k} \right] + E \left[\sum_{k=1}^{N(h-v)} e^{-\delta T_k} \sum_{j=N(h-v)+1}^{N(h+h'-v)} e^{-\delta T_j} \right] \right\} dF_{\tau^*}(v) \\ = E[Z_{n,t}^2(h)] \\ + E^2[X] \int_0^h e^{-2\delta v} \left\{ \int_{h-v}^{h+h'-v} e^{-\delta u} dm(u) + \int_0^{h-v} \int_{h-v-w}^{h+h'-v-w} e^{-\delta(u+2w)} dm(u) dm(w) \right\} dF_{\tau^*}(v).$$

□

Remark 1. (1) If the counting process is a homogeneous Poisson process, the memoryless property of the exponential inter-occurrence times implies the following identities,

- $E[Z_{n,t}(h)] = e^{\delta t} E[Z(t+h) - Z(t)],$
- $E[Z_{n,t}^2(h)] = e^{2\delta t} E[[Z(t+h) - Z(t)]^2],$
- $E[Z_{n,t}(h)Z_{n,t}(h+h')] = e^{2\delta t} E[[Z(t+h) - Z(t)][Z(t+h+h') - Z(t)]].$

(2) If $t_n = t$, then:

- $E[Z_{n,t}(h)] = E[Z(h)], E[Z_{n,t}^2(h)] = E[Z^2(h)],$
- $E[Z_{n,t}(h)Z_{n,t}(h+h')] = E[Z(h)Z(h+h')].$

(3) If $h \rightarrow \infty$, then we get the conditional asymptotic formulas for (i) and (ii), i.e.,

- $E[Z_{n,t}(\infty)] = E[X] \frac{L_{\tau^*}(\delta)}{1-L_{\tau}(\delta)},$
- $E[Z_{n,t}^2(\infty)] = E[X^2] \frac{L_{\tau^*}(2\delta)}{1-L_{\tau}(2\delta)} + 2E^2[X] \frac{L_{\tau}(\delta)}{1-L_{\tau}(\delta)} \frac{L_{\tau^*}(2\delta)}{1-L_{\tau}(2\delta)} \\ = \frac{L_{\tau^*}(2\delta)}{1-L_{\tau}(2\delta)} \left\{ E[X^2] + 2E^2[X] \frac{L_{\tau}(\delta)}{1-L_{\tau}(\delta)} \right\},$

where $L_{\tau}(\delta) = \int_0^{\infty} e^{-\delta v} dF_{\tau}(v)$ is the Laplace transform of the random variable τ .

Example 1. Consider the risk process $\{Z_{n,t}(h); h \geq 0\}$ such as given in Theorem 1, and assume that $\tau \sim \text{Erlang}(2,2)$, $X \sim \exp(1)$, $t = h = h' = 1$ and $\delta = 0.05$. Then, we obtain the following conditional expectations, standard deviations and autocorrelation functions of our conditional risk process for different values of the age $t - t_n$ in Table 1.

Table 1. Behavior of $Z_{n,1}(1)$ with respect to age $1 - t_n$.

$1 - t_n$	0	0.25	0.5	0.75	1
$E[Z_{n,1}(1)]$	0.73280	0.89454	0.97541	1.02393	1.05628
$E[Z_{n,1}^2(1)]$	1.76279	2.25139	2.49568	2.64226	2.73998
$\sigma[Z_{n,1}(1)]$	1.10715	1.20465	1.24268	1.26247	1.27446
$\rho[Z_{n,1}(1), Z_{n,1}(2)]$	0.66998	0.70132	0.71230	0.71774	0.72093

In this table, we observe that the first two conditional moments, standard deviation and autocorrelation function increase with the increase of the age process. This difference can be more apparent with a change of parameters. In other words, for this model, a more or less high age implies a more or less high risk for the next period.

For comparison, if you consider the non-conditional quantities corresponding to those calculated in the preceding table (actualized at time $t = 1$), then we get these middle range quantities:

$$\begin{aligned}
 E[e^{0.05}[Z(2) - Z(1)]] &= 0.97097, E[e^{0.1}[Z(2) - Z(1)]^2] = 2.48227, \\
 \sigma[e^{0.05}[Z(2) - Z(1)]] &= 1.24076, \\
 \rho[e^{0.05}[Z(2) - Z(1)], e^{0.05}[Z(3) - Z(1)]] &= 0.71177.
 \end{aligned}$$

Hence, for this model, this means that the risk evaluation without conditioning on the past experience of the insurance portfolio could lead to underestimating or overestimating the future risks and consequently to charging a premium that does not reflect the real risk.

In fact, the preceding behavior of our risk process with respect to the age process is not so surprising if we consider certain classes of distribution functions for the inter-occurrence time. In what follows, we analyze the behavior of the distribution function of $Z_{n,t}(h)$ with respect to the age process and the class of distributions to which the inter-occurrence time belongs. In order to do that, we will first recall two definitions, and we present an integral equation for the distribution function of $Z_{n,t}(h)$. The first definition is related to the usual stochastic order and the second one to the increasing (decreasing) failure rate family of distribution functions. We refer the interested reader to Müller and Stoyan (2002) or Shaked and Shanthikumar (2007).

Definition 1. Given two random variables X and Y , X is said to be preceding Y in the usual stochastic order, denoted as $X \leq_{st} Y$, if $\bar{F}_X(x) \leq \bar{F}_Y(x)$ for all $x \in \mathbf{R}$.

The preceding definition is equivalent to the inequality $E[w(X)] \leq E[w(Y)]$ for any non-decreasing function w such that the expectations exist.

Definition 2. A random variable X is said to have an increasing failure rate (IFR) if its survival function \bar{F}_X is log concave, and it is said to have a decreasing failure rate (DFR) if its survival function \bar{F}_X is log convex.

The preceding definition is equivalent to saying that X is IFR (DFR) if and only if, for each $x \leq x'$, we have $\{X - x | X > x\} \geq_{st} (\leq_{st}) \{X - x' | X > x'\}$. Note that if X is the lifetime of a device, then $\{X - x | X > x\}$ is the residual life of such a device with age x .

Now, an integral formula can be obtained for the distribution function of $Z_{n,t}(h)$, and that one will be solved (at least numerically) if we know the non-conditional distribution of $Z(t)$.

Lemma 1. Consider the risk process $\{Z(t); t \geq 0\}$ such as defined previously. Then, the conditional distribution function of $Z_{n,t}(h)$ is given, for $x \geq 0$, by:

$$F_{Z_{n,t}(h)}(x) = \bar{F}_{\tau^*}(h) + \int_0^h \int_0^{xe^{\delta v}} F_{Z(h-v)}(xe^{\delta v} - u) dF_X(u) dF_{\tau^*}(v).$$

Proof. Again, we condition on τ^* and X_{n+1} . Thus, we get:

$$\begin{aligned} F_{Z_{n,t}(h)}(x) &= \bar{F}_{\tau^*}(h) + \int_0^h \int_0^{xe^{\delta v}} P\left(ue^{-\delta v} + e^{-\delta v} \sum_{k=n+2}^{N(t+h)} e^{-\delta(T_k-v)} X_k \leq x | \tau^* = v\right) dF_X(u) dF_{\tau^*}(v) \\ &= \bar{F}_{\tau^*}(h) + \int_0^h \int_0^{xe^{\delta v}} P\left(\sum_{k=n+2}^{N(t+h)} e^{-\delta(T_k-v)} X_k \leq xe^{\delta v} - u | \tau^* = v\right) dF_X(u) dF_{\tau^*}(v) \\ &= \bar{F}_{\tau^*}(h) + \int_0^h \int_0^{xe^{\delta v}} P\left(\sum_{k=1}^{N(h-v)} e^{-\delta T_k} X_k \leq xe^{\delta v} - u\right) dF_X(u) dF_{\tau^*}(v) \\ &= \bar{F}_{\tau^*}(h) + \int_0^h \int_0^{xe^{\delta v}} F_{Z(h-v)}(xe^{\delta v} - u) dF_X(u) dF_{\tau^*}(v). \end{aligned}$$

□

Theorem 2. Consider the risk process $\{Z_{n,t}(h); h \geq 0\}$ such as defined previously. Then:

- If the inter-occurrence time is IFR (DFR), then $\bar{F}_{Z_{n,t}(h)}$ is a non-decreasing (non-increasing) function of the age $t - t_n$, i.e., $Z_{n,t}(h)$ increase (decrease) in stochastic order with the age $t - t_n$.
- If the inter-occurrence time is exponentially distributed, then $\bar{F}_{Z_{n,t}(h)}$ does not depend of the age $t - t_n$.

Proof. Lemma 1 yields:

$$F_{Z_{n,t}(h)}(x) = \int_0^\infty \left\{ 1_{]h,\infty[}(v) + 1_{[0,h]}(v) \int_0^{xe^{\delta v}} F_{Z(h-v)}(xe^{\delta v} - u) dF_X(u) \right\} dF_{\tau^*}(v).$$

Assuming that $w(x, h, v) = 1_{]h,\infty[}(v) + 1_{[0,h]}(v) \int_0^{xe^{\delta v}} F_{Z(h-v)}(xe^{\delta v} - u) dF_X(u)$, then:

$$w(x, h, v) = \begin{cases} P(Z(h-v) + X \leq xe^{\delta v}) & , v \in [0, h] \\ 1 & , v > h \end{cases}$$

and then, $F_{Z_{n,t}(h)}(x) = E[w(x, h, \tau^*)]$.

Now, when τ is IFR (DFR), we have for $t'_n \leq t_n \leq t$,

$$\{\tau - (t - t_n) | \tau > t - t_n\} \geq_{st} (\leq_{st}) \{\tau - (t - t'_n) | \tau > t - t'_n\}.$$

Since $w(x, h, v)$ is a non-decreasing function of v for each positive x and h , Definition 2 implies for $t'_n \leq t_n$:

$$E[w(x, h, \tau - (t - t^n) | \tau > t - t^n)] \geq (\leq) E[w(x, h, \tau - t^n) | \tau > t - t'_n]$$

Thus, assuming that $Z'_{n,t}(h)$ includes the last occurrence time t'_n , this last inequality and Definition 1 yield for any $x \in \mathbf{R}$:

$$F_{Z_{n,t}(h)}(x) \geq (\leq) F_{Z'_{n,t}(h)}(x) \Leftrightarrow \bar{F}_{Z_{n,t}(h)}(x) \leq (\geq) \bar{F}_{Z'_{n,t}(h)}(x) \Leftrightarrow Z_{n,t}(h) \leq_{st} (\geq_{st}) Z'_{n,t}(h).$$

Finally, if τ is exponentially distributed, it satisfies both the IFR and DFR requirements. Consequently, the distribution function of $Z_{n,t}(h)$ is independent of the age process, which is coherent with the memoryless property of the exponential distribution. \square

Remark 2. In Example 1, the inter-occurrence time τ follows an Erlang distribution with shape parameter $\alpha = 2$, which has an IFR distribution.

Corollary 1. If the inter-occurrence time τ is IFR (DFR), then $E[Z_{n,t}^k(h)]$ is a non-decreasing (non-increasing) function of the age $t - t_n$, for any $k \in \mathbb{N}^*$, such that the corresponding expectation exists.

Proof. From Theorem 2, and for $t'_n \leq t_n$, we have:

$$Z_{n,t}(h) \leq_{st} (\geq_{st}) Z'_{n,t}(h).$$

Since $w(x) = I_{[0,\infty[}(x)x^k$ is a non-decreasing function of x , Definition 1 yields:

$$E[Z_{n,t}^k(h)] \leq E[Z'_{n,t}^k(h)].$$

\square

Corollary 2. If the inter-occurrence time τ is IFR (DFR), then $M_{Z_{n,t}(h)}(s)$ is a non-decreasing (non-increasing) function of $s > 0$ ($s < 0$), for each $s \in \mathbb{R}$, such that the mgf exists.

Proof. Similarly to Corollary 1, we apply Definition 1 to the function $w(x) = e^{sx}$ and then get:

$$M_{Z_{n,t}(h)}(s) \leq (\geq) M_{Z'_{n,t}(h)}(s).$$

\square

Corollary 3. If the inter-occurrence time τ is IFR (DFR), then the Wang risk measure associated with our risk process $Z_{n,t}(h)$, i.e., the functional defined by:

$$\rho_g(Z_{n,t}(h)) = \int_0^\infty g(\bar{F}_{Z_{n,t}(h)}(x)) dx,$$

is a non-increasing (non-decreasing) function of the age $t - t_n$, where g is a non-decreasing function with $g(0) = 0$, $g(1) = 1$.

Proof. This is a direct consequence of Definition 1 and Theorem 2. \square

Remark 3. Let us recall that, for particular choices of the function g , we get certain classical risk measures such as VaR (value at risk), TVaR (tail value at risk), proportional hazard (PH)-transform and many other known distortion and spectral risk measures.

3. mgf of the Conditional Increments

Assume that the mgf of $Z(t)$ exists, i.e., that the distribution of X has a light tail. The next theorem gives an analytic expression for the mgf of $Z_{n,t}(h)$, from which all its conditional moments can be calculated in terms of the non-conditional recursive moments of $Z(t)$.

Theorem 3. Consider the risk process $\{Z(t); t \geq 0\}$ such as defined previously, and assume that the mgf of X , i.e., $M_X(s)$, exists on a subset $\Omega \in \mathbf{R}$ that contains the origin. Then, the mgf of the conditional increment process $\{Z_{n,t}(h); h \geq 0\}$ is given by:

$$M_{Z_{n,t}(h)}(s) = \bar{F}_{\tau^*}(h) + \int_0^h M_X(se^{-\delta v}) M_{Z(h-v)}(se^{-\delta v}) dF_{\tau^*}(v).$$

Proof. By conditioning on τ^* , we have:

$$\begin{aligned} M_{Z_{n,t}(h)}(s) &= E \left[E \left[e^{sZ_{n,t}(h)} | \tau^* \right] \right] \\ &= \int_h^\infty dF_{\tau^*}(v) + \int_0^h E \left[e^{sZ_{n,t}(h)} | \tau^* = v \right] dF_{\tau^*}(v) \\ &= \bar{F}_{\tau^*}(h) + \int_0^h E \left[\exp \left\{ se^{-\delta v} X_{n+1} + se^{-\delta v} \sum_{k=n+2}^{N(t+h)} e^{-\delta(\tau_{n+2} + \dots + \tau_k)} X_k \right\} | \tau^* = v \right] dF_{\tau^*}(v) \\ &= \bar{F}_{\tau^*}(h) + \int_0^h M_X(se^{-\delta v}) E \left[\prod_{k=n+2}^{N(t+h)} M_X \left(se^{-\delta v} \left(e^{-\delta(\tau_{n+2} + \dots + \tau_k)} \right) \right) | \tau^* = v \right] dF_{\tau^*}(v) \\ &= \bar{F}_{\tau^*}(h) + \int_0^h M_X(se^{-\delta v}) E \left[\prod_{k=1}^N M_X(se^{-\delta v}) \right] dF_{\tau^*}(v) \end{aligned}$$

As in [Léveillé and Garrido \(2001b\)](#), $E \left[\prod_{k=1}^{N(h-v)} M_X(se^{-\delta v} (e^{-\delta T_k})) \right] = M_{Z(h-v)}(se^{-\delta v})$, and then:

$$M_{Z_{n,t}(h)}(s) = \bar{F}_{\tau^*}(h) + \int_0^h M_X(se^{-\delta v}) M_{Z(h-v)}(se^{-\delta v}) dF_{\tau^*}(v).$$

□

Corollary 4. Consider the risk process $\{Z(t); t \geq 0\}$, such as defined in Theorem 3. Then, the m -th moment of the conditional increment process $\{Z_{n,t}(h); h \geq 0\}$ is given by:

$$M_{Z_{n,t}(h)}^{(m)}(0) = \sum_{j=0}^m \binom{m}{j} M_X^{(j)}(0) \int_0^h e^{-m\delta v} M_{Z(h-v)}^{(m-j)}(0) dF_{\tau^*}(v), \quad m \in \mathbf{N}^*,$$

where (see [Léveillé and Garrido \(2001b\)](#)),

$$M_{Z(u)}^{(k)}(0) = \sum_{i=0}^{k-1} \binom{k}{i} M_X^{(k-i)}(0) \int_0^u e^{-k\delta v} M_{Z(h-v)}^{(i)}(0) dm(v), \quad k \in \mathbf{N}^*, u > 0.$$

Proof. We easily verify that:

$$M_{Z_{n,t}(h)}^{(m)}(s) = \sum_{j=0}^m \binom{m}{j} M_X^{(j)}(se^{-\delta v}) \int_0^h e^{-m\delta v} M_{Z(h-v)}^{(m-j)}(se^{-\delta v}) dF_{\tau^*}(v), \quad m \in \mathbf{N}^*,$$

from which we get the result by evaluating the last identity at $s = 0$. □

Example 2. Assume that $t = h = 1$, $t_n = 0.5$, $\delta = 0.01$ and X and τ have the following probability density functions:

$$f_X(x) = (0.5e^{-x} + e^{-2x})1_{[0,\infty[}(x), f_\tau(t) = (0.02e^{-0.04t} + 0.05e^{-0.12t}\sinh(0.08t))1_{[0,\infty[}(t)$$

Then, the mgf of X is given by:

$$M_X(s) = \frac{0.5(4-3s)}{(1-s)(2-s)}, s < 1$$

and according to Wang et al. (2018), the mgf of $Z(t)$ is given by:

$$M_{Z(t)}(s) = \frac{s^2e^{-0.02t} - 3se^{-0.01t} + 2}{(1-s)(2-s)}s < 1.$$

The resulting integral equation of Theorem 3 yields:

$$M_{Z_{n,1}(1)}(s) = \frac{1 - F_\tau(1.5)}{1 - F_\tau(0.5)} + \frac{1}{1 - F_\tau(0.5)} \int_0^1 M_X(se^{-0.01v}) M_{Z(1-v)}(se^{-0.01v}) f_\tau(0.5 + v) dv.$$

As it is relatively hard to find an exact expression to the preceding integral, we will approximate $M_{Z_{n,1}(1)}(s)$ as follows:

- Firstly, we evaluate $M_{Z_{n,1}(1)}(s)$ at $s = -2, -1.8, -1.6, \dots, 0, 0.2, \dots, 0.8, 0.99999$. For each value of s , we use the adaptive quadrature method of Simpson with 100 partitions of the interval $[0, 1]$.
- Secondly, with the 16 points obtained, we then use the Thiele interpolation method to get an approximation of $M_{Z_{n,1}(1)}(s)$.

We thus obtain,

$$M_{Z_{n,1}(1)}(s) \approx \frac{A(s)}{B(s)},$$

where,

$$\begin{aligned} A(s) = & 5.610217992 - 13.45352888s - 22.26490450s^2 \\ & + 41.51418664s^3 + 14.68293130s^4 - 25.77608592s^5 \\ & - 6.269903420s^6 + 5.956565424s^7 + 7.293961274 \times 10^{-8}s^8 \end{aligned}$$

and:

$$\begin{aligned} B(s) = & 5.610217996 + 13.55065137s - 22.11155393s^2 \\ & - 42.01989981s^3 + 14.38336892s^4 + 26.24459595s^5 \\ & - 6.203308268s^6 - 6.096631995s^7. \end{aligned}$$

The corresponding Laplace transform can be inverted to give the (approximate) defective density function of $Z_{n,1}(1)$, i.e.,

$$\begin{aligned} f_{Z_{n,1}(1)}(x) \approx & \{0.02264150961e^{-2.000375828x} + 0.002402451389e^{-1.001576285x} \\ & + 0.009255658868e^{-0.9998435682x} - 1.130460919 \times 10e^{-0.3523520060x} \\ & + 6.782429960 \times 10^{-13}e^{0.6043858815x} \\ & - 7.710551954 \times 10^{-11}e^{1.366132116x} \cos(0.5392463904x) \\ & - 2.141819988 \times 10^{-11}e^{1.366132116x} \sin(0.5392463904x)\} 1_{[0,8.1966]}(x) \end{aligned}$$

and probability mass at $x = 0$, i.e., $P(Z_{n,1}(1) = 0) \approx 0.9770255943$.

How accurate is this approximation of the distribution of $Z_{n,1}(1)$? The answer will be given in Table 2, if we compare the exact and approximate values of the probability mass at $x = 0$ and the first two moments of $Z_{n,1}(1)$.

Table 2. Comparison of quantities related to the distribution of $Z_{n,1}(1)$

	$P(Z_{n,1}(1) = 0)$	$E[Z_{n,1}(1)]$	$E[Z_{n,1}^2(1)]$	$\alpha[Z_{n,1}(1)]$
Exact	0.9770254902	0.01732118193	0.02898943080	0.1693794777
Approximate	0.9770255943	0.01730588222	0.02886125039	0.1690022391
Absolute error	0.0000001041	0.00001529971	0.00012818041	0.0003772386
Relative error	0.0000001065	0.00088329480	0.00444126323	0.0022271801

We can conclude that this approximation of the distribution function of $Z_{n,1}(1)$ is satisfactory since the relative errors of the previous quantities are very small. The accuracy of our method can be improved in many ways by adding more interpolation points, by refining the partition in our adaptive quadrature method, and so on.

4. Conditional Increments in a Context with More Dependence

In this section, we examine our conditional increment process by adding to the previous assumptions that $\{(X_k, \tau_k); k \in N^*\}$ forms a sequence of i.i.d. random vectors where the components of each vector could be dependent. This new assumption could be interpreted such that if the inter-occurrence time τ_k is greater than a certain threshold, then the distribution of the next claim amount X_k could be modified. See [Albrecher and Teugels \(2006\)](#), [Asimit and Badescu \(2010\)](#) or [Woo and Cheung \(2013\)](#) for a discussion on this hypothesis.

For brevity's sake, we will focus our attention only on the first two moments of our conditional risk process. However, before presenting an expression for these first two conditional moments, we need a lemma that gives an integral expression for the first two non-conditional moments of our risk process, which takes into account our additional assumption. This lemma improves greatly the method used by [Bargès et al. \(2011\)](#) to get these moments for the compound Poisson process with discounted claims and generalize them to compound renewal sums with discounted claims.

Lemma 2. Consider our basic risk process $\{Z(t); t \geq 0\}$ such as defined in Section 1, except that we assume now that $\{(X_k, \tau_k); k \in N^*\}$ forms a sequence of i.i.d. random vectors with $F_{(X_k, \tau_k)}(x, y) = F_{(X, \tau)}(x, y)$. Then, the first two moments of our (non-conditional) risk process are given by:

$$\begin{aligned}
 (i) \quad E[Z(t)] &= \int_0^t e^{-\delta u} E[X|\tau = u] \left\{ 1 + \int_0^{t-u} e^{-\delta v} dm(v) \right\} dF_\tau(u), \\
 E[Z^2(t)] &= \int_0^t e^{-2\delta u} E[X^2|\tau = u] \left\{ 1 + \int_0^{t-u} e^{-2\delta v} dm(v) \right\} dF_\tau(u) \\
 (ii) \quad &+ 2 \int_0^t e^{-2\delta u} E[X|\tau = u] \left\{ E[Z(t-u)] + \int_0^{t-u} e^{-2\delta v} E[Z(t-u-v)] dm(v) \right\} dF_\tau(u).
 \end{aligned}$$

Proof. See [Léveillé and Hamel \(2018\)](#) and Appendix A. \square

Example 3. Consider the risk process $\{Z(t); t \geq 0\}$ such as defined in Lemma 2, and assume that $\tau \sim \text{Erl}(2, 2)$, $X \sim \exp(1)$, $\delta = 0.05$ and that the joint density function of the random vector (X, τ) is given by the Farlie–Gumbel–Morgenstern copula with dependence parameter $\theta = 1$ (see [Nelsen \(2006\)](#)), i.e.,

$$f_{X, \tau}(x, y) = 4ye^{-x-2y} \left\{ 1 + [-1 + 2e^{-x}] [-1 + (2 + 4y)e^{-2y}] \right\}.$$

Then, we obtain the following first two moments of $Z(t)$:

$$\begin{aligned}
 \bullet \quad E[Z(t)] &= (0.4298032364 + 0.7407031077t + 0.4999237927t^2 - 0.6584362139t^3)e^{-4.05t} \\
 &\quad - (2.821534801 \times 10^{-10} + 8.634146341 \times 10^{-10}t)e^{-2.05t} - 20e^{-0.05t} + 19.57019676.
 \end{aligned}$$

$$\begin{aligned}
\bullet \quad E[Z^2(t)] &= 0.0005344418631e^{-8.15t} \\
&+ (-5.628670301 \times 10^9 + 1.127823677t + 0.1685352370t^2)e^{-4.15t} \\
&+ (-1.585076044 \times 10^{12} - 4.818405190 \times 10^{10}t - 5.892141072 \times 10^8t^2 \\
&- 3.412320328 \times 10^6t^3 - 0.3325301684 \times 10^{-4}t^4 + 9.066666667 \times 10^{-7}t^5)e^{-4.1t} \\
&+ (1.605576585 \times 10^{12} - 3.154533279 \times 10^{10}t + 1.844758398 \times 10^8t^2 \\
&+ 1.706645888 \times 10^5t^3)e^{-4.05t} \\
&+ (-1.487187100 \times 10^{10} + 6.563090127 \times 10^8t - 9.201544226 \times 10^6t^2)e^{-4t} \\
&+ (1.007826244 \times 10^7 + 2.143574930 \times 10^5t + 0.1481056999 \times 10^{-5}t^2 \\
&- 0.1441068156 \times 10^{-5}t^3)e^{-2.1t} \\
&+ (-1.151963835 \times 10^7 + 2.913964353 \times 10^5t)e^{-2.05t} \\
&+ (1.441371845 \times 10^6 - 73937.78256t)e^{-2t} \\
&+ (510.9436234 - 8.141354387 \times 10^{-9}t + 5.516497390 \times 10^{-9}t^2)e^{-0.15t} \\
&+ (-460.7647363 + 7.294504087 \times 10^{-9}t - 5.063291141 \times 10^{-9}t^2)e^{-0.1t} \\
&+ (-428.5094693)e^{-0.05t} + (-0.005615672923t)e^{2t} \\
&+ 393.5612420 + 0.3070688311t - 0.1027233695t^2.
\end{aligned}$$

Theorem 4. Consider the risk process $\{Z_{n,t}(h); h \geq 0\}$ such as defined in Theorem 2, except that we assume now that $\{(X_k, \tau_k); k \in N^*\}$ forms a sequence of i.i.d. random vectors. Then, the first two moments of our conditional risk process are given by:

$$\begin{aligned}
(i) \quad E[Z_{n,t}(h)] &= \int_0^h e^{-\delta v} \{E[X|\tau = v + t - t_n] + E[Z(h - v)]\} dF_{\tau^*}(v), \\
(ii) \quad E[Z_{n,t}^2(h)] &= \int_0^h e^{-2\delta v} \{E[X^2|\tau = v + t - t_n] + 2E[X|\tau = v + t - t_n]E[Z(h - v)] \\
&+ E[Z^2(h - v)]\} dF_{\tau^*}(v),
\end{aligned}$$

where $E[Z(h - v)]$ and $E[Z^2(h - v)]$ are obtained from Lemmas 2 (i) and (ii).

Proof. (i) By conditioning on τ^* and using Lemma 2 (i), we have:

$$\begin{aligned}
E[Z_{n,t}(h)] &= E[E[Z_{n,t}(h)|\tau^*]] \\
&= E\left[e^{-\delta\tau^*} E[X_{n+1}|\tau^*] + e^{-\delta\tau^*} E\left[\sum_{k=n+2}^{N(t+h)} e^{-\delta(\tau_{n+2}+\dots+\tau_k)} X_k|\tau^*\right]\right] \\
&= \int_0^h e^{-\delta v} E[X|\tau = v + t - t_n] dF_{\tau^*}(v) \\
&\quad + \int_0^h e^{-\delta v} \int_0^{h-v} e^{-\delta u} E[X|\tau = u] \left[1 + \int_0^{h-v-u} e^{-\delta w} dm(w)\right] dF_{\tau}(u) dF_{\tau^*}(v) \\
&= \int_0^h e^{-\delta v} \{E[X|\tau = v + t - t_n] + E[Z(h - v)]\} dF_{\tau^*}(v)
\end{aligned}$$

(ii) By conditioning on τ^* and using Lemmas 2 (i) and (ii), we have:

$$\begin{aligned}
E[Z_{n,t}^2(h)] &= E[E[Z_{n,t}^2(h)|\tau^*]] = E\left[E\left[\left\{e^{-\delta\tau^*} X_{n+1} + e^{-\delta\tau^*} \sum_{k=n+2}^{N(t+h)} e^{-\delta(\tau_{n+2}+\dots+\tau_k)} X_k\right\}^2 \middle| \tau^*\right]\right] \\
&= E\left[E\left[e^{-2\delta\tau^*} X_{n+1}^2 + 2e^{-2\delta\tau^*} X_{n+1} \sum_{k=n+2}^{N(t+h)} e^{-\delta(\tau_{n+2}+\dots+\tau_k)} X_k + e^{-2\delta\tau^*} \left[\sum_{k=n+2}^{N(t+h)} e^{-\delta(\tau_{n+2}+\dots+\tau_k)} X_k\right]^2 \middle| \tau^*\right]\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
 E[Z_{n,t}^2(h)] &= E \left[e^{-2\delta\tau^*} E[X_{n+1}^2 | \tau^*] + 2e^{-2\delta\tau^*} E[X_{n+1} | \tau^*] E \left[\sum_{k=n+2}^{N(t+h)} e^{-\delta(\tau_{n+2} + \dots + \tau_k)} X_k | \tau^* \right] \right. \\
 &\quad \left. + e^{-2\delta\tau^*} E \left[\sum_{k=n+2}^{N(t+h)} e^{-\delta(\tau_{n+2} + \dots + \tau_k)} X_k \right]^2 | \tau^* \right] \\
 &= \int_0^h e^{-2\delta v} E[X_{n+1}^2 | \tau^* = v] dF_{\tau^*}(v) + 2 \int_0^h e^{-2\delta v} E[X_{n+1} | \tau^* = v] E \left[\sum_{k=1}^{N(h-v)} e^{-\delta T_k} X_k \right] dF_{\tau^*}(v) \\
 &\quad + \int_0^h e^{-2\delta v} E \left[\sum_{k=1}^{N(h-v)} e^{-\delta T_k} X_k \right]^2 dF_{\tau^*}(v) \\
 &= \int_0^h e^{-2\delta v} \{ E[X^2 | \tau = v + t - t_n] + 2E[X | \tau = v + t - t_n] E[Z(h-v)] \\
 &\quad + E[Z^2(h-v)] \} dF_{\tau^*}(v).
 \end{aligned}$$

□

Remark 4. If we remove the assumption of dependence between X_k and τ_k , then we retrieve the results of Theorems 1 (i) and 1 (ii). Thus, the first moment yields:

$$\begin{aligned}
 E[Z_{n,t}(h)] &= E[X] \int_0^h e^{-\delta v} dF_{\tau^*}(v) + E[X] \int_0^h e^{-\delta v} \int_0^{h-v} e^{-\delta u} \left[1 + \int_0^{h-v-u} e^{-\delta w} dm(w) \right] dF_{\tau}(u) dF_{\tau^*}(v) \\
 &= E[X] \int_0^h e^{-\delta v} \left[1 + \int_0^{h-v} e^{-\delta u} dm(u) \right] dF_{\tau^*}(v),
 \end{aligned}$$

and using Proposition 1, the second moment yields:

$$\begin{aligned}
 E[Z_{n,t}^2(h)] &= E[X^2] \int_0^h e^{-2\delta v} dF_{\tau^*}(v) + 2E[X] \int_0^h e^{-2\delta v} E[Z(h-v)] dF_{\tau^*}(v) \\
 &\quad + \int_0^h e^{-2\delta v} E[Z^2(h-v)] dF_{\tau^*}(v) \\
 &= E[X^2] \int_0^h e^{-2\delta v} \left[1 + \int_0^{h-v} e^{-2\delta u} dm(u) \right] dF_{\tau^*}(v) \\
 &\quad + 2E^2[X] \int_0^h e^{-2\delta v} \left[\int_0^{h-v} e^{-\delta u} dm(u) + \int_0^{h-v} \int_0^{h-v-w} e^{-\delta(u+2w)} dm(u) dm(w) \right] dF_{\tau^*}(v).
 \end{aligned}$$

Example 4. Consider the risk process $\{Z_{n,t}(h); h \geq 0\}$ such as given in Theorem 4, and consider the same assumptions such as given in Example 3. Then, we obtain the following conditional expectations of our risk process for different values of the age $t - t_n$ in Table 3.

Table 3. Behavior of $Z_{n,1}(1)$ with respect to age $1 - t_n$ and dependence between X_k and τ_k .

$1 - t_n$	0	0.25	0.5	0.75	1
$E[Z_{n,1}(1)]$	0.56324	0.75787	0.91893	1.05256	1.15882
$E[Z_{n,1}^2(1)]$	0.97900	1.52900	2.02490	2.46600	2.81680
$\sigma[Z_{n,1}(1)]$	0.81349	0.97706	1.08650	1.16538	1.21406

Again, we observe that the first two conditional moments increase with the increasing of the age of the process, but with greater variations than those obtained for the corresponding conditional moments in Table 1,

where X_k and τ_k are independent. On the other hand, the values of the standard deviation always increase with the increasing of the age of the process, but they are all relatively smaller than those obtained in Table 1.

Hence, in addition to conditioning on the past experience, we can conclude that the fact of adding dependence between X_k and τ_k has increased even more the risk of underestimating or overestimating the risks over the next period.

5. A Lundberg-Type Inequality, Conditional to the Past Experience

For several decades, the Lundberg inequality has been an “object of predilection” in the actuarial literature. Several models were considered in ruin theory for which Lundberg-type bounds were found. For the interested reader, excellent surveys on this subject can be found in Lin (2014) and Lin (2014).

A direction that has not been sufficiently explored is the dynamical context of an insurance company where the ruin probability can change from one period to another, if the past experience is taken into account. For an illustration of this idea, and in order to recover some past results on Lundberg-type inequalities, we will consider the following accumulated surplus process with constant interest rate:

$$U_\delta(t+h) = U_\delta(t)e^{\delta h} + c\bar{s}_{h\delta} - \{Z(t+h) - Z(t)\}e^{\delta(t+h)},$$

where $\bar{s}_{h\delta} = \frac{e^{\delta h} - 1}{\delta}$, δ is a force of interest without inflation and c is a constant premium.

Here, we also assume that $M_X(s)$ exists for $0 < s < \gamma$ and $\lim_{s \rightarrow \infty} M_X(s) = \infty$, $E[c\bar{s}_{\tau\delta} - X] > 0$ and that the surplus process is positive on $[0, t]$. Thus, by conditioning on the past experience, the occurrence time and the severity of the first claim in the interval $]t, \infty]$, we get:

$$\Psi(U_\delta(t) | \Sigma_{n,t}^+) = \int_0^\infty \left[\int_0^{U_\delta(t)e^{\delta v} + c\bar{s}_{v\delta}} \Psi(U_\delta(t)e^{\delta v} + c\bar{s}_{v\delta} - w) dF_X(w) + \bar{F}_X(U_\delta(t)e^{\delta v} + c\bar{s}_{v\delta}) \right] dF_{\tau^*}(v),$$

where $\Sigma_{n,t}^+ = \Sigma_{n,t} \cap \{U_\delta(v) \geq 0; 0 \leq v \leq t\}$.

Now, since after time τ^* , we have an ordinary renewal process, it can be proven (see Cai and Dickson (2003)) that a constant $\varsigma > 0$ exists, which is the unique solution of the following expression:

$$E[\exp\{-\varsigma(c\bar{s}_{v\delta} - X)\}] = 1,$$

from which we get for $U_\delta(t)e^{\delta v} + c\bar{s}_{v\delta} > w$,

$$\Psi(U_\delta(t)e^{\delta v} + c\bar{s}_{v\delta} - w) \leq \beta e^{-\varsigma(U_\delta(t)e^{\delta v} + c\bar{s}_{v\delta} - w)}, \beta^{-1} = \inf_{x \geq 0} \int_x^\infty [e^{\varsigma v} / (e^{\varsigma x} \bar{F}_X(x))] dF_X(v).$$

Hence, using the previous inequality and the definition of β , we get:

$$\begin{aligned} \Psi(U_\delta(t) | \Sigma_{n,t}^+) &\leq \int_0^\infty \left[\int_0^{U_\delta(t)e^{\delta v} + c\bar{s}_{v\delta}} \beta e^{-\varsigma(U_\delta(t)e^{\delta v} + c\bar{s}_{v\delta} - w)} dF_X(w) \right. \\ &\quad \left. + \beta e^{-\varsigma(U_\delta(t)e^{\delta v} + c\bar{s}_{v\delta})} \int_{U_\delta(t)e^{\delta v} + c\bar{s}_{v\delta}}^\infty e^{\varsigma w} dF_X(w) \right] dF_{\tau^*}(v) \\ &= \beta \int_0^\infty \left[\int_0^\infty e^{-\varsigma(U_\delta(t)e^{\delta v} + c\bar{s}_{v\delta} - w)} dF_X(w) \right] dF_{\tau^*}(v) \\ &= \beta M_X(\varsigma) E[e^{-\varsigma(U_\delta(t)e^{\delta \tau^*} + c\bar{s}_{\tau^*\delta})}]. \end{aligned}$$

Generally speaking, the calculations of this Lundberg-type bound have to be done numerically, but many software packages (such as Maple or Mathematica) can evaluate these quantities with a high degree of accuracy.

Now, let us examine some special cases for the preceding formula. For example, if $t - t_n = 0$, then we easily find one of the Lundberg-type inequalities of [Cai and Dickson \(2003\)](#), i.e.,

$$\begin{aligned}\Psi(U_\delta(t) | \Sigma_{n,t}^+) &\leq \beta M_X(\zeta) E \left[e^{-\zeta(U_\delta(t)e^{\delta\tau} + c\bar{s}_{\tau\delta})} \right] \\ &= \beta M_X(\zeta) e^{-\zeta U_\delta(t)} E \left[e^{-\zeta(\delta U_\delta(t) + c)\bar{s}_{\tau\delta}} \right] \\ &\leq \beta e^{-\zeta U_\delta(t)} E \left[e^{-\zeta(c\bar{s}_{v\delta} - X)} \right] \\ &= \beta e^{-\zeta U_\delta(t)}.\end{aligned}$$

If $t - t_n > 0$ and $\delta = 0$, then:

$$\Psi(U_0(t) | \Sigma_{n,t}^+) \leq \beta M_X(\zeta) \int_0^\infty e^{-\zeta(U_0(t) + cv)} dF_{\tau^*}(v) = \beta e^{-\zeta U_0(t)} E \left[e^{-\zeta(c\delta^* - X)} \right],$$

where ζ is the unique solution of $E[\exp\{-\zeta(c\delta - X)\}] = 1$.

If we add to the preceding inequality the assumption that $\tau \sim \exp(\lambda)$, then we obtain the classical Lundberg bound, i.e.,

$$\Psi(U_0(t) | \Sigma_{n,t}^+) \leq \beta e^{-\zeta U_0(t)} E \left[e^{-\zeta(c\delta - X)} \right] = \beta e^{-\zeta U_0(t)} \leq e^{-\zeta U_0(t)}.$$

Finally, assume that the inter-occurrence time of the claims is an IFR (DFR) distribution. Then, the following inequalities hold between the conditional and non-conditional Lundberg-type bounds, i.e.,

$$\beta M_X(\zeta) E \left[e^{-\zeta(U_\delta(t)e^{\delta\tau^*} + c\bar{s}_{\tau^*\delta})} \right] \geq (\leq) \beta M_X(\zeta) E \left[e^{-\zeta(U_\delta(t)e^{\delta\tau} + c\bar{s}_{\tau\delta})} \right].$$

Hereafter, we present a proof of these last two inequalities and an example that illustrates both cases.

Proof. Assume that τ is an IFR (DFR) distribution. In particular, the respective assumptions imply that:

$$\tau = \tau - 0 | \tau > 0 \geq_{st} (\leq_{st}) \tau^* = \tau - (t - t_n) | \tau > t - t_n, t > t_n.$$

Therefore, for the non-decreasing function $w(v) = -e^{-\zeta(U_\delta(t)e^{\delta v} + c\bar{s}_{v\delta})}$, we have:

$$E \left[-e^{-\zeta(U_\delta(t)e^{\delta\tau} + c\bar{s}_{\tau\delta})} \right] \geq (\leq) E \left[-e^{-\zeta(U_\delta(t)e^{\delta\tau^*} + c\bar{s}_{\tau^*\delta})} \right] \Leftrightarrow E \left[e^{-\zeta(U_\delta(t)e^{\delta\tau^*} + c\bar{s}_{\tau^*\delta})} \right] \geq (\leq) E \left[e^{-\zeta(U_\delta(t)e^{\delta\tau} + c\bar{s}_{\tau\delta})} \right]$$

The conclusion follows easily by multiplying both sides of the inequalities by $\beta M_X(\zeta)$. \square

Example 5. Consider the risk process such as defined at the beginning of this section, and assume that $\Sigma_{2,1} = \{N(1) = 2, T_1 = 0.1, T_2 = 0.25, X_1 = 1.5, X_2 = 2\}$, $\tau \sim \text{Erl}(2, 2)$, $X \sim \exp(1)$, $\delta = 0.05$, $c = 2$ and $U_\delta(0) = 5$. Then, we have:

$$U_{0.05}(1) = 3.661733537, \zeta = 0.630463424, \beta = 0.369536576,$$

which implies that, for this IFR distribution,

$$\begin{aligned}(0.369536576)M_X(0.630463424)E \left[e^{-0.630463424(3.661733537 e^{0.05\tau^*} + 2\bar{s}_{\tau^*0.05})} \right] &= 0.04881733328 \geq \\ (0.369536576)M_X(0.630463424)E \left[e^{-0.630463424(3.661733537 e^{0.05\tau} + 2\bar{s}_{\tau0.05})} \right] &= 0.03425337420.\end{aligned}$$

Now, keep the same assumptions except that now $\tau \sim \text{Erl}(0.5, 0.5)$. Then, for this DFR distribution, we have $\varsigma = 0.3712512013$, $\beta = 0.6287487988$, and we thus get:

$$\begin{aligned} (0.6287487988)M_X(0.3712512013)E\left[e^{-0.3712512013(3.661733537e^{0.05\tau^*} + 2\bar{s}_{\tau^*0.05})}\right] &= 0.1193205821 \leq \\ (0.6287487988)M_X(0.3712512013)E\left[e^{-0.3712512013(3.661733537e^{0.05\tau} + 2\bar{s}_{\tau0.05})}\right] &= 0.1572522651. \end{aligned}$$

These results are consistent with Theorem 2 and Corollary 2 where, for an IFR (DFR) distribution, the distribution function and the moments of our risk process increase (decrease) with the age process at time t . These families of distributions involve the increasing (decreasing) of the corresponding bound of the ruin probability.

6. Conclusions

By assuming first that the claims have heavy tails, formulas for the first two raw and joint moments were obtained for the increments of compound renewal sums with discounted claims, taking into account the past experience of the insurance portfolio. Then, by assuming that the claims have light tails, the mgf, moments and particular distribution functions of this conditional risk process were found. A situation where the severity of the claim is dependent on the time elapsed since the preceding claim was also studied in this context. Finally, a new “conditional” Lundberg-type bound was obtained for the ruin probability, and this upper bound could be larger or smaller than the non-conditional Lundberg bound if the inter-occurrence time has an IFR or DFR distribution.

We brought another point of view showing that ignoring the importance of the age process in an insurance portfolio could lead to greatly underestimating or overestimating the future risks over the next period. Our approach is more probabilistic than statistical and has to be considered complementary to the credibility theory, mainly because the age of the counting process is not considered explicitly in this last theory despite its importance.

Author Contributions: The three authors contributed to the development of the model. G.L. and I.-R.M. jointly wrote the paper.

Funding: This research was funded by “La Chaire d’actuariat de l’Université Laval”.

Acknowledgments: The authors thank the referees for their helpful comments.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Proof of Lemma 2: For the first identity, we have:

$$E[Z(t)] = E\left[E\left[Z(t) \mid T_1, T_2, \dots, T_{N(t)}, N(t)\right]\right] = E\left[\sum_{j=1}^{N(t)} e^{-\delta T_j} E[X_j \mid \tau_j]\right].$$

Conditioning on $N(t)$, we thus get:

$$\begin{aligned} E\left[\sum_{j=1}^{N(t)} e^{-\delta T_j} E[X_j \mid \tau_j]\right] &= E\left[E\left[\sum_{j=1}^{N(t)} e^{-\delta T_j} E[X_j \mid \tau_j]\right] \mid N(t)\right] \\ &= \sum_{n=1}^{\infty} \sum_{j=1}^n \int_0^t \int_v^t E[e^{-\delta u} E[X_j \mid \tau_j = u - v]] dF_{T_{j-1}, T_j \mid N(t)}(v, u \mid n) P(N(t) = n) \\ &= \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} \int_0^t \int_v^t E[e^{-\delta u} E[X_1 \mid \tau_1 = u - v]] P(N(t - u) = n - j) dF_{\tau_1}(u - v) dF_{T_{j-1}}(v) \\ &= \sum_{j=1}^{\infty} \int_0^t \int_v^t E[e^{-\delta u} E[X \mid \tau = u - v]] dF_{\tau}(u - v) dF_{\tau}^{*(j-1)}(v) \\ &= \int_0^t e^{-\delta u} E[X \mid \tau = u] dF_{\tau}(u) + \int_0^t \int_v^t e^{-\delta u} E[X \mid \tau = u - v] dF_{\tau}(u - v) dm(v). \end{aligned}$$

By permuting the order of integration of the last integral, we obtain the result.
For the second identity, we have,

$$\begin{aligned} E[Z^2(t)] &= E \left[E \left[\sum_{j=1}^{N(t)} e^{-2\delta T_j} X_j^2 \middle| T_1, T_2, \dots, T_{N(t)}, N(t) \right] \right] \\ &\quad + 2E \left[E \left[\sum_{j=1}^{N(t)-1} \sum_{k=j+1}^{N(t)} e^{-\delta(T_j+T_k)} X_j X_k \middle| T_1, T_2, \dots, T_{N(t)}, N(t) \right] \right] \\ &= E \left[\sum_{j=1}^{N(t)} e^{-2\delta T_j} E[X_j^2 | \tau_j] \right] + 2E \left[\sum_{j=1}^{N(t)-1} \sum_{k=j+1}^{N(t)} e^{-\delta(T_j+T_k)} E[X_j | \tau_j] E[X_k | \tau_k] \right]. \end{aligned}$$

Similarly to preceding identity, the first summation yields:

$$E \left[\sum_{j=1}^{N(t)} e^{-2\delta T_j} E[X_j^2 | \tau_j] \right] = \int_0^t e^{-2\delta u} E[X^2 | \tau = u] \left\{ 1 + \int_0^{t-u} e^{-2\delta v} dm(v) \right\} dF_\tau(u)$$

and the second summation yields,

$$\begin{aligned} E \left[\sum_{j=1}^{N(t)-1} \sum_{k=j+1}^{N(t)} e^{-\delta(T_k+T_j)} E[X_j | \tau_j] E[X_k | \tau_k] \right] &= E \left[E \left[\sum_{j=1}^{N(t)-1} \sum_{k=j+1}^{N(t)} e^{-\delta(T_j+T_k)} E[X_j | \tau_j] E[X_k | \tau_k] \middle| N(t) \right] \right] \\ &= \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \sum_{k=j+1}^n \int_0^t \int_0^{t-v} \int_0^{t-v-\gamma} e^{-\delta(z+u)} E[X_j | \tau_j = u-v] E[X_k | \tau_k = z-w] \\ &\quad \times dF_{T_{j-1}, T_j, T_{k-1}, T_k | N(t)}(v, u, w, z | n) P(N(t) = n) \\ &= \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \sum_{k=j+1}^n \int_0^t \int_0^{t-v} \int_0^{t-v-\gamma} e^{-\delta(z+u)} E[X | \tau = u-v] E[X | \tau = z-w] \\ &\quad \times P(N = n-k) dF_\tau dF_{T_{k-1-j}} dF_{T_{j-1}}(v) \\ &= \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \int_0^t \int_0^{t-v} \int_0^{t-v-\gamma} e^{-\delta(\alpha+2(\gamma+v))} E[X | \tau = \gamma] E[X | \tau = \alpha] \\ &\quad \times P(N(t-z) = n-j-1) dF_\tau(\alpha) dF_\tau(\gamma) dF_\tau^{*(j-1)}(v) \\ &\quad + \sum_{n=3}^{\infty} \sum_{j=1}^{n-2} \sum_{k=j+2}^n \int_0^t \int_0^{t-v} \int_0^{t-v-\gamma} \int_0^{t-v-\gamma-\beta} e^{-\delta(\alpha+\beta+2(\gamma+v))} E[X | \tau = \gamma] E[X | \tau = \alpha] \\ &\quad \times P(N(t-z) = n-k) dF_\tau(\alpha) dF_\tau^{*(k-1-j)}(\beta) dF_\tau(\gamma) dF_\tau^{*(j-1)}(v). \end{aligned}$$

Hence,

$$\begin{aligned} E \left[\sum_{j=1}^{N(t)-1} \sum_{k=j+1}^{N(t)} e^{-\delta(T_k+T_j)} E[X_j | \tau_j] E[X_k | \tau_k] \right] &= \sum_{j=1}^{\infty} \sum_{n=j+1}^{\infty} \int_0^t \int_0^{t-v} \int_0^{t-v-\gamma} e^{-\delta(\alpha+2(\gamma+v))} E[X | \tau = \gamma] E[X | \tau = \alpha] \\ &\quad \times P(N(t-z) = n-j-1) dF_\tau(\alpha) dF_\tau(\gamma) dF_\tau^{*(j-1)}(v) \\ &\quad + \sum_{j=1}^{\infty} \sum_{k=j+2}^{\infty} \sum_{n=k}^{\infty} \int_0^t \int_0^{t-v} \int_0^{t-v-\gamma} \int_0^{t-v-\gamma-\beta} e^{-\delta(\alpha+\beta+2(\gamma+v))} E[X | \tau = \gamma] E[X | \tau = \alpha] \\ &\quad \times P(N(t-z) = n-k) dF_\tau(\alpha) dF_\tau^{*(k-1-j)}(\beta) dF_\tau(\gamma) dF_\tau^{*(j-1)}(v) \\ &= \int_0^t \int_0^{t-\gamma} e^{-\delta(\alpha+2\gamma)} E[X | \tau = \gamma] E[X | \tau = \alpha] dF_\tau(\alpha) dF_\tau(\gamma) \\ &\quad + \int_0^t \int_0^{t-v} \int_0^{t-v-\gamma} e^{-\delta(\alpha+2(\gamma+v))} E[X | \tau = \gamma] E[X | \tau = \alpha] dF_\tau(\alpha) dF_\tau(\gamma) dm(v) \\ &\quad + \int_0^t \int_0^{t-\gamma} \int_0^{t-\gamma-\beta} e^{-\delta(\alpha+\beta+2\gamma)} E[X | \tau = \gamma] E[X | \tau = \alpha] dF_\tau(\alpha) dm(\beta) dF_\tau(\gamma) \\ &\quad + \int_0^t \int_0^{t-v} \int_0^{t-v-\gamma} \int_0^{t-v-\gamma-\beta} e^{-\delta(\alpha+\beta+2(\gamma+v))} E[X | \tau = \gamma] E[X | \tau = \alpha] dF_\tau(\alpha) dm(\beta) dF_\tau(\gamma) dm(v) \\ &= \int_0^t e^{-2\delta\gamma} E[X | \tau = \gamma] \left\{ \int_0^{t-\gamma} e^{-\delta\alpha} E[X | \tau = \alpha] \left\{ 1 + \int_0^{t-\gamma-\alpha} e^{-\delta\beta} dm(\beta) \right\} dF_\tau(\alpha) \right\} dF_\tau(\gamma) \\ &\quad + \int_0^t \int_0^{t-v} e^{-2\delta(\gamma+v)} E[X | \tau = \gamma] \left\{ \int_0^{t-v-\gamma} e^{-\delta\alpha} E[X | \tau = \alpha] \left\{ 1 + \int_0^{t-v-\gamma-\alpha} e^{-\delta\beta} dm(\beta) \right\} dF_\tau(\alpha) \right\} dF_\tau(\gamma) dm(v). \end{aligned}$$

Permuting the order of integration between γ and v in the second term, we then obtain:

$$\begin{aligned} E \left[\sum_{j=1}^{N(t)-1} \sum_{k=j+1}^{N(t)} e^{-\delta(T_k+T_j)} E[X_j|\tau_j] E[X_k|\tau_k] \right] &= \int_0^t e^{-2\delta\gamma} E[X|\tau=\gamma] E[Z(t-\gamma)] dF_\tau(\gamma) \\ &+ \int_0^t \int_0^{t-\gamma} e^{-2\delta(\gamma+v)} E[X|\tau=\gamma] E[Z(t-v-\gamma)] dm(v) dF_\tau(\gamma) \\ &= \int_0^t e^{-2\delta\gamma} E[X|\tau=\gamma] \left\{ E[Z(t-\gamma)] + \int_0^{t-\gamma} e^{-2\delta v} E[Z(t-v-\gamma)] dm(v) \right\} dF_\tau(\gamma). \end{aligned}$$

The result follows by adding the appropriate quantities. \square

References

- Albrecher, Hansjoerg, and Jozef L. Teugels. 2006. Exponential behavior in the presence of dependence in risk theory. *Journal of Applied Probability* 43: 257–73. [\[CrossRef\]](#)
- Asimit, Alexandru Vali, and Andrei Badescu. 2010. Extremes on the discounted aggregate claims in a time dependent risk model. *Scandinavian Actuarial Journal* 2: 93–104. [\[CrossRef\]](#)
- Bargès, Mathieu, Hélène Cossette, Stéphane Loisel, and Étienne Marceau. 2011. On the moments of the aggregate discounted claims with dependence introduced by a FGM copula. *Astin Bulletin* 41: 215–38.
- Bühlmann, Hans, and Alois Gisler. 2005. *A Course in Credibility Theory and Its Applications*. New York: Springer.
- Cai, Jun, and David C. M. Dickson. 2003. Upper bounds for ultimate ruin probabilities in the Sparre Andersen model with interest. *Insurance: Mathematics and Economics* 32: 61–71. [\[CrossRef\]](#)
- Garrido, José, and Ghislain Léveillé. 2014. Impact of interest and inflation on aggregate claims. In *Wiley StatsRef: Statistics Reference Online*. Hoboken: John Wiley & Sons, Ltd.
- Léveillé, Ghislain, and Franck Adékambi. 2011. Covariance of discounted compound renewal sums with a stochastic interest rate. *Scandinavian Actuarial Journal* 2: 138–53. [\[CrossRef\]](#)
- Léveillé, Ghislain, and José Garrido. 2001a. Moments of compound renewal sums with discounted claims. *Insurance: Mathematics and Economics* 28: 217–31.
- Léveillé, Ghislain, and José Garrido. 2001b. Recursive moments of compound renewal sums with discounted claims. *Scandinavian Actuarial Journal* 2: 98–110. [\[CrossRef\]](#)
- Léveillé, Ghislain, José Garrido, and Ya Fa Wang. 2010. Moment generating functions of compound renewal sums with discounted claims. *Scandinavian Actuarial Journal* 3: 165–84. [\[CrossRef\]](#)
- Léveillé, Ghislain, and Emmanuel Hamel. 2018. Compound trend renewal process with discounted claims. *Scandinavian Actuarial Journal*. submitted.
- Lin, X. Sheldon. 2014. Generalized Lundberg Approximations. In *Wiley StatsRef: Statistics Reference Online*. Hoboken: John Wiley & Sons, Ltd.
- Müller, Alfred, and Dietrich Stoyan. 2002. *Comparison Methods for Stochastic Models and Risks*. Chichester: Wiley.
- Nelsen, Roger B. 2006. *An Introduction to Copulas*, 2nd ed. Springer: New York.
- Shaked, Moshe, and George Shanthikumar. 2007. *Stochastic Orders*. New York: Springer.
- Wang, Ya Fa, José Garrido, and Ghislain Léveillé. 2018. The distribution of discounted compound PH-renewal processes. *Methodology and Computing in Applied Probability* 20: 69–96. [\[CrossRef\]](#)
- Woo, Jae-Kyung, and Eric C. K. Cheung. 2013. A note on discounted compound renewal sums under dependency. *Insurance: Mathematics and Economics* 52: 170–79. [\[CrossRef\]](#)
- Yang, Hailiang. 2014. Lundberg Inequality for Ruin Probability. In *Wiley StatsRef: Statistics Reference Online*. Hoboken: John Wiley & Sons, Ltd.



© 2018 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).