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Optimal Static Hedging of Variable Annuities with Volatility-Dependent Fees

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Abstract: Variable annuities (VAs) and other long-term equity-linked insurance products are typically difficult to hedge in the incomplete markets. A state-dependent fee tied with market volatility for VAs is designed to contribute the risk-sharing mechanism between policyholders and insurers. Different from prior research, we discuss several aspects on a fair valuation, fee-rate determination and hedging with volatility-dependent fees from the perspective of a VA hedger. A method of efficient hedging strategy as a benchmark compared to other strategies is developed in the stochastic volatility setting. We illustrate this method in guaranteed minimum maturity benefits (GMMBs), but it is also applicable to other equity-linked insurance contracts.

Keywords: variable annuity; volatility-dependent fee; efficient hedging; Heston model

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1. Introduction

A variable annuity (VA) is a tax-deferred and unit-linked insurance product, which provides various forms of guarantee riders to investors by allowing equity participation in a collective investment. With a variable annuity, investors make payments until their retirement and then begin receiving regular retirement benefit from the insurance company. VA guarantees can be classified into two broad types: guaranteed minimum death benefits (GMDBs) and guaranteed minimum living benefits (GMLBs). The GMLBs include guaranteed minimum maturity benefits (GMMBs), guaranteed minimum income benefits (GMIBs), and guaranteed minimum withdrawal benefits (GMWBs). While VA policyholders can benefit from the guarantee, a cost of insurance is charged in the form of a percentage fee on the account value, which could impact both the insurer's hedging performance and the policyholders' surrender decision on the policy.

In contrast to a fixed percentage fee which has been the insurer's dominant premium structure in VA markets, in the past decade, the actuarial literature begins to investigate the issues on policy design and risk management of VA products with alternative state-dependent fees initiated by the industry. Generally, the state-dependent fee structure allows VA insurers to align their premium rate with the market variables and helps reduce their hedging difficulty. To be specific, based on the fact that the embedded put option liability of VAs are in-the-money when the account value is low, Bernard et al. (2014) are the first to propose a barrier-type state-dependent structure where fees are only charged when the account value is below a threshold level. To better match the moneyness/riskiness of the embedded guarantee of VAs over time, Bernard and Moenig (2019) propose a time-dependent fee structure in a discrete model, where a higher constant fee rate is charged on the account in early years of the VA contract than the later years. They demonstrate that such a time-dependent structure could help lower the fee rates of VAs without reducing the insurer's profit, thus making VAs more attractive to potential investors. Landriault et al. (2021) generalize the fee rate proposal of Bernard et al. (2014) and investigate the impact on VAs with a high-water mark (HWM) fee structure, which is commonly used in the hedge fund industry by rewarding the VA insurer when the fund outperforms the past. They find that under appropriate market conditions, the HWM structure can benefit the welfare for both policyholders and the insurer. The above designs of state-dependent fees are established on the performance of the underlying fund process of VAs, which is triggered by a threshold level of account value or the operation time of the policy.

In May 2014, a VA contract called Polaris Choice IV was launched by American General Life in the U.S. market. In Polaris Choice IV, the premium for a variety of optional VA riders is adjusted based on a non-discretionary formula tied to the change in the Volatility Index (VIX), an index of market volatility reported by the Chicago Board Options Exchange (CBOE). In a nutshell, the non-discretionary formula is used to calculate the annual fee rate applicable after the first Benefit Year in the following form:

$$\text{Initial Annual Fee Rate} + [0.05\% \times (\text{Average Value of the VIX} - 20)], \tag{1}$$

where the initial annual fee rate is set to 1.1% for a policyholder aged from 65 to 85 at the inception of the contract. Since VIX is a measure of overall market volatility, the VIX-linked fee structure (1) can be regarded as state-dependent while it is charged in proportion to the level of VIX. We should note that the underlying fund and its corresponding volatility are negatively correlated due to the well-documented leverage effect, and hence the put option liability of VA guarantees is typically very expensive when the market is volatile. Such a volatility-dependent fee structure can provide a better alignment between the insurer's fee incomes and their put option liability embedded with VA guarantees. In line with the fee rate structure of Polaris Choice IV, Cui et al. (2017) model a guaranteed minimum maturity benefit with VIX-linked fees in a Heston-type stochastic volatility setting. Their numerical examples show that the VIX-linked fee reduces the sensitivity of the insurer's liability to market volatility when compared to a VA with the traditional fixed fee rate. It is noteworthy here that the fee structure proposed by Cui et al. (2017) is not directly tied to the average value of the VIX as used by Polaris Choice IV, but it is linked to the squared volatility of the market index for keeping the affinity of the modeling and thus producing analytical Heston-based formulas for VA valuation. Kouritzin and MacKay (2018) further assess the effectiveness of the VIX-linked fee structure in decreasing the sensitivity of the insurer's liability to volatility risk for a GMWB contract.

While most of the actuarial literature on state-dependent fees are mainly focused on the valuation and surrender analysis of VAs (e.g., Bauer et al. (2017); Bernard et al. (2014); Kirkby and Aguilar (2023); MacKay et al. (2017, 2023) and reference therein), there is quite a limited number of studies on the hedging problem of VAs with state-dependent fees. Delong (2014) considers an incomplete financial market by modeling the dynamics for the account value and the underlying investment asset with a general two-dimensional Lévy process. Under the quadratic optimization criterion based on a difference between the costs of a hedging portfolio and the insurer's liability, the author determines a self-financing strategy under the barrier-type state-dependent fees proposed by Bernard et al. (2014). To compare the various types of state-dependent structures, Wang and Zou (2021) propose a stochastic control framework for a representative insurer who seeks an optimal fee structure of VAs to maximize their business objective, which is defined as discounted "received management fees" minus "expenses for providing VA guarantees". Their solutions echo the theoretical result of Bernard et al. (2014) that the optimal fee structure is of a barrier-type and time dependent, and the insurer should charge fees only when the VA account value hits the reflection boundary from below. Although it is not a direct hedging strategy for VAs, the solution of the method by Wang and Zou (2021) can suggest an optimal fee structure which helps mitigate the insurer's hedging risk. Generally, an adoption of state-dependent fees causes a path dependency of the underlying fund process, which makes the hedging solution of VAs non-trivial. In particular, the intricacy of the hedging problem under volatility-dependent fees also arises from the fact that the presence of the stochastic volatility typically leads to an incomplete market in which perfect hedging strategies do not exist, depending on the available trading opportunities. Due to the scarcity of hedging approaches from the existing literature, this paper aims to fill this gap by proposing an optimal hedging strategy for VAs with volatility-dependent fees.

In an incomplete market, a hedger is faced with the problem of searching for strategies that reduce the risk as much as possible. In practice, a super-hedging strategy can often be too expensive. For this reason, Föllmer and Leukert (2000) investigate the possibility of investing less capital than the super-hedging price of the liability. This leads to a shortfall, the risk of which, measured by a suitable risk measure, should be minimized. For example, Kolkiewicz and Liu (2012) propose to hedge a version of the guaranteed minimum withdrawal benefit by using a European option whose payoff is determined by minimizing the mean-square hedging error. A common criticism of methods based on mean-square error is the fact that they treat gains and losses symmetrically, and the related shortfall risk which appears in that case should be minimized to protect the investor against the resulting loss. In the literature, alternative risk measures accepted by the investor on minimizing shortfall risk can be found in Föllmer and Leukert (1999, 2000), Cvitanic (2000); Pham (2002), Cong et al. (2013, 2014), among others. For path-dependent options, Kolkiewicz (2016) finds a general method of constructing static hedging strategies under the Black–Scholes setting to minimize the expected shortfall. This methodology is in line with the concept of the quantile hedging strategy consisting of super-hedging a modified claim φH, where H is the payoff of a contingent claim and φ is the solution of the statistical optimization problem based on the optimal randomized test (see Föllmer and Leukert (2000) for reference).

The objective of this paper is to extend the theoretical result of Kolkiewicz (2016) and develop a numerical approach of constructing static hedging strategies for path-dependent options that minimize, for a given time interval, the shortfall risk, which we define as the expectation of the shortfall weighted by some loss function. In particular, we are interested in extending the theoretical result of Kolkiewicz (2016) beyond the Black–Scholes framework with a Heston-type stochastic volatility of the market and hedging a VA rider tied to the volatility-dependent fees as proposed by Cui et al. (2017). The method consists of two main steps. In the first step, we formulate an optimal static hedging problem by identifying the risk for the GMMB liability with volatility-dependent fees under a certain risk measure, which we call the residual risk. At this stage, the initial price of the option and the hedging target are treated as given and we solve the problem numerically for each hedging period. In the second step, the hedger replicates this optimal static hedge by establishing an investment portfolio consisting of a non-risky bond and one or more risky securities, such as an underlying market index and numerous vanilla options. On the practical level, such an optimal hedging strategy could be determined by the approximate weights of hedger's replicating portfolio identified from a certain regression approach. The resulting optimal static hedging strategy, which is adapted to the stochastic volatility and allows a more general modeling of dependency among risk factors, can be viewed as a benchmark to compare other strategies, such as traditional dynamic hedging and short-dated static hedging.

The remainder of this paper is organized as follows. In Section 2, we introduce the model hypotheses and discuss the pricing condition for VA insurers on how to identify the structure of a volatility-dependent fee for a mitigation of their hedging difficulty. In Section 3, with a goal of minimizing the insurer's expected shortfall risk, we formulate the optimal hedging strategy for a GMMB under Heston-type volatility-dependent fees. In Section 4, we describe the numerical procedure for the proposed hedging strategy and illustrate it with examples. Section 5 concludes the paper.

2. Pricing a GMMB with Heston-Type Volatility-Dependent Fee

In this section, we discuss the problem of pricing a guaranteed minimum maturity benefit with volatility-dependent fees. Different from the derivation by Cui et al. (2017), in Section 2.1, we first demonstrate that the characteristic function for the underlying fund value of a GMMB deducted by a volatility-dependent fee can be found as a modified version of the one from Heston (1993), and then we present the pricing result for

GMMB in Section 2.2. In Section 2.3, we explore some hedging objectives of VA insurers who may use them to identify the exact structure of the volatility-dependent fee from a practical viewpoint.

To VA insurers, the inclusion of a state-dependent fee is expected to bring a more adequate coverage of the true hedging cost of VA guarantees. For instance, a state-dependent fee rate, $c_t(F_t, v_t, t)$, $t \geq 0$, can be generally tied to the time of the inception t, the value of the fund F_t and the level of market variance or squared volatility v_t. Specifically, the purpose of charging a volatility-dependent fee is to increase the fee income in order to compensate for the corresponding heightened cost of insurance occurring when the volatility of the equity index is high. As proposed by Cui et al. (2017), the variable fee is an increasing function of the squared volatility level in the sense that

$$c_t(v_t) \equiv c_t := a + bv_t, \tag{2}$$

where $a \geq 0$ is the base fee rate and $b \geq 0$ is the sensitivity parameter of the fee rate function c_t with respect to the squared volatility v_t.

In this paper, we focus on a theoretical development of the hedging strategy for a GMMB rider as described in Cui et al. (2017). If the policyholder is alive at the maturity T, she/he is entitled to receive the greater of the investment account value F_T and the guarantee level G_T at T, i.e., $\max(F_T, G_T)$. In consequence, the risk exposure of a GMMB to the insurer is the put option liability $(G_T - F_T)^+$, where the volatility-dependent fees are deducted from the account value at the rate of c_t.

2.1. Modeling Fund Dynamics with Volatility-Dependent Fees

We generalize a Heston-type volatility-dependent fee structure where the fee rate is a linear function of the current market variance as described in (2). Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $\{\mathcal{F}_t, t \geq 0\}$ generated by the equity index S_t, $t \geq 0$, and its squared volatility v_t, where $\mathbb{P}$ is the physical probability measure. In the framework of Heston (1993), the equity index S follows the dynamics

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{v_t} S_t d\widetilde{W}_t^x, & S_0 > 0, \\ dv_t = \kappa^*(\bar{v}^* - v_t)dt + \sigma\sqrt{v_t} d\widetilde{W}_t^v, & v_0 > 0, \end{cases} \tag{3}$$

where μ represents the physical return, and $\widetilde{W}_t^x$ and $\widetilde{W}_t^v$ are two correlated standard Brownian motions under $\mathbb{P}$ with correlation $\rho \in [-1, 1]$. The constant $\kappa^* > 0$, determines the speed of adjustment of the volatility towards its long-run mean $\bar{v}^* > 0$, and $\sigma > 0$ is the volatility of the volatility. The variance process v_t is strictly positive when the Feller condition, $2\kappa^*\bar{v}^* > \sigma^2$, is satisfied.

In our problem, we assume that a VA fund manager invests in a single market index S so that the fund performance of a GMMB fully tracks the equity index. Due to the fact that a volatility-dependent fee is charged from the fund for covering the insurance claim, we define $F_t(a, b) \equiv F_t$, which is continuously deducted by a fee at the rate of $c_t(v_t)$. Then, the instantaneous return of the VA account is equal to the one of the market index subtracting the instantaneous rate of fee deduction, i.e.,

$$\frac{dF_t}{F_t} = \frac{dS_t}{S_t} - c_t(v_t)dt,$$

and thus, we arrive at the following description of the dynamic of the log-price of the fund value, $x_t = \ln F_t$:

$$\begin{cases} dx_t = \left(\mu - c_t(v_t) - \frac{1}{2}v_t\right)dt + \sqrt{v_t} d\widetilde{W}_t^x, \\ dv_t = \kappa^*(\bar{v}^* - v_t)dt + \sigma\sqrt{v_t} d\widetilde{W}_t^v, \ v_0 > 0. \end{cases} \tag{4}$$

In the Heston model, the risk-neutral measure used for pricing purposes is obtained by specifying a constant volatility risk premium λ. Under the equivalent risk-neutral probability measure $\mathbb{Q}$, the fund value process (4) can be represented as

$$\begin{cases} dx_t = (\alpha - \beta v_t)dt + \sqrt{v_t}dW_t^x, \\ dv_t = \kappa(\bar{v} - v_t)dt + \sigma\sqrt{v_t}dW_t^v, \; v_0 > 0, \end{cases} \tag{5}$$

where r is a constant risk-free interest rate, $\kappa = \kappa^* + \lambda$, $\bar{v} = \kappa^*\bar{v}^*/(\kappa^* + \lambda)$. W_t^x and W_t^v are $\mathbb{Q}$–Brownian motions correlated by $dW_t^x dW_t^v = \rho dt$. Under the linear fee structure c_t with respect to the variance, the resulting dynamic system (5) follows a Heston stochastic volatility model with the modified rates of $\alpha = r - a$ and $\beta = b + \frac{1}{2}$. We thus can revise the pricing formula of Heston (1993) and obtain the characteristic function for the log-price of the fund x_T in the presence of volatility-dependent fees. It is noteworthy here that an equivalent representation for this characteristic function based on the Laplace transform technique can be found in Cui et al. (2017).

According to Heston (1993), the characteristic function for the log-price of the fund x_T, which is conditional on the initial states of log-price x_t and variance v_t, can be represented in the following way:

$$\phi_{\text{VA}}(u, t; a, b) := \phi_{\text{VA}}(u, x_t, v_t, \tau) = \exp(A(u, \tau) + B(u, \tau)x_t + C(u, \tau)v_t), \tag{6}$$

where $\tau = T - t$ and the functions

$$\begin{aligned} A(u, \tau) &= \int_0^\tau (\alpha iu - r)ds + \kappa\bar{v}\int_0^\tau C(u, s)ds = (\alpha iu - r)\tau + \kappa\bar{v}I_C(\tau), \\ B(u, \tau) &= iu, \\ C(u, \tau) &= \frac{-a_1 - C_1}{2a_2(1 - Ge^{-C_1\tau})}\left(1 - e^{-C_1\tau}\right), \end{aligned}$$

with the parameters $a_0 = \frac{1}{2}(iu)(iu - 2\beta)$, $a_1 = \rho\sigma(iu) - \kappa$, $a_2 = \frac{1}{2}\sigma^2$, $C_1 = \sqrt{a_1^2 - 4a_0a_2}$, $G = \frac{-a_1 - C_1}{-a_1 + C_1}$, and $I_C(\tau) = \frac{1}{2a_2}\left[(-a_1 - C_1)\tau - 2\ln\left(\frac{1 - Ge^{-C_1\tau}}{1 - G}\right)\right]$.

The characteristic function (6) leads to analytic formulas for the valuation of GMMB guarantees under the volatility-dependent fee.

2.2. Option Pricing in GMMB

In model (5), the dynamic of the fund value F_t in a guaranteed minimum maturity benefit depends on the market volatility and the state-dependent fee structure adopted by a VA policymaker. Given a deterministic maturity guarantee G_T, the payoff of the GMMB at maturity T can be written as

$$\max(F_T, G_T) = (F_T - G_T)^+ + G_T = F_T + (G_T - F_T)^+, \tag{7}$$

from which we can observe that the payoff of the GMMB can be represented either as a sum of a European call and the guaranteed amount or a sum of a European put and the fund value. Based on the above decomposition and the Fourier transform technique utilized by Heston (1993), in Proposition 1, we present a pricing formula for a GMMB.

Proposition 1. *In the Heston model, at time $t < T$, the price of a GMMB with the maturity payoff $\max(F_T, G_T)$ and the volatility-dependent fee structure in (2) has the form of*

$$P_t = C^*(t, T, F_t, v_t; a, b) + e^{-r(T-t)}G_T = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}[F_T(a, b)] + P^*(t, T, F_t, v_t; a, b), \tag{8}$$

where the respective time-t call and put option prices are given by

$$C^*(t, T, F_t, v_t; a, b) := F_t\mathcal{P}_1(t) - e^{-r(T-t)}G_T\mathcal{P}_2(t). \tag{9}$$

and

$$P^*(t,T,F_t,v_t;a,b) := e^{-r(T-t)}G_T\mathcal{P}_3(t) - F_t\mathcal{P}_4(t). \tag{10}$$

In (9), the probability functions

$$\begin{aligned}
\mathcal{P}_1(t) &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-iuk}\phi_{VA}(u-i,t;a,b)}{iu\phi_{VA}(-i,t;a,b)}\right]du, \\
\mathcal{P}_2(t) &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-iuk}\phi_{VA}(u,t;a,b)}{iu}\right]du, \\
\mathcal{P}_3(t) &= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-iuk}\phi_{VA}(u,t;a,b)}{iu}\right]du,
\end{aligned}$$

and

$$\mathcal{P}_4(t) = \phi_{VA}(-i,t;a,b)\left[\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-iuk}\phi_{VA}(u-i,t;a,b)}{iu\phi_{VA}(-i,t;a,b)}\right]du\right],$$

where ϕ_{VA} is the characteristic function given in (6). $\Re$ denotes the real part of a function. In (8), we define $\mathbb{E}_t^{\mathbb{Q}}$ as the expectation evaluated at time $t \in [0,T]$ under $\mathbb{Q}$. Then, the time-t fund value can be calculated as

$$e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}[F_T(a,b)] = \int_{-\infty}^{\infty} g_T(x)p_T(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iu(\ln s)}\phi_{VA}(u,t;a,b)duds. \tag{11}$$

In (11), $g_T(x) = g(\ln F_T(a,b)) = F_T(a,b)$. The inverse Fourier transform, $q_T(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\phi_{VA}(u,0;a,b)du$, is the probability density function of the log-price of the underlying fund $x_T = \ln F_T(a,b)$. In Proposition 1, we compute the respective prices of the embedded call and put options for a GMMB in (8). In addition, the corresponding put option price of $(G_T - F_T)^+$ can be alternatively derived using the put–call parity. It should be noted that the model hypotheses and pricing results presented in this section are consistent with the ones in Cui et al. (2017), where the authors derive the put option price as in (10).

2.3. Fee Rate Determination in GMMB

In what follows, we describe how VA insurers can determine the structure of a volatility-dependent fee (2) by specifying the levels of a and b. The insurer's objective is to minimize the expected shortfall of the difference between a GMMB liability and the fees to be collected throughout the lifetime of the contract.

Unlike standard exchange traded options, most insurance companies charge for the downside protection by deducting an ongoing fraction of invested assets instead of an upfront fee (Milevsky and Salisbury 2006). By (7), the insurer's risk is in the form of a put option, and the related potential liability is paid by the fees. Such a liability can be viewed as the insurer's residual risk that is paid by the fees financed from the fund throughout the lifetime of the contract:

$$P^*(T,T,F_T,v_T;a,b) \sim \mathrm{fee}(0,T,F_{0,T},v_{0,T};T), \tag{12}$$

which, with a proper selection of a and b, suggests that the expectation of their difference

$$\mathbb{E}[P^*(T,T,F_T,v_T;a,b) - \mathrm{fee}(0,T,F_{0,T},v_{0,T};T)] \longrightarrow 0.$$

In (12), $P^*(T,T,F_T,v_T;a,b) = (G_T - F_T)^+$ denotes the payoff of the GMMB put liability at maturity T. In (12), $\mathrm{fee}(0,T,F_{0,T},v_{0,T};T)$ denotes the accumulated value at time T of the fees collected over the entire period $[0,T]$, in which $F_{0,T}$ and $v_{0,T}$ are the paths of the fund and the variance over the time period $[0,T]$, respectively. The symbol "$\sim$" suggests a high degree of closeness between the amount of state-dependent fees, $\text{fee}(0, T, F_{0,T}, v_{0,T}; T)$, and the put payoff $P^*(T, T, F_T, v_T; a, b)$ at T.

The $\text{fee}(0, T, F_{0,T}, v_{0,T}; T)$ in (12) is collected for the purpose of covering the costs of the GMMB liability $P^*(T, F_T, v_T; a, b)$ at maturity T. Any mismatch between the cost of the put option liability and the fees leads to either overcharging the policyholder or increasing the hedging difficulty of the issuer when the fees are insufficient. Under the assumed form of the fee rate, $c_t = a + bv_t$, for any $\Delta \in (0, T)$, the time-Δ value of expected fees collected from Δ to time T can be represented as

$$\xi(\Delta, T, F_\Delta, v_\Delta; \Delta) := \mathbb{E}^{\mathbb{Q}}\left[\int_\Delta^T e^{-r(s-\Delta)}(a + bv_s)F_s ds | \mathcal{F}_\Delta\right]. \tag{13}$$

The entire fees collected from 0 to T, $\text{fee}(0, T, F_{0,T}, v_{0,T}; T)$, can be decomposed as

$$\text{fee}(0, \Delta, F_{0,\Delta}, v_{0,\Delta}; T) + \text{fee}(\Delta, T, F_{\Delta,T}, v_{\Delta,T}; T),$$

where $\text{fee}(0, \Delta, F_{0,\Delta}, v_{0,\Delta}; T)$ denotes the collected fees from time 0 to Δ and accumulated at time T at the risk-free rate r. Similarly, $\text{fee}(\Delta, T, F_{\Delta,T}, v_{\Delta,T}; T)$ denotes the fees financed from time Δ to T and accumulated to T. These fees can be represented as follows:

$$\text{fee}(0, \Delta, F_{0,\Delta}, v_{0,\Delta}; T) := \int_0^\Delta e^{r(T-s)}(a + bv_s)F_s ds, \tag{14}$$

and

$$\text{fee}(\Delta, T, F_{\Delta,T}, v_{\Delta,T}; T) := \int_\Delta^T e^{r(T-s)}(a + bv_s)F_s ds. \tag{15}$$

We need the above decomposition of $\text{fee}(0, T, F_{0,T}, v_{0,T}; T)$ for presenting the risk related to writing a GMMB contract with the state-dependent fees at any Δ, $\Delta \in [0, T]$. Later, we will use this when formulating a criterion for the fee rate determination.

Both the expected fees defined in (13) and the corresponding costs of the put liability in a GMMB depend on time, the fund value, and the market volatility. At time $\Delta \in [0, T]$, we are interested in the risk related to writing a GMMB contract, measured by $d(\Delta; a, b)$ as a difference between the price of put liability and the expected fees. According to the representation below, $d(\Delta; a, b)$ measures the degree of the expected fees collected for covering the cost of put option, or the expected loss at time Δ to the insurer. We thus can formulate some criterion on $d(\Delta; a, b)$ to determine the volatility-dependent fee structure. With the arguments of a and b, the expected loss $d(\Delta; a, b)$ is defined in the following way:

$$\begin{aligned}
& d(\Delta; a, b) := e^{-r(T-\Delta)}\mathbb{E}_\Delta^{\mathbb{Q}}[P^*(T, T, F_T, v_T; a, b) - \text{fee}(0, T, F_{0,T}, v_{0,T}; T)] \\
= \; & e^{-r(T-\Delta)}\mathbb{E}_\Delta^{\mathbb{Q}}\left[P^*(T, T, F_T, v_T; a, b) - \underbrace{\text{fee}(0, \Delta, F_{0,\Delta}, v_{0,\Delta}; T)}_{\text{known at } \Delta \text{ and realized under } \mathbb{P}} - \text{fee}(\Delta, T, F_{\Delta,T}, v_{\Delta,T}; T)\right] \\
= \; & P^*(\Delta, T, F_\Delta, v_\Delta; a, b) - \underbrace{\xi(\Delta, T, F_\Delta, v_\Delta; \Delta)}_{\text{expected fee from } \Delta \text{ to } T \text{ in money at } \Delta} - \underbrace{\text{fee}(0, \Delta, F_{0,\Delta}, v_{0,\Delta}; \Delta)}_{\text{simulated from the past market data under } \mathbb{P}} && (16) \\
= \; & C^*(\Delta, T, F_\Delta, v_\Delta; a, b) + G_T e^{-r(T-\Delta)} - F_\Delta - \text{fee}(0, \Delta, F_{0,\Delta}, v_{0,\Delta}; \Delta). && (17)
\end{aligned}$$

From (16), the insurer may assess the expected loss $d(\Delta; a, b)$ at future time Δ, at which he/she has received a varying level of the fees, $\text{fee}(0, \Delta, F_{0,\Delta}, v_{0,\Delta}; \Delta)$, depending on the past market experience during the period $[0, \Delta]$. In consequence, among the feasible pairs of (a, b) constrained by the initial budget the insurer could select the most favorable rate that leads to an optimization of the expected loss $d(\Delta; a, b)$, given the fees that have been realized with the past market performance during the period $[0, \Delta]$. We thus obtain the distribution of $d(\Delta; a, b)$ by simulating the paths of the underlying fund $F_{0,\Delta}$ and the variance $v_{0,\Delta}$ under the real-world probability measure $\mathbb{P}$. In addition, Equation (17) holds due to the put–call parity and the time–Δ call price $C^*(\Delta, T, F_\Delta, v_\Delta; a, b)$ is given in (9). Compared to (16), expression (17) reduces the computational burden without simulating the expected fees in (13), while $\text{fee}(0, \Delta, F_{0,\Delta}, v_{0,\Delta}; \Delta)$ has been realized from the single path of the fund jointly with the corresponding path of the variance by time $\Delta \in [0, T]$. In particular, at the inception of the contract, i.e., $\Delta = 0$, the expected loss-at-issue between the cost of put liability and the expected fees collected throughout the lifetime of the contract can be represented as

$$d(0; a, b) = P^*(0, T, F_0, v_0; a, b) - \xi(0, T, F_0, v_0; 0) = C^*(0, T, F_0, v_0; a, b) + G_T e^{-rT} - F_0. \tag{18}$$

For business operations, a hedging duration could be set on a fixed basis within the hedger's budget period (see reference in Friedman and DeCorla-Souza (2012)). Suppose that an insurer performs a hedging strategy on VA contracts within the budget period $[0, \Delta]$. To reduce the hedging difficulty, the insurer selects an optimal fee structure to meet his business target during the pricing stage. As mentioned in Wang and Zou (2021), such a criterion for the fee-rate determination can be subjective and the difference between expected "fee revenue" and "cost of liability" is maximized over the lifetime of the contract to compare a variety of state-dependent fee structures for policyholders with a mean-variance type preference. In this paper, we consider the alternative identification of the optimal fee structure by minimizing the hedging risk at or prior to the end of the budget period.

For instance, at time $\Delta \in [0, T]$, we want to determine the constants a and b in $c_t = a + bv_t$ by minimizing the hedging risk quantified by the expected shortfall $\mathbb{E}[d^+(\Delta; a, b)]$, where $d^+(\Delta; a, b) := \max(d(\Delta; a, b), 0)$. By the definition of $d(\Delta; a, b)$ in (16), this ensures that the issuers can optimally cover the ongoing GMMB liability by expecting a fair amount of ongoing fees at a fixed Δ, $\Delta \in [0, T]$. The optimization problem is governed by the global constraint $d(0; a, b) = 0$ and that leads to the following optimization problem:

$$(a^*, b^*) = \arg\min_{a,b} \mathbb{E}_\Delta^{\mathbb{Q}}\left[d^+(\Delta; a, b)\right] \tag{19}$$

with a nonlinear constraint

$$d(0; a, b) = 0. \tag{20}$$

It is important to note that the optimization in (19) is performed under the risk-neutral measure $\mathbb{Q}$, where the expectation operator $\mathbb{E}_\Delta^{\mathbb{Q}}$ on the shortfall risk of the expected loss $d^+(\Delta; a, b)$, as indicated in (16), is conditional on the past path of the fees, $\text{fee}(0, \Delta, F_{0,\Delta}, v_{0,\Delta}; \Delta)$, realized under $\mathbb{P}$. In the literature, such an expected shortfall is a coherent risk measure that has desirable theoretical properties (see, for example, Artzner et al. (1999) and Acerbi et al. (2001)).

We should also mention that the condition (20) suggests an exact matching or a zero expected loss-at-issue between the cost of GMMB liability and the fee expenses at origination of the contract. Thus, the expected present value of the fee expenses deducted from the VA fund can be represented in the following way:

$$\begin{aligned} P^*(0, T, F_0, v_0; a^*, b^*) &= \xi(0, T, F_0, v_0; 0) = F_0 - e^{-rT}\mathbb{E}^{\mathbb{Q}}[F_T(a^*, b^*)] \\ &= F_0 - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iu(\ln s)}\phi_{VA}(u, 0; a^*, b^*)\,du\,ds, \end{aligned} \tag{21}$$

which, by (11), is evaluated under $\mathbb{Q}$ and amounts to the insurer's initial budget limit for hedging a lifelong GMMB.

An alternative objective of the insurer could be minimizing the shortfall risk $d^+(\Delta; a, b)$ over a period of time but not limited to a specific Δ. This leads to formulating the optimization problem on averaging the expected shortfalls in the sense that

$$(a^*, b^*) = \arg\min_{a,b} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_\Delta^{\mathbb{Q}}\left[d^+(\Delta_i; a, b)\right] \tag{22}$$

with a nonlinear constraint

$$d(0; a, b) = 0,$$

where $\Delta_i \in [0, T]$ for $i = 1, 2, \cdots, n$.

We next determine the optimal pair of (a^*, b^*) by solving problem (19). As explained in the previous paragraphs, the VA insurer could be faced with minimizing the expected shortfall at different time periods over the lifetime of the contract. In the following, we explain the numerical procedure that we have used to determine the optimal pair of a^* and b^* for each fixed Δ:

(P1) Using the Euler scheme, simulate a path of the fund value F_t jointly with the variance process v_t, for $t \in [0, T]$ by discretizing the time with N intervals under $\mathbb{P}$. We fix the set of random numbers for simulating both the underlying fund and the variance paths under volatility-dependent fees with different levels of a and b.

(P2) For a fixed $\Delta \in [0, T]$, determine $d(\Delta; a, b)$ using the path of the fund value jointly with the path of the corresponding variance from the previous step. We approximate fee $(0, \Delta, F_{0,\Delta}, v_{0,\Delta}; \Delta)$ by integrating along the fund value and variance paths and evaluating the call option price with the current conditions at Δ according to (15).

(P3) Repeat Steps (P1)–(P2) M times and find $\mathbb{E}^{\mathbb{Q}}_{\Delta}[d^+(\Delta; a, b)]$ based on M numbers of $d(\Delta; a, b)$, as well as the Value-at-Risk (VaR) of $d(\Delta; a, b)$, which measures the profitability of the GMMB assessed at time Δ.

(P4) Use the Matlab "fmincon" (function-minimization-with-constraint) function to minimize $\mathbb{E}^{\mathbb{Q}}_{\Delta}[d^+(\Delta; a, b)]$ with the global constraint $d(0; a, b) = 0$ and thus obtain the optimal pair (a^*, b^*) at Δ.

(P5) Given the optimal pair of (a^*, b^*) at Δ, compute the corresponding $\mathbb{E}^{\mathbb{Q}}_{\Delta}[d^+(\Delta; a, b)]$ and the Value-at-Risk of $d(\Delta; a^*, b^*)$ based on M numbers of $d(\Delta; a^*, b^*)$.

We should emphasize that a selection for the rates of a and b based on a certain optimal criterion is at the insurer's discretion. Given the market parameters used in Section 4.2, we illustrate with an example and mainly focus on the assumption that the insurer wants to optimize his/her business objective with an optimal pair of a^* and b^*, by minimizing the expected shortfall $\mathbb{E}^{\mathbb{Q}}_{\Delta}[d^+(\Delta; a, b)]$ at a future time Δ. In Section 4.2, a similar optimization procedure will also be implemented to identify the optimal rates of a^* and b^* under the proposed criterion (22).

3. Efficient Hedging of Path-Dependent Options under the Heston Model

Based on the fee rate selected by some optimization criterion discussed in Section 2.3, in this section we determine an optimal hedging strategy for a GMMB contract with volatility-dependent fees under the assumption that the market index S follows the Heston model with stochastic volatility. We should note that the two optimization problems formulated in Sections 2.3 and 3 are independent: one is at the stage of creating the GMMB product (its fee structure), and the second is related to the hedging of this given product. In Section 3.1, we formulate the optimal hedging problem and present its solution in Section 3.2.

3.1. Static Hedging of a GMMB Contract

Due to the path-dependent nature of the contract, the insurer's risk at a future hedging time $T_h \leq T$, given by $(G_{T_h} - F_{T_h})^+$, can be decomposed in terms of the risk due to the uncertain terminal value of the equity index S_{T_h} and risk due to the uncertain shapes of the paths of the index and its volatility conditional on $S_{T_h} = s$. The latter can be represented as

$$L(s) := \left(G_{T_h} - F^s_{T_h}\right)^+, s \in \mathcal{R}^+, \tag{23}$$

with

$$F^s_{T_h} = s \cdot \exp\left(-\int_0^{T_h} c_s ds\right) = s \cdot \exp\left[-\left(aT_h + b\int_0^{T_h} v^s_t dt\right)\right], \tag{24}$$

where the integrated variance $\int_0^{T_h} v_t^s dt$ is conditional on $S_{T_h} = s$. We use the symbol v_t^s to account for the variance processes that evolves with the fixed terminal value of the underlying $S_{T_h} = s$. $F^s_{T_h}$ denotes the terminal fund value F_{T_h} conditional on the fixed underlying $S_{T_h} = s$, which jointly evolves with the variance path v_t^s in $[0, T_h]$.

The presence of the stochastic volatility of the equity index thus makes the market incomplete. The liability $L(S_{T_h})$ is path dependent because it is jointly linked with the index S and the path of the equity volatility, or equivalently, the path of the equity variance $v_{0,T}$ in $[0, T]$. Our objective is to hedge the path-dependent liability $L(S_{T_h})$ by using only properly selected path-independent European-style options on the equity index at maturity, S_{T_h}. Since the option is selected at time zero and held until maturity, the hedging is static in nature.

To determine an optimal hedge, the insurer could be interested in minimizing a particular loss function, such as the mean-square value of the difference between the overall liability $L(S_{T_h})$ and the payoff of hedging portfolio h, in the sense that

$$h_{ms} := \arg \inf_{h \in \mathcal{L}^2\left(S_{T_h}\right)} \mathbb{E}^{\mathbb{P}}\left[\left(L(S_{T_h}) - h(S_{T_h})\right)^2\right], \tag{25}$$

with $\mathcal{L}^2\left(S_{T_h}\right)$ denoting the set of measurable and square integrable functions of S_{T_h}. It is well known that the solution to (25) can be represented explicitly as

$$h_{ms}(s) := \mathbb{E}^{\mathbb{P}}\left[L(S_{T_h}) | S_{T_h} = s\right], \tag{26}$$

indicating that h_{ms} is mean-self financing, and thus the cost of setting it at time zero is the same as the cost of a put option liability (21). For more details about this strategy and its properties, we refer to Kolkiewicz and Liu (2012).

For the criterion of mean-square error, however, gains and losses are treated symmetrically and the insurer's budget constraint is ignored. From the perspective of risk management, a more natural approach is the one where we consider only the insurer's loss, in which case the payoff function h_{opt} of the optimal hedge solves the problem

$$h_{opt} = \arg \inf_h \mathbb{E}^{\mathbb{P}}\left[l_p\left(L(S_{T_h}) - h(S_{T_h})\right)\right], \tag{27}$$

where $l_p(x) = (x^+)^p$, $p \geq 1$, is a weight function. For $p = 1$, problem (27) corresponds to the minimization of the expected shortfall, but, in the general case, we minimize the expected shortfall weighted with the function l_p. The optimal payoff h_{opt} is subject to the constraint that the expected value of the payment $h_{opt}(S_{T_h})$ does not exceed the given budget V_0 at time T_h:

$$\mathbb{E}^{\mathbb{Q}}\left[h_{opt}\left(S_{T_h}\right)\right] \leq V_0. \tag{28}$$

3.2. Construction of Optimal Hedging Option

Here, we present a solution to the optimization problem (27) and (28). The approach we take to determine this solution follows the same steps as in Kolkiewicz (2016); however, in the current problem we assume that the volatility is stochastic and such that its dynamic can be described by the Heston model.

To define the set of admissible functions h, we first define the following set:

$$\mathcal{H}^0 := \{\text{functions } h \text{ on } \mathcal{R}^+ \text{ such that } h(s) \in \overline{\text{supp}(L(s))} \text{ for } s \in \mathcal{R}^+\},$$

with $\overline{A}$ denoting the closure of a set A and supp() denoting the support of a random variable. For a given initial capital V_0, we characterize admissible hedging options by defining the following set of admissible functions h

$$\mathcal{H} := \{h \in \mathcal{H}^0 : \mathbb{E}^{\mathbb{Q}}\left[h(S_{T_h})\right] \leq V_0\},$$

and the hedging strategy is bounded in the following sense:

$$\mathcal{H}_{h_L,h_U} := \{h \in \mathcal{H} : h_L(s) \le h(s) \le h_U(s) \text{ for } s \in \mathcal{R}^+\},$$

where h_L and h_U are exogenous functions that allow us to represent the problem of finding an optimal option as an optimization problem over functions that take values in a bounded interval.

Because of the general budget requirement such that $\mathbb{E}^{\mathbb{Q}}[h_L(S_{T_h})] \le V_0 \le \mathbb{E}^{\mathbb{Q}}[h_U(S_{T_h})] < \infty$, the selection of the respective lower and upper bounds of h_L and h_U does not form binding constraints in the optimization problem and hence it has no impact on the optimal h_{opt}. On the practical level, such a pair of h_L and h_U can be selected for GMMBs in mutiple ways. For instance, the expression of the put option liability (23) suggests that the fund value F_{T_h} charged with volatility-dependent fees is no greater than the equity index $S_{T_h} = s$. Thus, we can take the lower bound $h_L = \left(G_{T_h} - s\right)^+$ as the payoff of a standard European put option on the tradeable index. Meanwhile, the upper bound h_U can be set to the guarantee level of G_T, at which the put liability is capped for all $s \in \mathcal{R}^+$. Or, we can simply set two constants $h_L = 0$ and $h_U = G_T$ to ensure that the hedging strategy h is bounded over the support of the liability risk $L(S_{T_h})$. In consequence of either way, the bandwidth, $h_U - h_L$, is independent of the variability of $L(S_{T_h})$ caused by the integrated variance $\int_0^{T_h} v_t^s dt$. An alternative way to set the levels of h_L and h_U is based on the distributional quantiles of the overall liability $L(S_{T_h})$ so then they depend on the volatility risk. Conditionally on $S_{T_h} = s$, the variability of F_T^s and hence $L(s)$ is due only to the volatility-dependent fees written in (24) as a linear function of an integral of volatility. As a result, the level of the bandwidth, $h_U - h_L$, is dependent on the path of the integrated variance, which reflects the impact of charging volatility-dependent fees on the hedging risk (see the numerical results in Section 4.2).

For $\alpha_H > \alpha_L$, we take $h_U(s) = q^L_{\alpha_H}(s)$ and $h_L(s) = q^L_{\alpha_L}(s)$, in which $q^L_{\alpha_H}(s)$ and $q^L_{\alpha_L}(s)$ denote the respective α_H and α_L-quantiles of the residual risk $L(s)$. Under the Heston framework, we have the following characterization of the optimal static hedging option in Theorem 1, by extending the theoretical result of Kolkiewicz (2016) beyond the Black–Scholes model. The proof of the result stated in Theorem 1 is similar to the one in Kolkiewicz (2016) and, therefore, only the main steps of the proof, together with additional explanations that are pertinent to the stochastic volatility model, are presented in Appendix A.

Theorem 1. *For a GMMB liability whose price of the underlying S follows a Heston model, the solution to the optimization problem in* (27) *can be represented in the form*

$$h_{opt}(s) = h_L(s) + \widetilde{\gamma}(s)[h_U(s) - h_L(s)], \tag{29}$$

where

$$\widetilde{\gamma}(s) = 1 - l_e\left(s, c \cdot (h_U(s) - h_L(s))^{1-p} \frac{d\mathbb{Q}}{d\mathbb{P}}(s)\right) \tag{30}$$

with a continuous Radon–Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$. With its distribution of the liability F_L on $S_{T_h} = s$, the inverse function l_e takes the value in $[0,1]$, which is given by

$$l_e(s,y) = \frac{h_U(s) - F_L^{-1}(1-y)}{h_U(s) - h_L(s)} 1_{[1-\alpha_H, 1-\alpha_L]}(y) + 1_{(1-\alpha_L,\infty)}(y),\ y \in \mathcal{R}. \tag{31}$$

Then, a unique c is selected based on the budget constraint

$$\mathbb{E}^{\mathbb{Q}}\left[h_{opt}(S_{T_h})\right] = V_0. \tag{32}$$

In Theorem 1, the Radon–Nikodym derivative, $d\mathbb{Q}/d\mathbb{P}$, is determined in the Heston model. The above theorem shows that in our problem the optimal hedging option h_{opt} is a continuous function for all loss functions with $p \ge 1$. This smoothness property of h_{opt}

implies that we would be able to approximate it closely on bounded intervals by piecewise linear functions, which can be represented in the form

$$\beta_0 s + \beta_1 + \sum_{i=1}^{m} \alpha_i (\gamma_i - s)^+,$$

for suitably chosen constants β_0, β_1, α_i and γ_i, $i = 1, \cdots, m$. In other words, we will be able to approximate closely the optimal static hedging strategy for a GMMB liability if there are sufficiently many vanilla options on the market.

4. Implementation of the Method and Numerical Examples

In this section, a framework that admits the Heston-type stochastic volatility is considered for hedging the GMMB contract with volatility-dependent fees. It extends the results of hedging path-dependent options from the Black–Scholes framework to the Heston model in the spirit of Kolkiewicz (2016). In Section 4.1, we characterize an exact simulation based on the method of Broadie and Kaya (2006) for constructing the conditional residual risk described in (23). Then, we describe a numerical procedure to implement the construction of the optimal hedging strategy using Theorem 1. Numerical examples are presented in Section 4.2.

In practice, one challenge to implement the theoretical result in Theorem 1 is how to find the distribution of conditional residual risk $L(s)$ that we have defined in (23). In the Black–Scholes setting, a Brownian bridge technique can be developed so that it is possible to obtain an analytical form of the optimal static hedge for the path-dependent options or simulate the distribution of $L(s)$ with the Brownian bridge technique, conditional on a fixed benchmark value, i.e., $S_{T_h} = s$ (see Kolkiewicz (2016), for example). In the Heston framework, the bridge techniques for approximating the distribution of $L(s)$ can be hardly achieved due to the complexity of joint distribution between the equity process and its stochastic volatility, and thus the simulation method is expected to be applied. Another challenge for constructing the optimal hedging strategy in the stochastic volatility setting is to find the Radon–Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$ that reconciles the distributions of the equity index S under both the physical and risk-neutral measures, which is desirable for determining the optimal ratio $\widetilde{\gamma}$ by the Neyman–Pearson lemma with the Heston model.

4.1. Sampling Method and Numerical Procedures

The objective of this section is to address the above challenges and then illustrate Theorem 1 with some numerical examples. To this end, in this section we characterize an exact simulation method based on the approach formulated by Broadie and Kaya (2006) for constructing the conditional residual risk described in (23). We also describe a numerical procedure to approximate the optimal hedging strategy stated in Theorem 1.

Finding the optimal hedging option characterized in Theorem 1 requires two main components: the residual risks $L(s), s \in R^+$, and the derivative $d\mathbb{Q}/d\mathbb{P}$. To find the residual risks we use the following method:

(1) By using the method of Broadie and Kaya (2006), we simulate values from the joint distribution of S_{T_h} and $\int_0^{T_h} v_s ds$.

(2) We group the simulated pairs by partitioning the range of S_{T_h} into a certain number of bins. For instance, we take sufficient numbers of terminal underlying S_{T_h} located in a small interval $[s, s+\delta)$, $\delta \to 0$. and jointly collect the corresponding conditional integrated variances $\int_0^{T_h} v_t^s dt$ for $S_{T_h} \in [s, s+\delta)$.

(3) For each bin, we estimate $F^s_{T_h}$, and hence the liability $L(s)$, by using (24).

We should note that the sampling method of Broadie and Kaya (2006) not only enables us to characterize the path dependency of the overall liability $L(S_{T_h})$ driven by the integrated variance and the underlying equity index, but also it works out a theoretically exact simulation for computing efficiency and accuracy of the modeling.

To find the Radon–Nikodym derivative, we use the work of Drăgulescu and Yakovenko (2002), where the authors derived an analytical solution for the probability density function of equity index returns based on the Heston model. To be specific, by setting $y_t = \ln(S_t/S_0) - \mu t$, the Heston-type stochastic volatility can be represented in terms of the centered log-return y_t and the variance v_t such that $dy_t = -\frac{v_t}{2}dt + \sqrt{v_t}d\widetilde{W}_t^x$. In consequence, the time-dependent transition probability $P_t[y, v|y_0 = 0, v_0]$ evolves with the Fokker-Planck equation:

$$\frac{\partial}{\partial t}P = \kappa^* \frac{\partial}{\partial v}[(v - \bar{v}^*)P] + \frac{1}{2}\frac{\partial}{\partial y}(vP) + \rho\sigma\frac{\partial^2}{\partial y \partial v}(vP) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(vP) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial v^2}(vP),$$

which, by time t, yields the probability density function of centered returns y (D–Y formula) given by

$$f_Y(y) := P_t(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[i\xi y + F_t(\xi)]d\xi, \tag{33}$$

where

$$F_t(\xi) = \frac{\kappa^*\bar{v}^*}{\sigma^2}\Gamma t - \frac{2\kappa^*\bar{v}^*}{\sigma^2}\ln\left[\cosh\frac{\Omega t}{2} + \frac{\Omega^2 - \Gamma^2 + 2\kappa^*\Gamma}{2\kappa^*\Omega}\sinh\frac{\Omega t}{2}\right],$$

with $\Gamma = \kappa^* + i\rho\sigma\xi$ and $\Omega = \left[\Gamma^2 + \sigma^2\left(\xi^2 - i\xi\right)\right]^{1/2}$.

Denote by f and g the respective probability density functions for the centered return y and the underlying equity index S. By the change in variable, we obtain $g_S(s) = f_Y(y(s))/s$. Under the Heston framework, we thus express the continuous Radon–Nikodym derivative by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(s) = \frac{g_S^{\mathbb{Q}}(s)}{g_S^{\mathbb{P}}(s)} = \frac{f_Y^{\mathbb{Q}}(y(s))}{f_Y^{\mathbb{P}}(y(s))}, \text{ for } s \in \mathcal{R}^+.$$

In the following, we present the numerical procedure on how to determine the optimal hedging option h_{opt} numerically for the GMMB liability defined in (23):

(S1) Determine the optimal pair of (a^*, b^*) constrained by the criterion (19) or (22). For the selected pair of (a^*, b^*) determined in the pricing stage of a GMMB contract, calculate the price of a GMMB liability as the initial hedging budget by (21) and that $V_0 = P^*(0, T_h, F_0, v_0; a^*, b^*)e^{rT_h}$ at the hedging time T_h.

(S2) Use the exact simulation of Broadie and Kaya (2006) to simulate sufficient sets of S_{T_h}, $\int_0^{T_h} v_s ds$ and F_{T_h}. Then, take sufficient numbers of terminal underlying S_{T_h} in $[s, s+\delta)$ and collect the corresponding conditional integrated variances $\int_0^{T_h} v_t^s dt$ for $S_{T_h} \in [s, s+\delta)$. Finally, obtain the corresponding numbers of $F_{T_h}^s$ under the fee rate structure of $c_t = a + bv_t$ by (24) for the approximate distribution of $L(s) = \left(G_{T_h} - F_{T_h}^s\right)^+$ in $[s, s+\delta)$, $\delta \to 0$. The mean-square hedging option h_{ms} can be approximated using the following unbiased and asymptotically consistent sample means

$$h_{ms}(s) \approx \frac{1}{M}\sum_{i=1}^{M}\left(G_{T_h} - F_{T_h}^s\right)^+.$$

(S3) For selected values α_L and α_H from the interval $(0, 1)$ such that $\alpha_L < \alpha_H$, find the corresponding empirical quantiles of the distributions of $L(s)$ at each $S_{T_h} = s$. Then, obtain the lower and upper bounding functions h_L and h_U as the corresponding quantiles of $L(s)$, $s \in \mathcal{R}^+$.

(S4) To determine h_{opt} in Theorem 1, we need to approximate $l_e(s, y)$ and $\widetilde{\gamma}(s)$. By constructing the mesh of points

$$\mathcal{M} := \{(s_m(i), z_m(j)) : s_m(i) \;=\; s_{\min} + i\frac{\Delta_s}{K_s}, i = 0, \cdots, K_s$$
$$z_m(j) \;=\; -h_U(s_m(i)) + j\frac{\Delta_z(i)}{K_z}, j = 0, \cdots, K_z\},$$

where $\Delta_s := s_{\max} - s_{\min}$ and $\Delta_z(i) := h_U(s_m(i)) - h_L(s_m(i))$.

(i) Select $\hat{s}(l)$, $l = 1, \cdots, k$, in each small interval $(s_m(i-1), s_m(i)]$ for approximating the distribution at $s = s_m(i)$. Then, obtain a sufficient number of residual risks of $L(s_m(i)) = \left(G_{T_h} - F_{T_h}^{\hat{s}(l)}\right)^+ | S_T = s_m(i)$ in each small interval $(s_m(i-1), s_m(i)]$. From these points, use a kernel density estimator to find an estimate $\hat{s}_i^L$ of the density of $L(s_m(i))$, which will be used to evaluate the conditional expectation of $\mathbb{E}^{\mathbb{P}}\left[(L(s_m(i)) + z_m(j))^+ | S_{T_h} = s_m(i)\right]$ for each equally spaced mesh of points $z_m(j)$ from $\mathcal{M}$. This gives the approximation of $\hat{g}_0(s_m(i), z_m(j); 1)$. For the bounds $h_L(s_m(i))$ and $h_U(s_m(i))$, take the α_L and α_H-quantiles of the distribution of $L(s_m(i))$ for evaluating $g(s_m(i), z_m(j); p)$ over $\mathcal{M}$. Note that the functions $\hat{g}_0$ and g are both defined in (5.6) from Kolkiewicz (2016).

(ii) Based on relation (5.7) in Kolkiewicz (2016), use a central finite difference to approximate $\hat{l}_e(s, y)$ in (31) by inverting the derivative of $\hat{g}$.

(iii) Use the bisection method to fit the value of c by the budget constraint in (32). $\hat{\tilde{\gamma}}(s)$ and $\hat{h}_{opt}(s_m(i))$ are then determined by all the elements obtained by the aforementioned steps with the Radon–Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$.

(S5) Repeat the above processes and get all $h_{opt}(s_m(i))$ for $i = 1, \cdots, K_s$.

4.2. Numerical Examples

In this section, we present a numerical result of Theorem 1 with the optimal rate of volatility-dependent fees determined by the pricing criteria in Section 2.3. We use the following base parameters: $S_0 = F_0 = G_T = G_{T_h} = 100$, $\rho = -0.3$, $\kappa^* = 3$, $v_0 = \bar{v}^* = 0.04$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.4$, $\lambda = -0.15$, $\alpha_L = 0.05$, $\alpha_H = 0.95$, $K_s = K_z = 150$ and $T = 30$. The parameter of the weight function p is set to be one for hedging the GMMB with a non-parametric expected shortfall risk.

Based on the Monte Carlo method with $M = 10^6$ repetitions and $N = 100$ intervals, in Table 1 we identify the optimal pairs of (a^*, b^*) with a T-year GMMB for the insurer who wants to minimize the shortfall risk $\mathbb{E}_{\Delta}^{\mathbb{Q}}[d^+(\Delta; a^*, b^*)]$ at a specific $\Delta \in [0, T]$, under the condition that the embedded put option liability is fairly priced at time zero. We also present the Value-at-Risk of the expected loss $d(\Delta; a^*, b^*)$, which, as indicated in Section 2.3, reflects the level of insurer's profitability at time Δ by offering the GMMB rider.

Table 1. Optimal pairs of (a^*, b^*) that solve the problem (19) with the base parameters.

$\boldsymbol{\Delta}$	**Optimal** (a^*, b^*)	$\mathbb{E}^{\mathbb{Q}}[d^+(\Delta; a^*, b^*)]$	$\mathbf{VaR}_{0.90}(d(\Delta; a^*, b^*))$	$\mathbf{VaR}_{0.95}(d(\Delta; a^*, b^*))$	$\mathbf{VaR}_{0.99}(d(\Delta; a^*, b^*))$
0.3	$(0, 0.6471)$	0.2822	0.7493	0.7755	0.9621
0.6	$(0, 0.6472)$	0.0967	0.3757	0.6283	1.1457
0.9	$(0.0206, 0.1283)$	0.0389	−0.0080	0.2863	0.8696
1.2	$(0.0214, 0.1093)$	0.0103	−0.5704	−0.2422	0.3841
1.5	$(0.0213, 0.1125)$	0.0018	−1.1967	−0.8605	−0.2170
1.8	$(0.0197, 0.1513)$	0.0002	−1.8782	−1.5261	−0.8785
2.1	$(0.0235, 0.0571)$	0	−2.7251	−2.3735	−1.7085
2.4	$(0.0169, 0.2220)$	0	−3.3442	−2.9519	−2.2616
2.7	$(0.0075, 0.4579)$	0	−3.3409	−2.9116	−1.9572
3.0	$(0.0041, 0.5448)$	0	−4.0911	−3.4009	−2.2567
3.3	$(0.0039, 0.5483)$	0	−4.9313	−4.1798	−2.8882

With an optimal pair of a^* and b^*, in Figure 1 we are interested in the distributional characteristics of $d(\Delta; a^*, b^*)$ over different levels of Δ, which evolves with the corresponding variance of the fund whose distributions are presented in Figure 2. In Figure 2, we observe that the distribution of the variance v_Δ appears to be stationary over the selected levels of Δ. We also discover from both Figure 1 and Table 1 that the expected shortfall and its corresponding VaRs on the loss-at-time Δ, $d(\Delta; a^*, b^*)$, turn smaller and the latter one becomes negative for a larger Δ, while the distribution of the underlying fund shifts to the right. This observation indicates that the insurer could suffer an expected deficit in the early years of the contract and then profit from the growth of his fee revenue in proportion to the fund value and a decrease in the cost of put option liability in the later years. Moreover, with a smaller optimal base rate a^* and the corresponding larger sensitivity parameter b^*, we find that a fair volatility-dependent fee rate can be more sensitive to the market volatility in the early or later years of the contract, while both parameters of optimal (a^*, b^*) are more stable for a certain period, i.e., when $\Delta \in [0.9, 1.8]$.

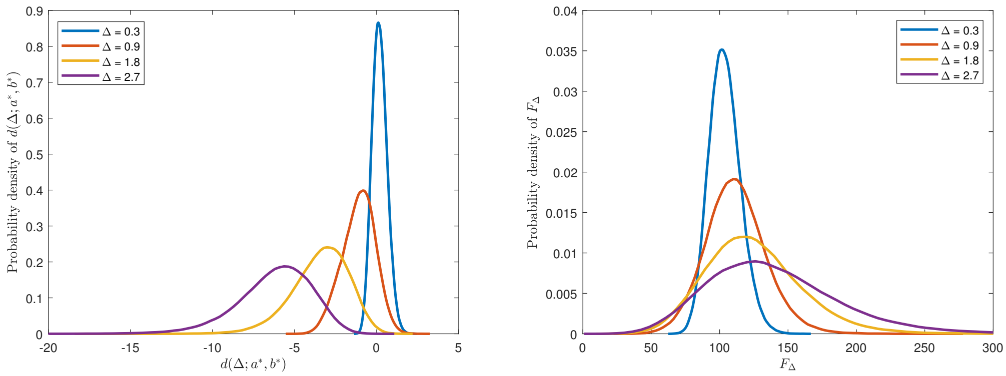

Figure 1. The respective probability densities of the expected loss $d(\Delta; a^*, b^*)$ (**left panel**) and the fund value F_Δ under $\mathbb{P}$ (**right panel**).

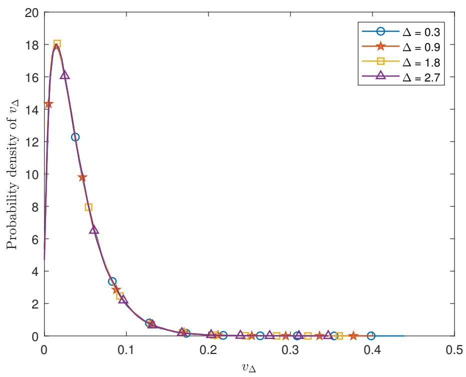

Figure 2. The probability densities of the time–Δ variance v_Δ measured under $\mathbb{P}$.

As an alternative criterion discussed in Section 2.3, during a period of time the insurer could consider minimizing average of the expected shortfalls, $\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_\Delta^{\mathbb{Q}}[d^+(\Delta_i; a, b)]$, as described in (22). For example, by a subjective choice of Δ_i over the time set $\mathcal{T} = \{0.3, 0.9, 1.8, 2.7\}$, we show the feasibility of such an approach with the optimal pair of $a^* = 0.0178$ and $b^* = 0.1993$, while the corresponding minimum of the expected average shortfall is equal to 0.3308.

For illustrative purposes, we select a pair of $a^* = 0.0075$ and $b^* = 0.4579$ satisfying condition (20), assuming that the insurer wants to minimize the shortfall risk at $\Delta = 2.7$. With this pre-selected fee structure of $c_t = 0.0075 + 0.4579 v_t$, the insurer decides to hedge the GMMB liability for a specific time interval. In the following analyses, we compare the hedging results between the proposed optimal static hedging strategy and the benchmark mean-square approach. In particular, we are interested in the hedging performance of the proposed strategy based on a variety of market conditions, such as equity return μ and the length of hedging period T_h chosen by the hedger. Correspondingly, Figures 3–6 display the resulting optimal hedging strategy for GMMB with volatility-dependent fees (left panels), which, by Theorem 1, is derived from the distributions of underlying equity index under both risk-neutral and physical probability measures (right panels). For illustrative purposes, we demonstrate the optimal hedging strategy in the range where the probability density of the underlying equity index is near and above zero. The following Scenarios 1–4 are considered in our analyses:

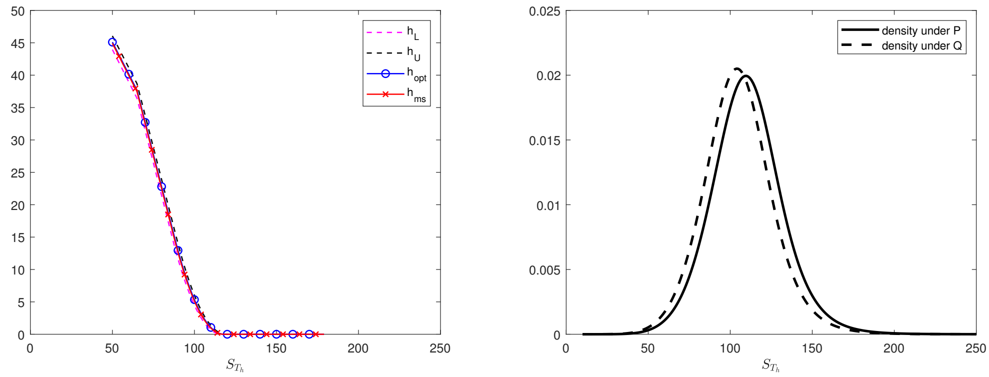

Figure 3. The optimal static hedging option conditional on $S_{T_h} = s$ (**left panel**) and the probability densities of the underlying S_{T_h} with the Heston-type stochastic volatility (**right panel**) for Scenario 1.

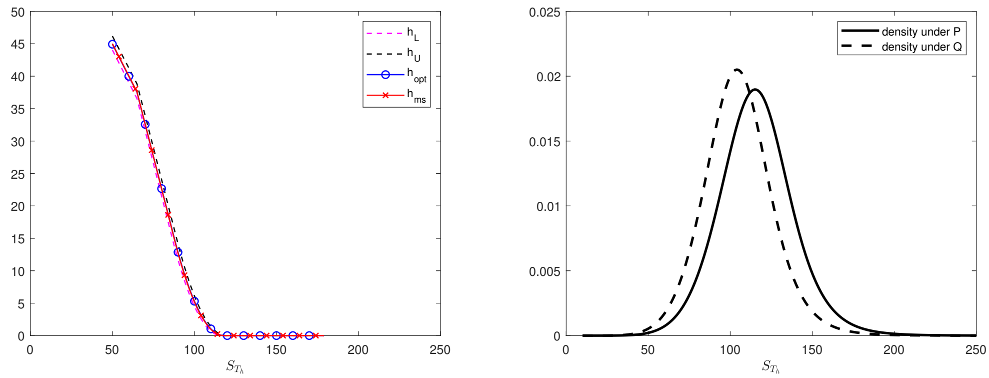

Figure 4. The optimal static hedging option conditional on $S_{T_h} = s$ (**left panel**) and the probability densities of the underlying S_{T_h} with the Heston-type stochastic volatility (**right panel**) for Scenario 2.

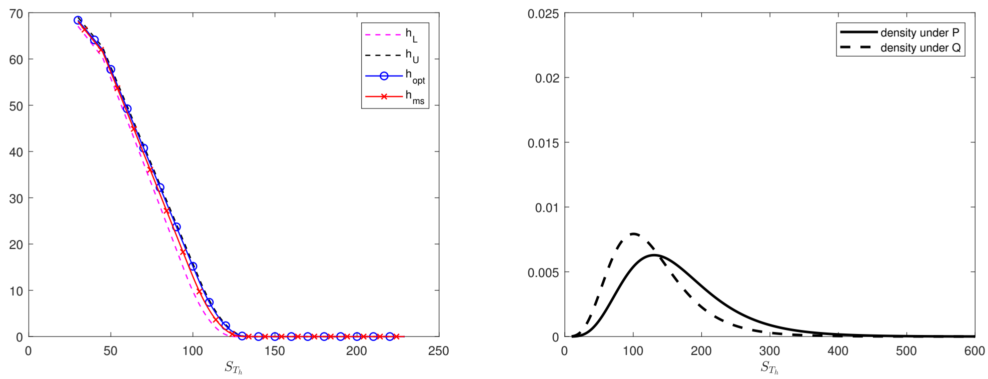

Figure 5. The optimal static hedging option conditional on $S_{T_h} = s$ (**left panel**) and the probability densities of the underlying S_{T_h} with the Heston-type stochastic volatility (**right panel**) for Scenario 3.

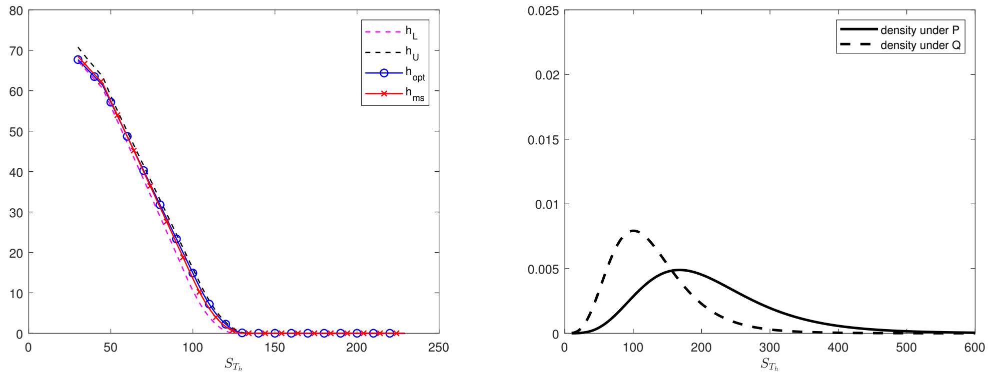

Figure 6. The optimal static hedging option conditional on $S_{T_h} = s$ (**left panel**) and the probability densities of the underlying S_{T_h} with the Heston-type stochastic volatility (**right panel**) for Scenario 4.

Scenario 1: when $\mu = 10\%$ and $T_h = 1$, the expected shortfalls of h_{opt} and h_{ms} are in the respective sizes of 0.0746 and 0.1178, while the corresponding standard deviations are 0.0890 and 0.1421, respectively, with $c = 0.2150$ and $V_0 = 2.1712$.
Scenario 2: when $\mu = 15\%$ and $T_h = 1$, the expected shortfalls of h_{opt} and h_{ms} are in the respective sizes of 0.1157 and 0.1215, while the corresponding standard deviations are 0.1396 and 0.1463, respectively, with $c = 0.2150$ and $V_0 = 2.1712$.
Scenario 3: when $\mu = 10\%$ and $T_h = 5$, the expected shortfalls of h_{opt} and h_{ms} are in the respective sizes of 0.0249 and 0.1168, while the corresponding standard deviations are 0.0535 and 0.2506, respectively, with $c = 0.0825$ and $V_0 = 15.6066$. Note that a longer hedging maturity $T > \Delta$ is conjectured for examining the hedging performance outside the hedger's initial budget period that was used to set the level of c_t in Section 2.3.
Scenario 4: when $\mu = 15\%$ and $T_h = 5$, the expected shortfalls of h_{opt} and h_{ms} are in the respective sizes of 0.1006 and 0.1138, while the corresponding standard deviations are 0.2681 and 0.2393, respectively, with $c = 0.0825$ and $V_0 = 15.6066$.

In our problem, the only source of path-dependency of the GMMB liability when it is conditional on a fixed equity index $S_{T_h} = s$ is the associated volatility-dependent fees charged with the path of variance during the period $[0, T_h]$. Given a level of variance $v_t \in [0, 0.09]$, the rate of $c_t = 0.0075 + 0.4579v_t$ changes between 0.75% and 4.8711% around its average level of $\bar{c} = 0.0075 + 0.4579\bar{v}^* = 2.5816\%$. Since the fee is charged in a relatively low amount at a fixed $S_{T_h} = s$, it leads to a narrower bandwidth in Figures 3 and 4 between $h_L(s)$ and $h_U(s)$ for a shorter hedging period $T_h = 1$. For a hedger who targets the shortfall risk at a larger maturity $T_h = 5$, we observe from Figures 5 and 6 that the bandwidth between h_L and h_U stays in a wider range, indicating that the varying level of volatility-dependent fees has a greater impact on the risk of the GMMB liability.

The numerical results show that the optimal hedging option h_{opt} outperforms its counterpart, the mean-square one h_{ms}. For instance, when $\mu = 0.10$ and $T_h = 1$, the expected shortfalls from h_{opt} and h_{ms} are 0.0746 and 0.1178, respectively, which is a reduction of 36.67% in shortfall risk. The standard deviation of the shortfall risk from h_{opt} is reduced by 37.36% when compared to h_{ms}. Depending on the model parameters, we find that the optimal hedging option can reduce the expected shortfall by an amount ranging from 4% to 46%. Due to its limited variation of the conditional liability caused by the volatility-dependent fees, in all scenarios the hedging strategies share a similar payoff to a standard European put option liability for GMMBs. On top of the fact that an optimal pair of a^* and b^* identified in the pricing stage can already reduce the potential expected shortfall based on a certain insurer's criterion, as noted in Section 3.2, a desirable replication of the hedging strategy by standard European options can be another reason for the relatively small size of the expected shortfall risks resulted from both h_{opt} and h_{ms} in Figures 3–6.

5. Concluding Remarks

The main objective of this paper is to extend a theoretical result that characterizes an optimal static hedging strategy for a path-dependent GMMB liability in the incomplete Heston market. In this paper, we demonstrate the feasibility of the optimal hedging method, which can be considered a benchmark when compared to other strategies with the introduction of stochastic volatility and/or the other more complex path-dependent feature embedded with VAs. We have discussed several aspects on the valuation, fee determination, and static hedging strategy for GMMB in a logical sequence, so that it is convenient for VA hedgers to practice the strategy with a volatility-dependent fee design. The resulting optimal static hedging strategy also enriches the existing literature about the hedging approach for VAs with state-dependent fees.

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Abbreviations

The following abbreviations are used in this manuscript:

VA	Variable Annuity
GMDB	Guaranteed Minimum Death Benefit
GMLB	Guaranteed Minimum Living Benefit
GMMB	Guaranteed Minimum Maturity Benefit
GMIB	Guaranteed Minimum Income Benefit
GMWB	Guaranteed Minimum Withdrawal Benefit
U.S.	United States
VIX	Volatility Index
CBOE	Chicago Board Options Exchange
D–Y	Drăgulescu–Yakovenko
VaR	Value-at-Risk

Appendix A. Proof of Theorem 1

The proof is similar to the one presented in Kolkiewicz (2016) in the context of hedging path-dependent options under the Black–Scholes framework, and here only main steps are shown in the stochastic volatility setting.

In the spirit of Föllmer and Leukert (2000), we want to characterize the optimization of the static hedging strategy as the solution based on the statistical optimal randomized test. For a function $\gamma(S_{T_h}) \in [0,1]$, the optimization problem of each admissible static hedging strategy, with its assumed form of $h = h_L + \gamma(h_U - h_L)$, is equivalent to determining the optimal ratio $\widetilde{\gamma}$ in the sense that

$$\begin{aligned}
\widetilde{\gamma}(S_{T_h}) &= \arg\min_{\gamma(S_{T_h})\in[0,1]} \mathbb{E}\left[\mathbb{E}\left[\left(\left(L(S_{T_h}) - h_L(S_{T_h}) - \gamma(S_{T_h})\left(h_U(S_{T_h}) - h_L(S_{T_h})\right)\right)^+\right)^p \middle| S_{T_h}\right]\right] \\
&= \arg\min_{\gamma(S_{T_h})\in[0,1]} \mathbb{E}^{\mathbb{P}}\left[\left(h_U(S_{T_h}) - h_L(S_{T_h})\right)^p g(S_{T_h}, 1-\gamma(S_{T_h}); p)\right],
\end{aligned} \tag{A1}$$

where the differentiable function $g(s,z;p)$ is convex and non-decreasing in $z \in [0,1]$, which is given by (4.2) in Kolkiewicz (2016).

For a a new probability measure $\widetilde{\mathbb{Q}}$ on $S_{T_h} = s \in \mathcal{R}^+$ defined by $d\widetilde{\mathbb{Q}} = const \cdot (h_u(s) - h_L(s))d\mathbb{Q}$, we select the bounds h_L and h_U such that $\mathbb{E}^{\mathbb{Q}}[h_L(S_{T_h})] \leq V_0$ and

$V_0 \le \mathbb{E}^{\mathbb{Q}}[h_U(S_{T_h})] < \infty$. This allows us to rewrite the budget constraint $\mathbb{E}^{\mathbb{Q}}\left[h(S_{T_h})\right] \le V_0$ for problem (A1) as

$$\mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\widetilde{\gamma}(S_{T_h})\right] \le \widetilde{H}_0 := \frac{V_0 - \mathbb{E}^{\mathbb{Q}}[h_L(S_{T_h})]}{\mathbb{E}^{\mathbb{Q}}[h_U(S_{T_h}) - h_L(S_{T_h})]} \in [0,1]. \tag{A2}$$

Using the method of Karlin (2003), Kolkiewicz (2016) derives the equivalent condition for $\widetilde{\gamma}$ to be optimal as

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}}\left[\left(h_U(S_{T_h}) - h_L(S_{T_h})\right)^p g_z\left(S_{T_h}, 1 - \widetilde{\gamma}(S_{T_h}); p\right)\widetilde{\gamma}(S_{T_h})\right] \\ \ge \quad & \mathbb{E}^{\mathbb{P}}\left[\left(h_U(S_{T_h}) - h_L(S_{T_h})\right)^p g_z\left(S_{T_h}, 1 - \widetilde{\gamma}(S_{T_h}); p\right)\gamma(S_{T_h})\right], \end{aligned} \tag{A3}$$

where $g_z(s,z;p)$ is the derivative of the function g in (A1), which is strictly increasing in $z \in [0,1]$. Under a new integrable measure $\widetilde{\mathbb{P}}$ defined by $d\widetilde{\mathbb{P}} = const \cdot (h_U(s) - h_L(s))^p \cdot g_z(s, 1-\widetilde{\gamma}(s); p)d\mathbb{P}$, the problem of finding the maximum of the right-hand side of (A3) with respect to γ, subject to the constraint (A2), can be recognized as looking for the most powerful test for the hypothesis $\widetilde{\mathbb{Q}}$ against the alternative $\widetilde{\mathbb{P}}$ at the level $\varphi := \widetilde{H}_0$, where the optimal test $\widetilde{\gamma}$ can be structured in line with the Neyman–Pearson lemma in terms of the likelihood ratio

$$\Lambda(s) := \frac{d\widetilde{\mathbb{P}}}{d\widetilde{\mathbb{Q}}} = const \cdot (h_U(s) - h_L(s))^{p-1} \cdot g_z(s, 1-\widetilde{\gamma}(s); p)\frac{d\mathbb{P}}{d\mathbb{Q}}.$$

Since g_z is strictly increasing in $z \in [0,1]$, for a constant c, we obtain that (1) the optimal test $\widetilde{\gamma} = 1$ on the set $\{\Lambda(s) > c\} = \{s : (h_U(s) - h_L(s))^{p-1} \cdot g_z(s,0;p)\frac{d\mathbb{P}}{d\mathbb{Q}} > c\}$; (2) $\widetilde{\gamma} = 0$ on the set $\{\Lambda(s) > c\} = \{s : (h_U(s) - h_L(s))^{p-1} \cdot g_z(s,0;p)\frac{d\mathbb{P}}{d\mathbb{Q}} > c\}$; (3) the optimal test $\widetilde{\gamma}$ is satisfied with the level condition on the set $\{\Lambda(s) = c\} = \{s : (h_U(s) - h_L(s))^{p-1} \cdot g_z(s, 1-\widetilde{\gamma}(s); p)\frac{d\mathbb{P}}{d\mathbb{Q}} = c\}$. Then, as demonstrated in Kolkiewicz (2016), the optimal test $\widetilde{\gamma}$ can be summarized in the following way:

$$\widetilde{\gamma}(s) := 1 - l_e\left(s, c \cdot (h_U(s) - h_L(s))^{1-p}\frac{d\mathbb{Q}}{d\mathbb{P}}\right), \tag{A4}$$

where the constant c is chosen to satisfy the budget constraint $\mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\widetilde{\gamma}(S_{T_h})\right] = \widetilde{H}_0$.

In the stochastic volatility setting, we would like to make the following comments on the theoretical result presented in (A4). First, the expression (A4) is derived based on the invertibility of the function g_z, which needs to be approximated by a kernel density when the conditional loss $L(s)$ is simulated in our numerical results. Second, since in the incomplete Heston market there exists infinitely many equivalent martingale measures, the Radon–Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$ could be derived in multiple forms. In this paper, we utilize the theoretical result based on the D–Y formula, which leads to the Radon–Nikodym derivative as a continuous function of the underlying index $S \in \mathcal{R}^+$. Finally, the uniqueness of the constant c can be demonstrated based on the invertibility of g_z and the continuity of $d\mathbb{Q}/d\mathbb{P}$. The proof of this uniqueness follows the derivation by Kolkiewicz (2016) and it is omitted here.

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