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Advantages of Accounting for Stochasticity in the Premium Process

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Abstract: In this paper, we study a risk model with stochastic premium income and its impact on solvency risk management. It is assumed that both the premium arrival process and the claim arrival process are modelled by homogeneous Poisson processes, and that the premium amounts are modelled by independent and identically distributed random variables. While this model has been studied in the existing literature and certain explicit results are known under more restrictive assumptions, these results are relatively difficult to apply in practice. In this paper, we investigate the factors that differentiate this model and the classical risk model. After reviewing various known results of this model, we derive a simulation approach for obtaining the probability of ultimate ruin based on importance sampling, which does not require specific distributions for the premium and the claim. We demonstrate this approach first with examples where the distribution of the sampling random variable can be identified. We then provide additional examples where we use the fast Fourier transform to obtain an approximation of the sampling random variable. The simulated results are compared with the known results for the probability of ruin. Using the simulation approach, we apply this model to a real-life auto-insurance data set. Differences with the classical model are then discussed. Finally, we comment on the suitability and impact of using this model in the context of solvency risk management.

Keywords: risk model; Poisson process; importance sampling; fast Fourier transform



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1. Introduction

The studies of an insurer's cashflow date back to the early 1900s. To model the cashflow of an insurer, it suffices to consider the premium income and the claim payments arriving over time. In the most basic setting, investing is not considered, the rate at which the insurer collects premiums is assumed to be constant, and the claim arrivals are modelled by a *homogeneous Poisson process*. This model is known as the *classical risk model* or the *compound-Poisson risk model*. Under this model, the surplus process of an insurer is given by

$$U(t) = u + ct - \sum_{k=1}^{N(t)} Y_k, \quad t \geq 0, \quad (1)$$

where $u \geq 0$ is the initial surplus of the insurer, $c > 0$ is the continuous premium rate, $N(t)$ is a homogeneous Poisson process that counts the number of claim arrivals up to time t , and Y_k is the amount of the k th claim. The mathematical properties of a homogeneous Poisson process make the analysis of this model tractable. Consequently, many theoretical results under this model have been obtained through probabilistic arguments. An important advancement in this field was made by [Gerber and Shiu \(1998\)](#), where the analysis of various quantities of interest, such as the probability of ultimate or finite-time ruin, the

Laplace transform of the time of ruin, the distribution of the deficit at ruin, etc., are captured in a systematic way by the *discounted penalty function*, also known as the *Gerber–Shiu function*,

$$m(u) = \mathbb{E} \left[e^{-\delta T} w(U(T-), |U(T)|) \mathbf{1}(T < \infty) | U(0) = u \right], \quad u \geq 0,$$

where $T = \inf\{t : U(t) < 0\}$ is the random time of ruin, the penalty function, w , is a bi-variate function of the surplus just before ruin and the deficit at ruin, and $\delta \geq 0$ is the discount factor. By setting different values of w and δ , the Gerber–Shiu function reduces to various quantities of interest. See, for example, [Asmussen and Albrecher \(2010\)](#), Chapter XII for a summary.

The classical risk model (1) is mathematically convenient as it is tractable, but its restrictive assumptions limit its applicability to real-world insurance portfolios. To address this, researchers have introduced various stochastic processes to enhance the model's accuracy. An early attempt was [Andersen \(1957\)](#), where the author replaced the homogeneous Poisson process by a renewal process, allowing for arbitrary inter-arrival time distributions and providing more flexibility than the exponential inter-arrival times of the Poisson process. However, renewal processes still cannot capture time inhomogeneity observed in claim arrivals.

To model time-dependent behaviours, stochastic processes incorporating periodicity and non-stationarity have been considered. [Asmussen and Rolski \(1994\)](#) introduced a risk model with a periodic claim arrival process to capture seasonal fluctuations, [Morales \(2004\)](#) demonstrated simulation methods for calculating ruin probabilities in this context, and [Miao et al. \(2023a\)](#) considered a risk model where both premiums and claims exhibit seasonalities. More recently, Cox processes, where the claim arrival intensity itself is stochastic, have gained attention. Notable examples include the shot-noise Cox process, studied by [Dassios and Jang \(2003\)](#), [Albrecher and Asmussen \(2006\)](#), [Dassios et al. \(2015\)](#), and [Avanzi et al. \(2021b\)](#), which models sudden surges in claim rates due to external events. The Markov-modulated Poisson process (e.g., [Asmussen \(1989\)](#); [Lu and Li \(2005\)](#); [Avanzi et al. \(2021a\)](#)) allows the model to switch between different regimes with distinct claim arrival rates. A non-stationary Cox process incorporating trends and cycles over time was considered in [Albrecher et al. \(2021\)](#). These stochastic processes enhance claim modelling by capturing complex patterns like time inhomogeneity, regime switching, and external shocks.

Modifications have also been proposed for premium income by replacing deterministic premium income with a stochastic premium process. This accounts for uncertainties due to economic conditions or policyholder behaviour. This stochastic premium model remains mathematically tractable and has been studied extensively. [Boikov \(2003\)](#) derived integral equations for ultimate and finite-time survival probabilities under this model, providing closed-form expressions in special cases. [Labbé and Sendova \(2009\)](#) explored the expected discounted penalty function, establishing renewal and integral equations. [Albrecher et al. \(2010\)](#) offered alternative derivations using exponential claim sizes before generalising to arbitrary distributions. Additionally, [Vidmar \(2018\)](#) investigated models with stochastic dependence between premiums and claim arrivals, reflecting scenarios where economic factors simultaneously influence both income and outflows.

While the stochastic premium model is not new and a great deal of risk theory results are known, the existing results are often from a theoretical point of view. Moreover, the results are mostly given in a non-explicit form, often involving integral equations and Laplace transforms. These results, while providing mathematical insights into the model, are relatively hard to implement in practice. Explicit results may be obtained by imposing assumptions on the distributions of premiums and claims, such as the examples in [Labbé and Sendova \(2009\)](#). However, these assumptions are often quite restrictive and may not be suitable for real-life situations. Another approach is to explore numerical approximations for specific quantities. [Temnov \(2004\)](#) derived an expression for the probability of ruin and developed a numerical approximation procedure based on Fourier transforms. This

approach does not impose assumptions on premiums and claims, and hence may be applied to more general cases. The convolution formula for the probability of ruin has the same form as the classical model and is given by

$$\psi(u) = q \sum_{k=0}^{\infty} (1-q)^k (1 - F_h^{*k}(u)),$$

where $q = \psi(0)$ and F_h^{*k} is the k th fold self-convolution of a cumulative distribution function, F_h . Although this formula is succinct and works for all distributions, neither the value of q nor F_h are known. Both of these quantities need to be evaluated numerically through a chain of approximations. Multiple steps of truncation are also involved in the process. These steps increase the complexity of the computations and introduce the potential for accumulated numerical errors.

In this paper, we aim to investigate the impact of applying this model in practice in the solvency risk management context. We first present a simulation-based method to obtain the probability of ultimate ruin. As discussed in [Asmussen and Binswanger \(1997\)](#), this is a rare event simulation with an infinite time horizon. A common technique to solve this issue is applying Esscher transforms to the sample random variables. This technique finds many applications in finance and insurance; see, for example, [Bühlmann et al. \(1998\)](#); [Elliott et al. \(2005\)](#); and [Chuang and Brockett \(2014\)](#). Taking advantage of the fact that the drop of surplus upon each claim arrival follows a compound-geometric distribution, we derive a simulator based on Esscher transforms. The simulation approach does not require specific distributions for premiums and claims and can be applied to a wide range of real-life scenarios. In comparison with the numerical approximation approach, the simulation approach does not require one to evaluate the probability of ruin for all admissible values of the initial surplus. It also involves fewer steps and truncations. Using the simulation approach, we first give two examples with simple distributions where the exact distribution of the sampling random variable can be identified. We then present a Fourier transform-based algorithm to deal with more general cases with other distributions. The simulated results are compared with known explicit results to highlight the differences from the classical model. We then investigate whether applying this model in practice requires significantly different resources to manage solvency risk. To this end, we fit the model to a real-life auto-insurance data set. We compare this model with the classical model. We discover that the classical model is a good approximation to the stochastic premium model under certain circumstances. However, the differences between the two models could be more significant for other lines of insurances. Moreover, the changing physical environment may deteriorate the claim experience for an established pool of policies. One such example is the significantly increased claims related to extreme weather events from catastrophic insurances over the past few years. As we show using a modified and hypothetical version of the data set, simply increasing the premium amount to counter the increased claims is not sufficient from a solvency risk management point of view. These results indicate that the stochastic premium model has practical value, and that the differences from the classical model vary depending on the characteristics of the specific insurance product that is studied.

This paper is structured as follows. Some useful mathematical tools and their properties are reviewed in Section 2. In Section 3, we introduce formally the risk model with stochastic premium income and briefly summarise known results in the existing literature. A simulator based on importance sampling is derived in Section 4. An approximation method based on the fast Fourier transform is also presented in this section. Examples are given to illustrate the simulation approach. In Section 5, we apply the stochastic premium model to a real-life auto-insurance data set and comment on the differences it makes. We then compare the performance of the model under different scenarios using modified and hypothetical versions of the auto-insurance data set. We comment on the application of the model in a changing environment and its impacts. Conclusions are given in Section 6.

2. Preliminaries

In this section, we present some useful mathematical results that will be used later in the paper.

2.1. Moment Generating Function and Cumulant Generating Function

One convenient way to represent a random variable is through its generating functions. Let X be a random variable. Its *moment generating function* (m.g.f.), M_X , and its *cumulant generating function* (c.g.f.), Γ_X , are defined by

$$\begin{aligned} M_X(t) &= \mathbb{E}[e^{tX}], \\ \Gamma_X(t) &= \ln M_X(t). \end{aligned} \quad (2)$$

These two functions are not defined for all distributions. For example, they are not defined for the Cauchy distribution. Notice that it is possible that the expectation (2) converges to infinity for some $t \in \mathbb{R}$, i.e., the generating functions are only defined on a subset of $\mathbb{R}$. We thus require the following definition.

Definition 1. The domain of a function, f , denoted by $\mathcal{D}(f)$, is the subset of $\mathbb{R}$, where the value of the function is finite, i.e.,

$$\mathcal{D}(f) = \{t \in \mathbb{R} : |f(t)| < \infty\}.$$

Moment generating functions and cumulant generating functions are convenient because they provide a systematic way to calculate the moments of a random variable. If there exists $\epsilon > 0$, such that $(-\epsilon, \epsilon) \subseteq \mathcal{D}(\Gamma_X)$, then we have the following convenient results

$$\begin{aligned} \Gamma_X(0) &= \ln \mathbb{E}[\exp(0 \cdot X)] = 0, \\ \Gamma'_X(t) &= \frac{M'_X(t)}{M_X(t)} = \frac{\mathbb{E}[X \exp(tX)]}{\exp(\Gamma_X(t))} = \mathbb{E}[X \exp(tX - \Gamma_X(t))] \quad \text{for all } t \in \mathcal{D}(\Gamma_X), \\ \Gamma'_X(0) &= \mathbb{E}[X]. \end{aligned} \quad (3)$$

2.2. Importance Sampling and Esscher Transform

Importance sampling is an important simulation technique. It is often used to achieve variance reduction in rare event simulations.

Definition 2 (Importance sampling). Suppose X is a $\mathbb{R}$ -valued random variable defined on the measurable space $(\Omega, \mathcal{F})$. Let $\mathbb{P}$ and $\mathbb{Q}$ be two measures on the same measurable space. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function. Then we have

$$\mathbb{E}_{\mathbb{P}}[h(X)] = \int_{\Omega} h(X(\omega)) d\mathbb{P}(\omega) = \int_{\Omega} h(X(\omega)) \cdot \frac{d\mathbb{P}(\omega)}{d\mathbb{Q}(\omega)} \cdot d\mathbb{Q}(\omega) = \mathbb{E}_{\mathbb{Q}}[h(X) \cdot \mathcal{L}] \quad (4)$$

where $\mathcal{L}(\omega) = d\mathbb{P}(\omega)/d\mathbb{Q}(\omega)$ is the Radon–Nikodym derivative and is called the likelihood ratio.

If the random variable in Definition 2 has a probability density function (p.d.f.) f under measure $\mathbb{P}$ and has a p.d.f. g under measure $\mathbb{Q}$, then the likelihood ratio is f/g , i.e., Equation (4) becomes

$$\mathbb{E}[h(X)] = \int h(x) \cdot f(x) dx = \int h(x) \cdot \frac{f(x)}{g(x)} \cdot g(x) dx = \mathbb{E}\left[h(\hat{X}) \cdot \frac{f(\hat{X})}{g(\hat{X})}\right] \quad (5)$$

where $\hat{X}$ has the p.d.f. g .

A major benefit of importance sampling is variance reduction. Namely, suppose we are interested in estimating $h(X)$ by simulation. We may simulate X under the probability

measure $\mathbb{P}$, calculate $h(X)$, repeat this procedure n times and calculate the average of these n different realisations. To reduce the variance of the estimator, we simply increase the number of simulations, n . This approach is called the *crude Monte Carlo method* or the *naïve Monte Carlo method*. Under certain circumstances, this method is not efficient. These situations usually involve simulating some rare events associated with very small probabilities. A number of variance-reduction techniques exist for rare event simulation, see [Bucklew \(2013\)](#) for more details. When using importance sampling, we simulate X under the probability measure $\mathbb{Q}$. The new estimator is $h(X) \cdot \mathcal{L}$. We may choose an appropriate $\mathbb{Q}$ such that the variance of the estimator is smaller than that of the crude estimator.

The key to importance sampling is to specify a new probability measure under which the random variable X has an appropriate distribution. If the original distribution is *absolutely continuous*, i.e., the p.d.f. exists, then it suffices to specify the new p.d.f. under the new measure. The importance sampling for this case is given by Equation (5). We introduce next a method to define a new probability measure.

Definition 3 (Esscher transform). *Let X be a $\mathbb{R}$ -valued random variable defined on a measure space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose X has the m.g.f. M_X and the c.g.f. Γ_X . The Esscher transform of the measure $\mathbb{P}$, denoted by $\mathbb{P}_\theta$, is given by the likelihood ratio*

$$\frac{d\mathbb{P}_\theta}{d\mathbb{P}}(x) = \frac{\exp(\theta x)}{M_X(\theta)} = \exp(\theta x - \Gamma_X(\theta)), \quad (6)$$

where θ is the parameter of the Esscher transform.

In this paper, we denote random variables related to an Esscher-transformed measure by a tilde together with the transformation parameter given in a pair of parentheses. It is easy to show that the m.g.f. of the transformed distribution, denoted by $M_{\tilde{X}(\theta)}$, is

$$M_{\tilde{X}(\theta)}(t) = \int_{\Omega} e^{tX} d\mathbb{P}_\theta = \int_{\Omega} e^{tX} \frac{d\mathbb{P}_\theta}{d\mathbb{P}} d\mathbb{P} = \frac{\int_{\Omega} e^{(\theta+t)X} d\mathbb{P}}{e^{\Gamma_X(\theta)}} = \frac{M_X(\theta+t)}{M_X(\theta)}. \quad (7)$$

Furthermore, if a random variable, X , has a p.d.f. f , then the p.d.f. of the transformed distribution, denoted by $f_{\tilde{X}(\theta)}$, is given by

$$f_{\tilde{X}(\theta)}(x) = \frac{f(x) \exp(\theta x)}{M_X(\theta)}.$$

Some common distributions are invariant under the Esscher transform. For example, if X is exponentially distributed with parameter λ , then its Esscher transform with parameter θ is also an exponential distribution with parameter $\tilde{\lambda} = \lambda - \theta$. If X is geometrically distributed with parameter p , then its Esscher transform with parameter θ also follows a geometric distribution with parameter $\tilde{p} = 1 - (1 - p) \exp(\theta)$. For more details on the Esscher transform and its applications, see, for example, [Asmussen and Glynn \(2007\)](#).

If the likelihood ratio of the importance sampling is defined by Equation (6), then we have

$$\begin{aligned} \mathbb{E}_{\mathbb{P}_\theta}[X] &= \int_{\Omega} X d\mathbb{P}_\theta = \int_{\Omega} X \cdot \frac{d\mathbb{P}_\theta}{d\mathbb{P}} \cdot d\mathbb{P} \\ &= \int_{\Omega} X \cdot \exp(\theta X - \Gamma_X(\theta)) d\mathbb{P} \\ &= \mathbb{E}_{\mathbb{P}}[X \exp(\theta X - \Gamma_X(\theta))]. \end{aligned} \quad (8)$$

By Equations (3) and (8), we have

$$\Gamma'_X(\theta) = \mathbb{E}_{\mathbb{P}_\theta}[X]. \quad (9)$$

Equation (9) establishes a relationship between the expected value under the new probability measure and the c.g.f. under the original probability measure. This relationship is utilised to derive the estimator of the probability of ultimate ruin in Section 4.

Lemma 1 extends Equation (8) to a random number of random variables.

Lemma 1. Let $X_1, X_2, \dots, X_n$ be independent and identically distributed random variables with a common c.g.f., Γ , then

$$\mathbb{E}[f(X_1, X_2, \dots, X_n)] = \mathbb{E}_{\mathbb{P}_\theta} \left[f(X_1, X_2, \dots, X_n) \exp \left(-\theta \sum_{i=1}^n X_i + n\Gamma(\theta) \right) \right].$$

2.3. Fourier Transform and Fast Fourier Transform

The Fourier transform is a useful tool that is used extensively in science and engineering. It studies how a function may be approximated by a sum of trigonometric functions.

Definition 4 (Fourier transform). Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be an integrable function, the Fourier transform of f , denoted by $\mathfrak{F}f$, is given by

$$\mathfrak{F}f(\omega) = \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx,$$

where $\omega \in \mathbb{R}$. The inverse Fourier transform of a function g , denoted by $\mathfrak{F}^{-1}g$, is defined by

$$\mathfrak{F}^{-1}g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) e^{-i\omega x} d\omega.$$

Remark 1. There are different definitions of the Fourier transform and its inverse. Generically, the Fourier transform can be defined as

$$\mathfrak{F}f(\omega) = \frac{1}{A} \int_{-\infty}^{\infty} f(x) e^{iB\omega x} dx,$$

where A and B are two constants. For possible combinations of these two constants and their properties, see Osgood (2019), Chapter 2.4. We choose $A = B = 1$ because under this definition the Fourier transform of a probability function is the characteristic function of the distribution.

The discrete Fourier transform is utilised in many practical applications. Suppose instead of a continuous function, f , we have $n < \infty$ equally spaced samples, i.e.,

$$f_0 = f(x_0), \quad f_1 = f(x_1), \quad f_2 = f(x_2), \quad \dots, \quad f_{n-1} = f(x_{n-1}).$$

where $x_k - x_{k-1} = x_j - x_{j-1}$ for all $k, j \in \mathbb{N}^+$, $k, j \leq n-1$.

Definition 5 (Discrete Fourier transform). For a finite sequence $(f_0, f_1, \dots, f_{n-1})$, the discrete Fourier transform and its inverse are defined as

$$\begin{aligned} (\mathfrak{F}f)_k &= \sum_{j=0}^{n-1} f_j e^{-i2\pi jk/n}, \\ f_k &= \frac{1}{n} \sum_{j=0}^{n-1} (\mathfrak{F}f)_j e^{i2\pi jk/n}. \end{aligned}$$

The discrete Fourier transform, which operates on a finite discrete sequence, may be viewed as a discretisation of a Fourier transform, which operates on a continuous integrable function. As the number of samples increases, the discrete Fourier transform converges to

the Fourier transform. The discrete Fourier transform is of great practical value because it can be implemented through a highly efficient algorithm, called the *fast Fourier transform*. A direct implementation of the discrete Fourier transform has a computational complexity of $O(n^2)$, while an implementation using the fast Fourier transform has a computational complexity of $O(n \ln n)$. For more details, see, for example, [Cooley and Tukey \(1965\)](#); [Brigham \(1988\)](#); and [Osgood \(2019\)](#), Chapter 7.

In risk theory, Fourier transforms are often used to evaluate the probability function of a compound distribution of the form $S = \sum_{i=1}^N X_i$, where N is a non-negative integer-valued random variable, and X_i s are independent and identically distributed positive random variables. To obtain the exact density function or mass function of the compound distribution, one needs to evaluate the self-convolution of X_i , which is often cumbersome if possible at all. Some approximation methods exist. Examples of these approximations include normal approximation, gamma approximation, and Esscher approximation. For a review on this topic, see [Hardy \(2006\)](#). Under certain conditions, exact numerical methods also exist, notably the Panjer recursion, see [Klugman et al. \(2012\)](#), Chapter 9.6; and [Sundt and Jewell \(1981\)](#).

Using the discrete Fourier transform, one may obtain an approximation of the probability mass function of the compound distribution $S = \sum_{i=1}^N X_i$. We assume that the compounded random variable X_i has mass at equally spaced points $x_0, x_1, \dots, x_{n-1}$ with probabilities $q_0, q_1, \dots, q_{n-1}$, respectively. Notice that we assume that the support of the distribution is discrete. If the distribution of the compounded random variable X_i is continuous, then it is discretised first. Under this assumption, the compound random variable is discrete. For all $k = 0, 1, 2, \dots, n-1$, denote

$$\begin{aligned} q_k &= \mathbb{P}\{X_1 = x_k\}, \\ s_k &= \mathbb{P}\{S = x_k\}. \end{aligned}$$

Let $(\mathfrak{F}q)_k$ and $(\mathfrak{F}s)_k$ be the k th expression of the discrete Fourier transform of the sequences $\{q_k\}_{k=0}^{n-1}$ and $\{s_k\}_{k=0}^{n-1}$, respectively. These two sequences are approximations of the characteristic functions of X_1 and S . To obtain an approximation of the probability mass function of S , one first obtains the sequence $\mathfrak{F}q$, then calculates

$$(\mathfrak{F}s)_k = P_N[(\mathfrak{F}q)_k],$$

where P_N is the probability generating function (p.g.f.) of N . Finally, the mass function of S is

$$s_k = \mathfrak{F}^{-1}(\mathfrak{F}s)_k.$$

For more details of this algorithm, see, for example, [Albrecher et al. \(2017\)](#), Chapter 6.4; and [Asmussen and Glynn \(2007\)](#), Chapter 2e. We also note that this algorithm yields an approximation of the actual mass probability. The error introduced by the approximation is studied in [Grübel and Hermesmeier \(1999\)](#), where a procedure to reduce the approximation error is also presented.

3. A Risk Model with Stochastic Premium Income

One way to extend the classical model (1) is to replace the deterministic premium income by a stochastic process. The surplus process is defined by

$$U(t) = u + \sum_{k=1}^{H(t)} X_k - \sum_{k=1}^{N(t)} Y_k, \quad (10)$$

where $u \geq 0$ is the initial surplus, the premium counting process $H(t)$ is a homogeneous Poisson process with rate μ , the claim counting process, $N(t)$, is a homogeneous Poisson process with rate λ , $X_k > 0$ is the amount of the k th premium arrival, and $Y_k > 0$ is the size

of the k th claim arrival. We assume that $H(t)$, $N(t)$, X_k s, and Y_k s are mutually independent. We make the following assumptions on the surplus process.

Assumption 1. The relative security loading is positive, i.e., $\mu\mathbb{E}[X_1] > \lambda\mathbb{E}[Y_1]$.

Assumption 2. The m.g.f. of X_1 and Y_1 , denoted by $M_X(t)$ and $M_Y(t)$, respectively, exist. Furthermore, for all $\gamma \in \mathbb{R}$, there is a $t_{X,\gamma} \in \mathcal{D}(M_X)$ and a $t_{Y,\gamma} \in \mathcal{D}(M_Y)$, such that $M_X(t_{X,\gamma}) > \gamma$ and $M_Y(t_{Y,\gamma}) > \gamma$.

Since the classical model has been thoroughly studied, it often serves as the benchmark model for comparison. We are interested in how much *additional* risk the stochastic premium model (10) can capture compared with the classical model (1). To this end, we need to change only the premium process of these two models while holding other factors unchanged.

Definition 6 (Comparable risk models). Two risk models of the general form

$$U_k(t) = u_k + A_k(t) - B_k(t), \quad k = 1, 2,$$

where $A_k(t)$, which is the total premium amount collected up to time t , and $B_k(t)$, which is the total claim amount paid up to time t , are comparable if $\mathbb{E}[A_1(t)] = \mathbb{E}[A_2(t)]$ and $\mathbb{E}[B_1(t)] = \mathbb{E}[B_2(t)]$ for all $t \geq 0$.

The classical model (1) and the stochastic premium model (10) are comparable if $c = \mu\mathbb{E}[X_1]$, while N and Y_i s are the same in these two models.

Some ruin theory results have been derived for the stochastic premium model using both probabilistic arguments and through the Gerber–Shiu function. The next theorem is a collection of known results pertinent to the contents of this paper.

Theorem 1. For the stochastic premium model (10), define the time of ruin

$$T := \inf\{t \geq 0 : U(t) < 0 | U(0) = u\}.$$

The probability of ruin is $\psi(u) = \mathbb{P}\{T < \infty | U(0) = u\}$. Let $\varphi(u) = 1 - \psi(u)$ be the survival probability. Then

1. (Boikov 2003) The survival probability satisfies

$$(\lambda + \mu)\varphi(u) = \lambda \int_0^u \varphi(u - x) dF(x) + \mu \int_0^\infty \varphi(u + x) dG(x),$$

where F is the cumulative distribution function of the claim size random variable Y_1 , and G is the cumulative distribution function of the premium amount random variable X_1 .

2. (Boikov 2003) Let R^* be a positive root of

$$\mu \left[\mathbb{E}e^{-RX_1} - 1 \right] + \lambda \left[\mathbb{E}e^{RY_1} - 1 \right] = 0, \quad (11)$$

then by a martingale argument we have

$$\psi(u) \leq e^{-R^*u}.$$

3. (Temnov 2004) Let $\psi_{CL}(u)$ be the probability of ruin under the classical model (1) with $c = \mu\mathbb{E}[X_1]$, then $\psi(u) > \psi_{CL}(u)$ for all $u \geq 0$.

4. (Temnov 2004) The classical model may be expressed as a limiting case of the stochastic premium model. The ruin probability satisfies

$$\lim_{\substack{\mu \rightarrow \infty \\ \mu \mathbb{E}[X_1] = c}} \psi(u) = \psi_{CL}(u).$$

for all $u \geq 0$.

For the proof of these results, the reader is referred to the original papers.

4. A Simulation Approach

In this section, we derive a simulation approach to obtain the probability of ruin. We first obtain an expression of the probability of ruin using a change-of-measure argument. The derivation is similar to Pham (2007); Asmussen and Albrecher (2010); and Grandell (2012).

In the classical model, the surplus increases linearly between two claims and drops at the arrival of a claim. In the stochastic premium model, the surplus has an upward jump at the arrival of a premium and has a downward drop at the arrival of a claim. Figure 1 gives a sample trajectory of the surplus process under the stochastic premium model.

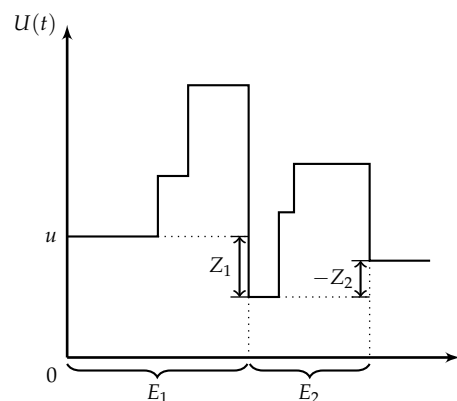


Figure 1. A sample trajectory of the surplus process.

Let E_k be the time between the $(k-1)$ st claim arrival and the k th claim arrival. (Given the Poisson nature of claim arrivals, we may assume that the 0th arrival is at 0 without loss of generality.) Since $N(t)$ is a homogeneous Poisson process with rate λ , we know that E_k s are independent and identically distributed exponential random variables with parameter λ . Let Z_k be the k th drop in the surplus process, i.e.,

$$Z_k = U\left(\sum_{j=1}^{k-1} E_j\right) - U\left(\sum_{j=1}^k E_j\right) = Y_k - \sum_{i=1}^{\xi_k} X_i, \quad (12)$$

where ξ_k is the number of premium arrivals between the $(k-1)$ st claim and the k th claim, which may be expressed as

$$\xi_k = H\left(\sum_{j=1}^k E_j\right) - H\left(\sum_{j=1}^{k-1} E_j\right).$$

Since the premium arrival process, H , is a homogeneous Poisson process with intensity μ , we know that ξ_k follows a Poisson distribution, given E_k ,

$$\xi_k | E_k \sim \text{Poisson}(\mu \cdot E_k).$$

Using the m.g.f., it is easy to prove the following result.

Lemma 2. ξ_k is geometrically distributed with probability of success $\lambda/(\lambda + \mu)$.

Proof. The m.g.f. of ξ_k , denoted by $M_{\xi}(t)$, may be obtained by using the law of total expectation,

$$\begin{aligned} M_{\xi}(t) &= \mathbb{E}[e^{t\xi_k}] = \mathbb{E}[\mathbb{E}[e^{t\xi_k} | E_k]] = \mathbb{E}[e^{\mu(\exp(t)-1) \cdot E_k}] \\ &= M_E(\mu(e^t - 1)) = \frac{\lambda}{\lambda - \mu(e^t - 1)} \\ &= \frac{\lambda/(\lambda + \mu)}{1 - [\mu/(\lambda + \mu)]e^t}. \end{aligned} \quad (13)$$

Expression (13) is the m.g.f. of a geometric distribution with probability of success $\lambda/(\lambda + \mu)$. By the uniqueness of the m.g.f., we establish the result. $\square$

Lemma 2 indicates that the total premium amount between the two claim arrivals in Equation (12) has a compound geometric distribution. This property is convenient since a number of methods exist for estimating the p.d.f. of this compound distribution.

As in the classical model, ruin can only happen when there is a claim arrival. Consequently, it suffices to consider the embedded discrete-time surplus process defined at each claim arrival, namely,

$$U_n = u - \sum_{k=1}^n Z_k, \quad (14)$$

where U_n is the surplus at the n th claim arrival and Z_k is the drop of surplus defined in Equation (12). The discrete time of ruin, defined as the number of claim arrivals at time of ruin, is then

$$\tau := \inf\{k \in \mathbb{N}^+ : U_k < 0\} = \inf\{k \in \mathbb{N}^+ : \sum_{j=1}^k Z_j > u\}. \quad (15)$$

A Lundberg-type upper bound for the probability of ruin is given in Theorem 2.

Theorem 2. Suppose that a stochastic premium model satisfies Assumptions 1 and 2. Then the probability of ultimate ruin satisfies

$$\psi(u) \leq e^{-R^*u},$$

where R^* is the positive root of the equation

$$\Gamma_Z(R) = \Gamma_Y(R) + \ln P_{\xi}(M_X(-R)) = 0, \quad (16)$$

where Γ_Z and Γ_Y are the c.g.f. of Z_1 and Y_1 , respectively, P_{ξ} is the p.g.f. of ξ_1 , and M_X is the m.g.f. of X_1 .

Proof. We first prove that Equation (16) has a root in $\mathcal{D}(\Gamma_Z) \cap (0, \infty)$. Assumption 1 indicates that

$$\mathbb{E}[Z_1] = \mathbb{E}[Y_1] - \mathbb{E}[\xi_1] \cdot \mathbb{E}[X_1] = \mathbb{E}[Y_1] - \frac{1 - \lambda/(\lambda + \mu)}{\lambda/(\lambda + \mu)} \cdot \mathbb{E}[X_1] = \frac{\lambda\mathbb{E}[Y_1] - \mu\mathbb{E}[X_1]}{\lambda} < 0.$$

By definition, the c.g.f. satisfies

$$\begin{aligned} \Gamma_Z(0) &= 0, \\ \Gamma'_Z(0) &= \mathbb{E}[Z_1] < 0, \end{aligned}$$

and therefore, there exists $\epsilon_1 > 0$, such that $\Gamma_Z(\epsilon_1) < 0$. By Assumption 2, $\Gamma_Z(R)$ goes to infinity when R approaches the endpoint of its domain. Consequently, there exists $\epsilon_2 > 0$, such that $\Gamma_Z(\epsilon_2) > 0$. By the continuity of Γ_Z , we may conclude that there exists a positive root to Equation (16) in the interval $(\epsilon_1, \epsilon_2) \subseteq \mathcal{D}(\Gamma_Z) \cap (0, \infty)$. Figure 2 gives an illustration of Equation (16). By the convexity of the c.g.f., it is easy to show that $\Gamma'_Z(R^*) > 0$.

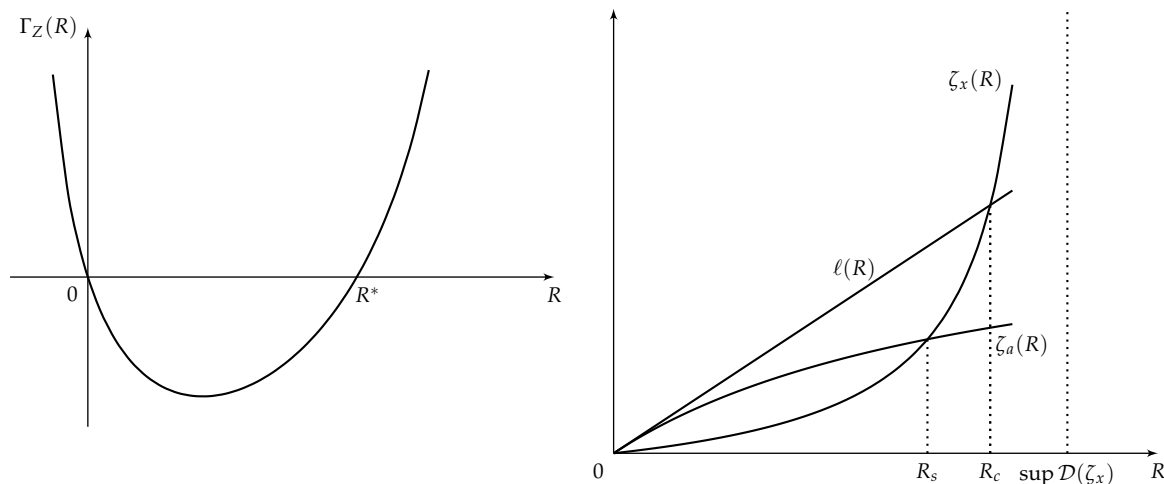


Figure 2. (Left): an example of Equation (16); (right): a comparison of adjustment coefficients under the classical model and under the stochastic premium model.

Consider a new probability measure defined by an Esscher transform with parameter R^* . By Equation (9), we have

$$\mathbb{E}_{R^*}[Z] = \Gamma'_Z(R^*) > 0, \quad (17)$$

i.e., the random walk, $\sum_{j=1}^k Z_j$, in Equation (15) has a positive drift after the Esscher transform and, consequently, for any positive u , the event $\{\tau < \infty\}$ is certain. By Lemma 1, and observing that the time of ruin, τ , is a stopping time with respect to the σ -algebra generated by $\{Z_1, Z_2, Z_3, \dots\}$, the probability of ultimate ruin under the original probability measure, $\mathbb{P}$, may be expressed as

$$\begin{aligned} \psi(u) &= \mathbb{P}(\tau < \infty) = \mathbb{E}[\mathbb{1}_{\tau < \infty}] \\ &= \mathbb{E}_{R^*} \left[\mathbb{1}_{\tau < \infty} \cdot \exp \left(-R^* \sum_{j=1}^{\tau} Z_j + \tau \Gamma_Z(R^*) \right) \right] = \mathbb{E}_{R^*} \left[\exp \left(-R^* \sum_{j=1}^{\tau} Z_j \right) \right]. \end{aligned} \quad (18)$$

By the definition of the time of ruin (15), we know that

$$\sum_{j=1}^{\tau} Z_j > u \geq 0,$$

and substituting the summation in Equation (18) by u , we have

$$\psi(u) \leq \mathbb{E}_{R^*}[\exp(-R^* \cdot u)] = e^{-R^* \cdot u}.$$

□

Remark 2. Equation (16) is equivalent to Equation (11). Using Lemma 2, Equation (16) may be rewritten as

$$\ln M_Y(R) + \ln \left(\frac{\lambda / (\lambda + \mu)}{1 - [\mu / (\lambda + \mu)] M_X(-R)} \right) = 0.$$

Exponentiating the equation yields

$$M_Y(R) = \frac{1 - [\mu/(\lambda + \mu)]M_X(-R)}{\lambda/(\lambda + \mu)} = \frac{\lambda + \mu - \mu M_X(-R)}{\lambda},$$

which produces Equation (11) after rearranging.

Corollary 1. The upper bound in Theorem 2 is greater than or equal to the Lundberg upper bound for the comparable classical model.

Proof. It suffices to show that the adjustment coefficient of the stochastic premium model is smaller than or equal to the adjustment coefficient of the comparable classical model. We first prove the existence of the roots. The Lundberg equations, expressed in terms of moment generating functions, for the classical model (1) and the comparable stochastic premium model (10) are given by

$$\text{Classical:} \quad \lambda[M_Y(R) - 1] = cR, \quad (19)$$

$$\text{Stochastic premium:} \quad \lambda[M_Y(R) - 1] = -\mu[M_X(-R) - 1]. \quad (20)$$

For notational convenience, denote by

$$\zeta_x(R) := \lambda[M_Y(R) - 1], \quad \ell(R) := cR, \quad \zeta_a(R) := -\mu[M_X(-R) - 1],$$

and observe that ζ_x is an increasing, continuous and convex function, ℓ is an increasing, continuous and linear function, and ζ_a is an increasing, continuous and concave function, which are demonstrated by their respective derivatives

$$\begin{aligned} \zeta'_x(R) &= \lambda M'_Y(R) > 0, & \zeta''_x(R) &= \lambda M''_Y(R) > 0, \\ \ell'(R) &= c, & \ell''(R) &= 0, \\ \zeta'_a(R) &= \mu M'_X(-R) > 0, & \zeta''_a(R) &= -\mu M''_X(-R) < 0. \end{aligned}$$

We have further

$$\begin{aligned} \zeta_x(0) &= \zeta_a(0) = \ell(0) = 0, \\ \ell'(0) &= \zeta'_a(0) = \mu \mathbb{E}X_1 > \lambda \mathbb{E}Y_1 = \zeta'_x(0), \end{aligned} \quad (21)$$

meaning that 0 is a root to both Equation (19) and Equation (20), and that there exists a sufficiently small $\epsilon_1 > 0$, such that, for all $R \in (0, \epsilon_1)$, $\zeta_x(R) < \zeta_a(R)$ and $\zeta_x(R) < \ell(R)$. Equation (21) also implies that $\ell(R) > \zeta_a(R)$ for all $R > 0$. Consider Equation (20), we have

$$\lim_{R \rightarrow \infty} \zeta_a(R) = -\mu \lim_{R \rightarrow \infty} \{\mathbb{E}[\exp(-RX)] - 1\} = \mu.$$

Since ζ_a is monotonically increasing, we have for all $R > 0$

$$\zeta_a(R) < \mu.$$

Assumption 2 implies that there exists $\epsilon_2 \in \mathcal{D}(M_Y)$, such that $\zeta_x(\epsilon_2) > \mu > \zeta_a(\epsilon_2)$. It follows that Equation (20) has a root in (ϵ_1, ϵ_2) . Similarly, since $\zeta_x(R)$ is monotonically increasing, Assumption 2 implies that

$$\lim_{R \rightarrow \sup \mathcal{D}(M_Y)} \zeta_c(R) = \infty.$$

It follows that there exists $\epsilon_3 \in \mathcal{D}(M_Y)$, such that

$$\ell(\epsilon_3) = c\epsilon_3 < \zeta_x(\epsilon_3).$$

We may conclude that Equation (19) has a root in the interval (ϵ_1, ϵ_3) . We finished the proof of the existence of the roots. Denote the root to Equations (19) and (20) by R_c and R_s , respectively. We have $\zeta_a(R_s) = \zeta_x(R_s) < \ell(R_s)$, which implies that $R_s \in (0, R_c)$, i.e., $R_s < R_c$. The proof is finished. Figure 2 illustrates Equations (19) and (20). $\square$

We showed in the proof of Theorem 2 that ruin is certain under the new probability measure defined by the Esscher transform with parameter R^* . This enables us to simulate the probability of ruin. To this end, we need to simulate the Esscher-transformed random variable Z_k . Since Z_k is a linear combination of two random variables, the following lemma holds.

Lemma 3. Let $X_1, X_2, \dots, X_n$ be independent random variables, $a_1, a_2, \dots, a_n$ be a sequence of real numbers, and $X = a_1X_1 + a_2X_2 + \dots + a_nX_n$. Assume the p.d.f. of X exists. The Esscher transform of X with parameter $\theta \in \mathcal{D}(M_X)$ has the same distribution as $a_1\tilde{X}_1(a_1\theta) + a_2\tilde{X}_2(a_2\theta) + \dots + a_n\tilde{X}_n(a_n\theta)$, where $\tilde{X}_k(a_k\theta)$ s are independent random variables, and $\tilde{X}_k(a_k\theta)$ is the Esscher transform of X_k with parameter $a_k\theta$.

Proof. By the independence of a_nX_n s, we have

$$\begin{aligned} M_X(t) &= \mathbb{E}\left[\exp\left(t \cdot \sum_{k=1}^n a_k X_k\right)\right] = \mathbb{E}\left[\prod_{k=1}^n \exp(a_k t X_k)\right] \\ &= \prod_{k=1}^n \mathbb{E}[\exp(a_k t X_k)] = \prod_{k=1}^n M_{X_k}(a_k t), \end{aligned}$$

where M_{X_k} is the m.g.f. of X_k . By Equation (7), the m.g.f. under the Esscher-transformed measure is

$$M_{\tilde{X}(\theta)}(t) = \frac{M_X(\theta + t)}{M_X(\theta)} = \prod_{k=1}^n \frac{M_{X_k}(a_k\theta + a_k t)}{M_{X_k}(a_k\theta)}. \quad (22)$$

On the other side of the equation, the m.g.f. of an individual random variable, X_k , after the Esscher transform with parameter $a_k\theta$ is

$$M_{\tilde{X}_k(a_k\theta)}(t) = \frac{M_{X_k}(a_k\theta + t)}{M_{X_k}(a_k\theta)}.$$

Consequently, the m.g.f. of a_kX_k under the measure $\mathbb{P}_\theta$ is

$$\mathbb{E}[\exp(t \cdot a_k \tilde{X}_k(a_k\theta))] = M_{\tilde{X}_k(a_k\theta)}(a_k t) = \frac{M_{X_k}(a_k\theta + a_k t)}{M_{X_k}(a_k\theta)},$$

which is the k th term in Equation (22). By the uniqueness of the m.g.f., the Esscher-transformed random variable X with parameter θ has the same distribution as $a_1\tilde{X}_1(a_1\theta) + a_2\tilde{X}_2(a_2\theta) + \dots + a_n\tilde{X}_n(a_n\theta)$. $\square$

Lemma 3 indicates that the transformed random variable Z_k may be simulated by simulating the Esscher-transformed distributions of Y_k and $\sum_{i=1}^{\tilde{\zeta}_k} X_i$ separately, and the parameter of the Esscher transform applied to the former distribution is R^* , as defined in Equation (16), while the parameter used for the latter distribution is $-R^*$. This immediately allows us to conduct the simulation study for some special cases. The algorithm for these cases is:

- (i) Calculate the root, R^* , to Equation (16).

- (ii) Identify the distribution of the sampling random variables $\tilde{Z}_k(R^*) = \tilde{Y}_k(R^*) - \left(\widetilde{\sum_{i=1}^{\xi_k} X_i}\right)^{(-R^*)}$ by applying the Esscher transform with parameter R^* to the distribution of Y_k and with parameter $-R^*$ to the distribution of $\sum_{i=1}^{\xi_k} X_i$.
- (iii) Simulate $\tilde{Z}_k(R^*)$ until $\sum \tilde{Z}_k(R^*) > u$. Return $\exp(-R^* \cdot \sum \tilde{Z}_k(R^*))$.
- (iv) Repeat step (iii) sufficiently many times. The probability of ruin is the average of the return values.

Example 1 (exponential). Suppose that the premiums are exponentially distributed with mean $1/a$, and the claims are exponentially distributed with mean $1/b$. This special case has been studied in Boikov (2003), where the author derived the exact ruin probability as

$$\psi(u) = \frac{(a+b)\lambda}{(\lambda+\mu)b} e^{-R^*u}. \quad (23)$$

Consider the simulation approach. We simulate the transformed distribution of Z_k by simulating those of Y_k and $\sum_{i=1}^{\xi_k} X_i$ separately. It is known that the Esscher transform of an exponential distribution is an exponential distribution. Hence, we have $\tilde{Y}_k(R^*) \sim \exp(b - R^*)$. On the other hand, consider the m.g.f. of the compound geometric random variable $\sum_{i=1}^{\xi_k} X_i$,

$$M(t) = \mathbb{E}\left[\exp\left(t \sum_{i=1}^{\xi_k} X_i\right)\right] = P_{\xi}(M_X(t)) = \frac{p}{1 - (1-p) \cdot \frac{a}{a-t}} = \frac{a-t}{a-t/p}, \quad (24)$$

where P_{ξ} is the p.g.f. of ξ_1 , M_X is the m.g.f. of X_1 , and $p = \lambda/(\lambda + \mu)$ is the probability of success of ξ_1 . The m.g.f. after the Esscher transform with parameter $-R^*$ is then

$$\begin{aligned} \frac{M(-R^* + t)}{M(-R^*)} &= \frac{(a + R^* - t)(ap + R^*)}{(ap + R^* - t)(a + R^*)} = \frac{(a + R^*) - t}{(a + R^*) \frac{(ap + R^* - t)}{ap + R^*}} \\ &= \frac{(a + R^*) - t}{(a + R^*) - \frac{t}{(ap + R^*)/(a + R^*)}}. \end{aligned} \quad (25)$$

Observing that Equation (25) has the same form as Equation (24), we conclude that the Esscher transformed compound geometric-exponential distribution is still a compound geometric-exponential distribution, with the compounding random variable being $\text{Ge}(\tilde{p} = \frac{ap + R^*}{a + R^*})$, and the compounded random variable being $\exp(\tilde{a} = a + R^*)$.

Now, all the components of the transformed surplus process may be simulated. Suppose $a = 0.01$, $b = 0.002$, $\mu = 10,000$, $\lambda = 1000$. It may be calculated that the adjustment coefficient $R^* = (b\mu - a\lambda)/(\lambda + \mu)$. The theoretical probability of ruin by Equation (23) is then

$$\psi(u) = \frac{6}{11} \exp\left(-\frac{1}{1100}u\right).$$

We conduct the simulation study using 100,000 sample paths. The simulation results are plotted against the theoretical result in Figure 3, and one may see that they are perfectly aligned.

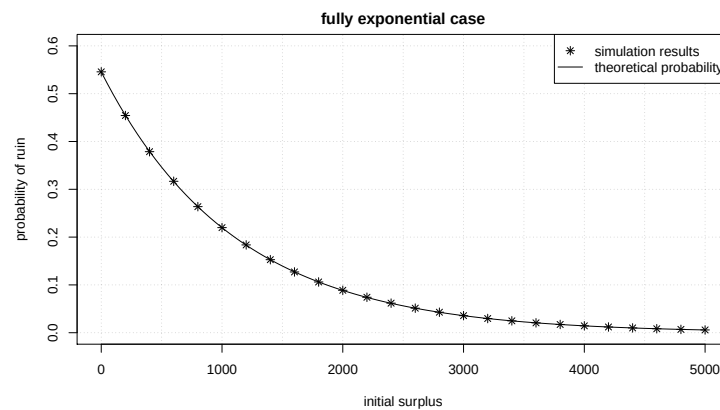


Figure 3. Simulated probability of ruin vs. theoretical probability of ruin when both premiums and claims follow exponential distributions.

In the next example, we consider the case where the premium is a fixed amount, a . In some jurisdictions, the premiums for obligatory liability insurance are fixed or tiered, which may be modelled by the next example. We have the following lemma about this model.

Lemma 4. *If the premium is a constant, a , for each policy, and the claim size distribution is exponential with mean $1/b$, then the probability of ultimate ruin is*

$$\psi(u) = \frac{b - R^*}{b} \exp(-R^*u) \quad (26)$$

where the adjustment coefficient, R^* , is the root of the equation

$$\frac{\lambda b}{b - R} = \lambda + \mu - \mu \exp(-aR). \quad (27)$$

Proof. We first derive an integral equation satisfied by the survival probability $\varphi(u) = 1 - \psi(u)$. Considering the number of events in a small time interval $(0, \Delta)$, similarly to Boikov (2003), we have

$$\begin{aligned} \varphi(u) = & (1 - \lambda\Delta)(1 - \mu\Delta)\varphi(u) \\ & + \lambda\Delta(1 - \mu\Delta) \int_0^u b \exp(-bx) \varphi(u - x) dx + (1 - \lambda\Delta)\mu\Delta\varphi(u + a) + o(\Delta). \end{aligned}$$

Rearranging the terms of the latter equation yields

$$\begin{aligned} (\lambda + \mu)\Delta\varphi(u) = & \lambda\Delta \int_0^u b \exp(-bx) \varphi(u - x) dx + \mu\Delta\varphi(u + a) \\ & + \lambda\mu\Delta^2 \left[\varphi(u) - \varphi(u + a) - \int_0^u b \exp(-bx) \varphi(u - x) dx \right] + o(\Delta). \end{aligned}$$

Dividing by Δ and letting $\Delta \rightarrow 0$ yields

$$(\lambda + \mu)\varphi(u) = \mu\varphi(u + a) + \lambda \int_0^u b \exp(-bx) \varphi(u - x) dx. \quad (28)$$

One may verify that

$$\varphi(u) = 1 - \frac{b - R^*}{b} \exp(-R^*u) \quad (29)$$

is the solution to Equation (28), where the adjustment coefficient, R^* , is determined by Equation (27). $\square$

Example 2 (fixed premium amount per policy). The simulation for this case is straightforward. The random variable $Z_k = Y_k - a\zeta_k$ is a linear combination of random variables. By Lemma 3, the transformed Z_k may be simulated by $\tilde{Z}_k(R^*) = \tilde{Y}_k(R^*) - a\tilde{\zeta}_k(-aR^*)$, where $\tilde{Y}_k(R^*) \sim \exp(b - R^*)$, and $\tilde{\zeta}_k(-aR^*) \sim \text{Ge}(\tilde{p})$, where $\tilde{p} = 1 - (1 - p)\exp(-aR^*)$.

Suppose $a = 100, b = 0.002, \mu = 10,000, \lambda = 1000$, it can be calculated that $R^* \approx 9.516623 \times 10^{-4}$. A comparison of the simulated ruin probability and the theoretical ruin probability is given in Figure 4. Again, it is the case that they are aligned.

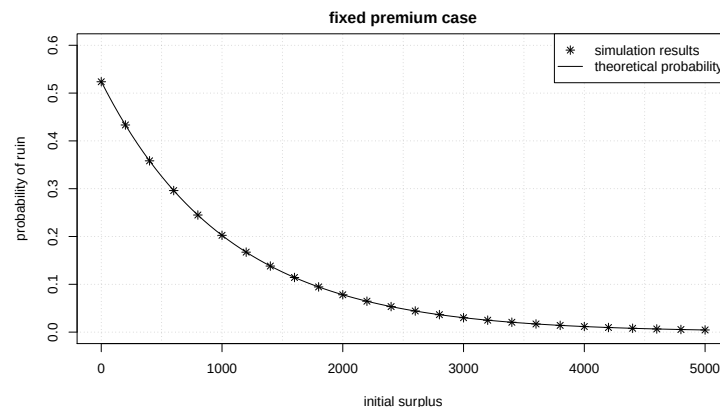


Figure 4. Simulated ruin probability vs. theoretical ruin probability when premiums are fixed amounts.

The simulations in Examples 1 and 2 may be performed since the exact distribution of the Esscher transformed random variables $\tilde{Z}_k(R^*)$ may be identified. Specifically, the transformation of the compound random variable $\sum_{i=1}^{\zeta_k} X_i$ is known to be another compound random variable. This allows us to simulate it by simulating the compounding random variable and the compounded random variables sequentially. Notice that, in this case, one does not need the p.d.f. of $\sum_{i=1}^{\zeta_k} X_i$ to conduct the simulation. With arbitrary X_k , the Esscher-transformed distribution may not be identifiable. In this case, only the m.g.f. of the new distribution is known, which is given by Equation (7). Some techniques exist for simulating random variables from moment generating functions. For example, McLeish (2014) discusses the use of the saddlepoint approximation to simulate a random variable from its m.g.f. Various possible different approaches are also compared therein. However, this type of approximation often relies on the assumption that the distribution is close to a known distribution, such as a normal distribution, and its accuracy varies in different settings. Taking advantage of the fact that the $\sum_{i=1}^{\zeta_k} X_i$ is a compound geometric distribution, we next present an approximation method based on the fast Fourier transform, which allows us to conduct the simulation for any appropriate premium-size distribution.

For the cases where the exact distribution of the Esscher transform of $\sum_{i=1}^{\zeta_k} X_i$ cannot be identified, the algorithm is:

- (i) Choose a sufficiently large truncation point, Π . Choose equally spaced $x_0 = 0, x_1, x_2, \dots, x_n = \Pi$, calculate

$$q_i = \begin{cases} 0 & i = 0, \\ \mathbb{P}(x_{i-1} < X_1 \leq x_i), & i = 1, 2, \dots, n. \end{cases}$$

- (ii) Obtain a discretisation of $\sum_{i=1}^{\zeta_k} X_i$ using the fast Fourier transform.
 - (a) Obtain $\mathfrak{F}q$, the fast Fourier transform of q_i s.
 - (b) Calculate $(\mathfrak{F}f)_i = p/[1 - (1 - p)(\mathfrak{F}p)_i]$ as the approximation of the characteristic function.
 - (c) Invert the fast Fourier transform to obtain $f_i = \mathfrak{F}^{-1}(\mathfrak{F}f)_i$. f_i s are the discretisation of the compound distribution.

- (iii) Calculate the positive root, R^* , to Equation (16).
- (iv) Simulate $\tilde{Z}_k(R^*)$ until $\sum \tilde{Z}_k(R^*) > u$, where $\left(\sum_{i=1}^{\tilde{c}_k} X_i\right)^{(-R^*)}$ is approximated by a discrete random variable, G , with $\mathbb{P}(G = x_i) = f_i \exp(-x_i R^*) / M(R^*)$. Return $\exp(-R^* \cdot \sum \tilde{Z}_i(R^*))$.
- (v) Repeat step (iv) sufficiently many times. The probability of ruin is the average of the return values.

The approximation approach has two main sources of error. The first source of error is discretisation. One may prove that the discretised random variable converges to the original random variable in distribution as the grid of the discretisation becomes finer, i.e., as $x_1 - x_0$ approaches 0. The second source of error is the fast Fourier transform. To apply the fast Fourier transform, one needs to truncate the original distribution at the chosen truncation point, Π . Due to the periodic nature of the Fourier transform, the p.d.f. beyond point Π is “wrapped around” and appears in $f_0, f_1, \dots, f_n$. This error is usually termed the *aliasing error*. For this reason, one should choose a sufficiently large Π , such that the p.d.f. beyond the truncation point is minimal to reduce the error. For demonstration, consider a compound Poisson-logarithmic distribution. It is known that this compound distribution is a negative binomial distribution. We obtain the mass function of the compound distribution using the fast Fourier transform with a different truncation point, Π . Since Poisson distribution is in the $(a, b, 0)$ class, we may also obtain the mass function using Panjer’s recursion. The results are plotted in Figure 5. We observe that the aliasing error decreases as the truncation point, Π , increases. Both the fast Fourier transform and Panjer’s recursion can yield highly accurate approximations. However, it is known that the computation complexity of the fast Fourier transforms is $O(n \log n)$, while the computation complexity of Panjer’s recursion is $O(n^2)$. When there are many points to evaluate, the fast Fourier transform has a computational advantage over Panjer’s recursion. For a more detailed comparison between the fast Fourier transform and Panjer’s recursion and an algorithm for reducing the aliasing error, see Embrechts and Frei (2009).

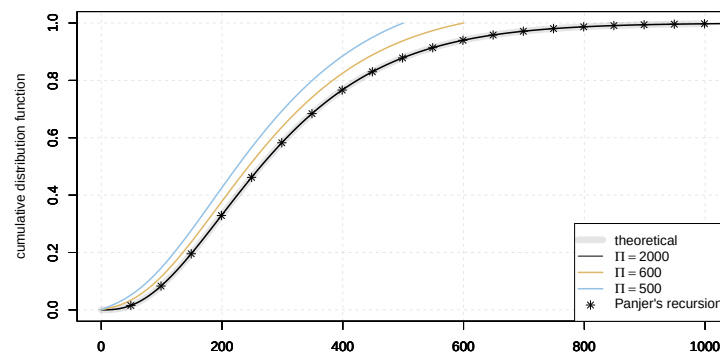


Figure 5. Fast Fourier transform with aliasing error.

Example 3. In this example, we consider gamma claim sizes and exponential premium sizes. For the claim process, let the intensity of $N(t)$ be $\lambda = 1000$ and let $Y_k \sim \text{Ga}(\alpha, \alpha/100)$, so that $\lambda \mathbb{E}[Y_1] = 100,000$; for the premium process, let the intensity of $H(t)$ be μ and let $X_k \sim \exp(\mu/200,000)$, so that $\mu \mathbb{E}[X_1] = 200,000$, which implies a relative security loading of 1. We vary two key parameters of the model:

- α : this parameter controls the shape of the claim size distribution. Notice that, since the expected value of the claim size distribution is kept constant, the distribution has a heavier tail with a smaller α .
- μ : we let $\mu \rightarrow \infty$ to observe the limiting behaviour of the model.

For each claim size distribution, we also consider the ruin probability under the classical model with the premium rate $c = 200,000$. Notice that this probability is not a function of μ . The truncation point, Π , used in the simulation is 1,000,000, at which point the survival function of the distribution of the compound random variable $\sum_{i=1}^{\zeta_k} X_i$ is sufficiently small. The simulation results with 100,000 simulated sample paths with $\mu = 100, 500, 3000, 100,000$ and $\alpha = 0.25, 1, 10$ are given in Figure 6.

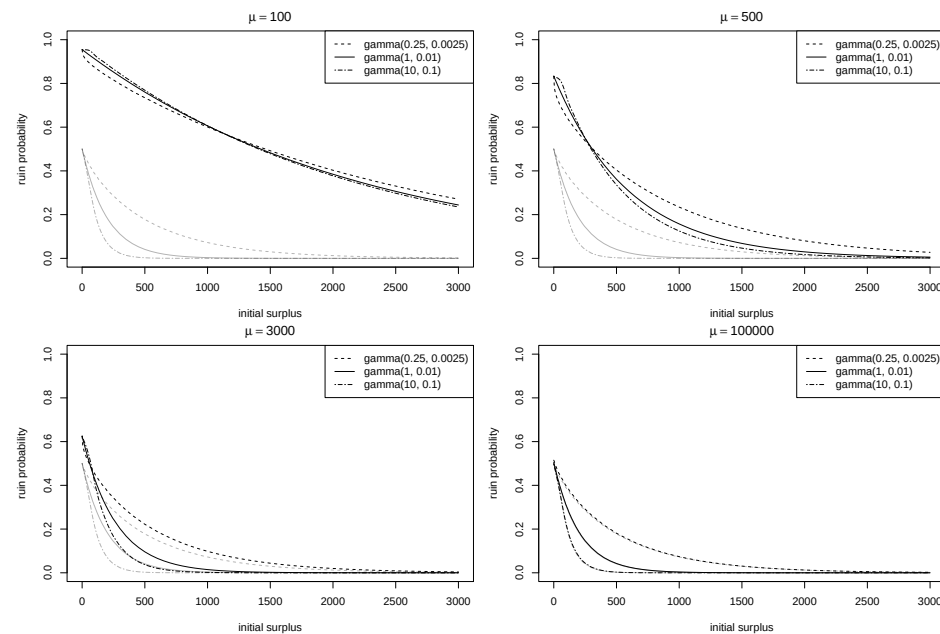


Figure 6. Simulated ruin probability with gamma distributed claim sizes.

Remark 3. Various known behaviours of the stochastic premium model may be observed in Example 3. For each gamma distribution, we see that the ruin probability under the stochastic premium model is greater than the ruin probability under the classical model for all u , regardless of μ . This is result 3 in Theorem 1. We also observe that, as μ increases, the ruin probability under the stochastic premium model converges to the ruin probability under the classical model, consistent with result 4 in Theorem 1.

Remark 4. Some cases in Example 3 are extreme and unlikely to happen in practice. Notice that the expected number of premium arrivals in one period of time is $\mathbb{E}[H(1)] = \mu$. For an established insurer whose business does not vary dramatically on a year-to-year basis, μ is also approximately the portfolio size. Since all the claims are from the portfolio, the average number of claims per period of time from each policy is $\mathbb{E}[N(1)]/\mu = \lambda/\mu$, which means that the average number of claims per policy is 10 when $\mu = 100$ and is 0.01 when $\mu = 100,000$. Based on some industry studies, such as [APCIA et al. \(2020\)](#), the claim frequency per exposure varies based on the physical environment and covered peril. As a result, we expect the impact of adopting the stochastic premium model to differ among different insurance portfolios.

5. An Example Using a Real-Life Data Set

We demonstrate the stochastic premium model with a real-life auto-insurance data set. The data set is provided by an industry partner who wishes to remain anonymous. The data set contains enough information for fully specifying a risk model. For a more detailed description of the data set, see [Miao et al. \(2023b\)](#).

We model the size and the frequency of premiums and claims separately. For the premium amount, it is determined that a 3-point mixture of normal distributions approximates the distribution well. The data set contains information on an insurance policy that is a combination of compulsory liability coverage and optional coverage. The premium for the

compulsory liability coverage is determined by the class of the insured vehicle and past claim experience. The premium for the additional coverage is calculated using information that is more specific to the driver and the vehicle, such as the value of the vehicle and the region where the vehicle is mainly driven. This justifies the usage of mixture distribution: the three normal distributions are likely a result of three classes of vehicles, while within each class the premium varies due to the bonus–malus system and optional coverage. Using the maximum likelihood estimate, we obtain the parameters as

$$f(x) = \frac{\sum_{k=1}^3 a_k f_k(x)}{0.997322}, \quad x > 0, \quad (30)$$

where f_k s are normal density functions with

k	a_k	mean	std
1	0.10	1410	227
2	0.41	2764	560
3	0.49	4367	1716

Notice that the distribution is truncated at 0, hence the normalising constant. The left panel of Figure 7 presents the histogram of the observed premium amounts with the fitted density function.

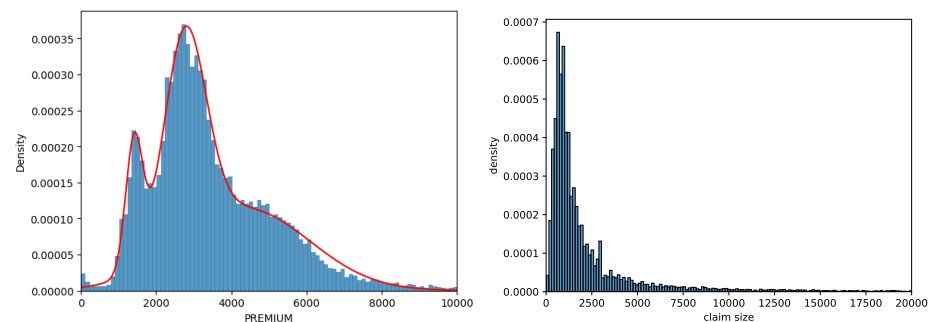


Figure 7. (Left): histogram of the observed premium amounts with fitted density function. (Right): empirical distribution of the claim sizes.

The data set reveals distinct temporal patterns in the premium arrival process, including seasonality and growth trends. As noted in Miao et al. (2023b), these patterns alter the rate of premium arrivals, with fluctuations driven by recurring events such as policy renewals, periodic public events, and economic factors, along with long-term growth influenced by market expansion and demographic shifts. In this example, we simply use the average annual premium arrival as the intensity of $H(t)$, meaning that we assume premium arrivals occur at a steady rate throughout the year, disregarding the detailed seasonality and growth patterns. This simplification allows for easier analysis and computation. In this case, it is

$$\mu = 17,992,$$

which completes the modelling of the premium arrival process.

The claim amounts exhibit heavier tails than the premium amounts. This poses certain difficulties as the moment generating function of heavy-tailed distributions may not exist. For this example, we use the empirical distribution instead of trying to find a heavy-tailed parametric model. The right panel of Figure 7 illustrates the empirical distribution of the claim sizes.

A key quantity to determine the relationship between the premium arrival process and the claim arrival process is the claim rate per exposure. To this end, we calculate

the number of claims as well as the *in-force exposure* on each day. The daily claim rate is calculated as the ratio of the two and is given in Figure 8. The daily claim rate is then converted to the annual rate. More specifically, each exposure generates 0.314201 claims per year. As a result, we have

$$\lambda = 0.314201 \times 17,992 = 5653$$

and the claim process is also fully specified.

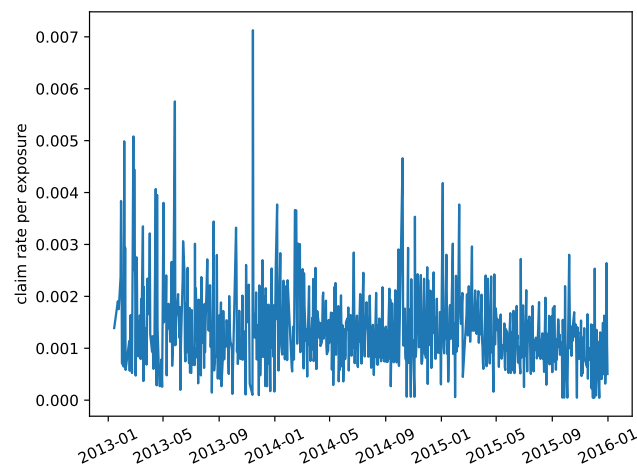


Figure 8. Daily claim rate per exposure.

We also examine the classical risk model to show the differences the stochastic premium model makes. The comparable classical risk model (1) has the premium rate

$$c = \mu \mathbb{E}[X_1] = 17,992 \times 3424.71 = 61,617,382.32.$$

Probabilities of ruin are simulated under both models with a different initial surplus. The results are given in Figure 9 as the blue and the orange curves.

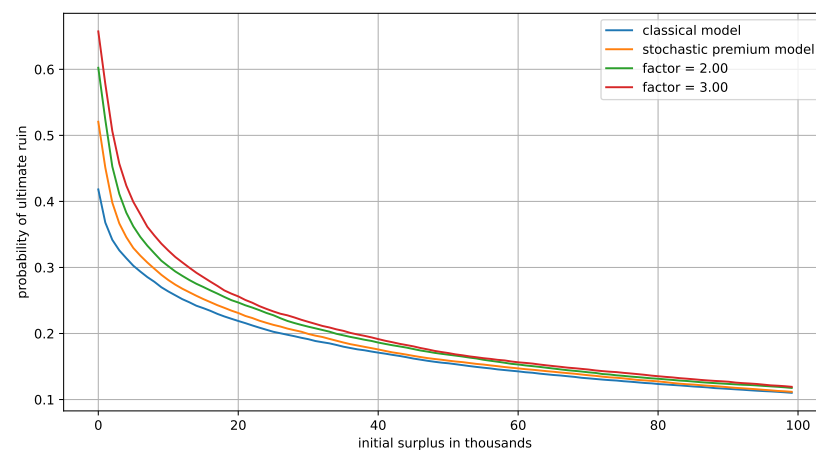


Figure 9. Comparison of probability of ruin under the classical model and the stochastic premium model using the auto-insurance data set.

Consistent with Theorem 1, the probability of ruin is higher under the stochastic premium model than the classical model. The required reserves for the insurance company to control the probability of ruin with different risk levels are given in Table 1 in the column $\eta = 1$. We note that the stochastic premium model is a more conservative model that yields a higher reserve requirement, although in this specific case the average number of claims

per exposure is not high, and therefore the classical model and the stochastic premium model are not extremely different.

Table 1. Initial surplus required to limit the probability of ruin.

Risk Level	Classical Model	Stochastic Premium Model			
		$\eta = 0.01$	$\eta = 1$	$\eta = 2$	$\eta = 3$
0.05	252,253	252,753 (+0.20%)	256,757 (+1.79%)	264,765 (+4.96%)	269,770 (+6.94%)
0.03	348,849	349,850 (+0.29%)	357,858 (+2.58%)	359,860 (+3.16%)	363,864 (+4.30%)
0.01	546,547	547,047 (+0.09%)	553,053 (+1.19%)	558,058 (+2.11%)	565,065 (+3.39%)
0.005	665,916	667,167 (+0.19%)	671,171 (+0.79%)	676,176 (+1.54%)	683,183 (+2.59%)

Application of the Model under Other Scenarios

An interesting comparison we can make is allowing the average claim rate per exposure to vary. In the example using the real-life auto-insurance data set, we multiply the average claim rate per exposure by a constant, η . We also scale the premium size distribution accordingly such that the relative security loading remains constant. The modified risk process is then $\check{U}_\eta(t) = u + \sum_{k=1}^{H(t)} \check{X}_{\eta,k} - \sum_{k=1}^{\check{N}_\eta(t)} Y_k$, where $\check{X}_{\eta,k} \stackrel{d}{=} \eta X_k$, and $\check{N}_\eta(t)$ is a Poisson process with intensity $\eta\lambda$. We choose $\eta = 0.01, 2$, and 3 to demonstrate the difference it makes. Under the classical model, the probability of ruin depends only on the relative security loading and the claim size distribution. Consequently, the results under the classical model remain constant regardless of the value of η . The results under the stochastic premium model are simulated using the method in Section 4, and the level of initial surplus required to control the probability of ruin is given in Table 1. We observe that the difference between the classical model and the stochastic premium model increases as η increases. Moreover, for each η , the percentage difference is larger if the insurance company has higher risk tolerance.

As noted in [Werner and Modlin \(2010\)](#), it is a standard practice in ratemaking to adjust future premiums based on historical claim data. When insurers observe a higher-than-expected number of claims per exposure, they typically respond by raising premiums to cover the increased costs. However, from a risk management perspective, merely increasing premiums does not fully address the risk of insolvency. As illustrated by the results in Table 1, a higher average claim rate not only raises expected liabilities but also amplifies the relative uncertainty of the premium income. This means that higher reserves are necessary to maintain the same risk level. For practitioners, accounting for uncertainties in both the timing and amount of premiums and claims yields a better description of their insolvency risk. Compared with the classical model, the stochastic premium model more accurately reflects real-world conditions. This makes it a more realistic tool for evaluating insolvency risk and is one advantage of the stochastic premium model.

In practice, the changes in the physical environment often alter the actual claim experience. For example, [Richardson and Hartman \(2022\)](#) used dynamic linear models to study the impact of the COVID-19 pandemic on the auto-insurance industry. It was discovered that the frequency of claims was lower while the severity was higher during the pandemic, likely due to fewer miles driven. Since the claim rate is lower, it is likely that the difference between the classical model and the stochastic premium model is smaller. This is one example where an external factor affects the claim rate per exposure.

Another better studied example is the increasing frequency of claims for property and casualty insurance. Due to climate change, extreme weather events are becoming more frequent and more severe, leading to a heavy burden on communities across the globe. Its impacts on insurance practices have attracted more attention in recent years. Using Canada as an example, insurers are incurring more losses due to climate change in recent years. Figure 10 illustrates the total inflation-adjusted loss and the number of catastrophic events

over the past four decades (source: [Insurance Bureau of Canada \(2023\)](#)). Both the total loss and the number of events has increased significantly in this time period.

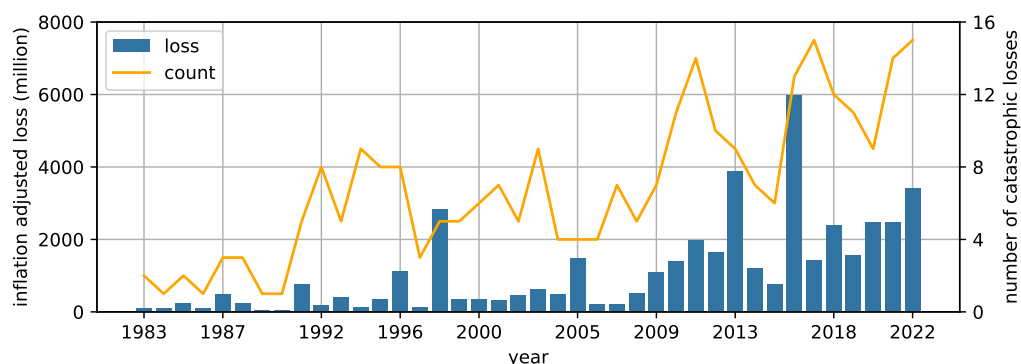


Figure 10. Total annual inflation-adjusted catastrophic losses and number of catastrophic events in Canada.

One way an extreme weather event could impact insurance companies is by causing a large volume of claims to arrive at the same time. An early example of this is Hurricane Andrew, which struck the Bahamas and part of the US in 1992. The large losses caused at least 16 insurance companies to go bankrupt. This prompted regulatory changes that imposed stricter solvency requirements on insurers. Nowadays, it is a common practice for insurers to manage their exposure to catastrophe losses through more prudent risk selection and the use of reinsurance contracts. This focuses on the tail distribution of the aggregate claim.

The results presented in this paper reveal another possible impact that climate change may have on insurance companies, namely an increased claim rate in day-to-day non-catastrophic claims. As meteorological studies point out, climate change also alters the patterns of more localised events, such as flash floods, hail, and landslides; see, for example, [Kleinen and Petschel-Held \(2007\)](#); [Raupach et al. \(2021\)](#); and [Gariano and Guzzetti \(2016\)](#). These events are likely to increase the claim rate per policy, but not likely to result in insolvency in the short term. The results of this paper suggest that, in addition to adjustments to the premium rate, an insurer needs to have a higher level of reserves to achieve the same risk level. Naturally, climate change would alter the demand for insurance, as well as claim size distribution. This requires the insurers to re-evaluate their insolvency risk using the best estimates that take into consideration the new patterns in premiums and claims.

The stochastic premium model considers the uncertainty in both premiums and claims. As demonstrated in this paper, this may contribute to a better understanding of an insurer's insolvency risk. By adopting the stochastic premium model, insurance companies can better withstand the pressures associated with uncertain future cashflow, which in turn may contribute to a better financial resilience and stability. Insolvency of insurers can destabilise the market and leave policyholders vulnerable, such as in the catastrophic scenario after Hurricane Andrew. The ability to proactively adjust reserves and premiums using more sophisticated risk models not only protects the insurer's solvency but also provides policyholders with more secure and reliable coverage. This is a potential societal benefit of the proposed method in this paper.

6. Conclusions

In this paper, we investigate the application of the stochastic premium model. We first derive a simulation approach to obtain the probability of ruin. A new probability measure is defined through the Esscher transform under which ruin is certain. The simulation is carried out under the new probability measure, and the probability of ruin under the original probability measure is obtained. For cases where the sampling distribution is not identifiable, we take advantage of the fact that the drop of surplus may be expressed

as a compound geometric distribution and use the fast Fourier transform as an approximation. Some theoretical results pertinent to this model are reviewed and demonstrated through examples.

By using special cases whose theoretical probability of ruin is known, we discover that the simulation approach is fast and is capable of producing highly accurate results. The simulation approach may be readily extended to more general cases as well. We also discover that the differences between the stochastic premium model and the classical model vary based on a number of factors, such as the heaviness of tail of the distributions and the claim frequency per exposure. This indicates that the impact of adopting the stochastic premium model varies among different insurance portfolios and that the applicability of this model in practice is worth further investigation.

We demonstrate the impact of using the model in practice through a real-life auto-insurance example. After fitting the model to the data, we use the probability of ruin as a risk measure to compare the performance of the stochastic premium model and the classical model. We also present the amount of initial surplus required to control the insolvency risk as a more intuitive and practical measure. We discover that, albeit the results are consistent with the known results in theory, the practical differences are marginal for this specific data set. This is due to the fact that the claim rate per exposure is relatively low. We explore other hypothetical scenarios using modified versions of the data set. More specifically, we investigate whether a worsening claim experience further differentiates the two models. We discover that, in a changing physical environment where the claim rate is higher, simply increasing the premium to counter the increased claims is not sufficient for insurers to control their insolvency risk adequately. As the claim rate increases, the risk associated with the uncertainty of premium income becomes more pronounced. As a result, insurers need to have more surplus to account for this increased risk. As external factors such as climate change, COVID-19, and urbanisation alter the demand for insurance and the claim experience for each policy, the insolvency risk may be re-evaluated using the latest estimates that account for changes in both premiums and claims. This highlights the potential practical value of the stochastic premium model in an ever-evolving environment.

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Data Availability Statement: The programs used for the simulation study can be provided upon request. All examples presented in Section 4 may be replicated using the program. The real-life auto insurance data set used in Section 5 is confidential and not available to the public.

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