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Asymptotic Expected Utility of Dividend Payments in a Classical Collective Risk Process

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Abstract: We find the asymptotics of the value function maximizing the expected utility of discounted dividend payments of an insurance company whose reserves are modeled as a classical Cramér risk process, with exponentially distributed claims, when the initial reserves tend to infinity. We focus on the power and logarithmic utility functions. We also perform some numerical analysis.

Keywords: utility function; risk process; asymptotics

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1. Introduction

The problem of identifying the optimal dividend strategy of an insurance company was introduced in the seminal paper of De Finetti (1957) and mathematically formalized by Gerber ([1979] 2012). Since then, many authors have analyzed various scenarios for which they proposed optimal dividend strategies.

Gerber ([1979] 2012) assumed that the reserve process $R = (R_t)_{t\geq 0}$ of an insurance company follows a classical Cramér-Lundberg risk process given by

$$R_t = x + \mu t - \sum_{i=1}^{N_t} Y_i, \tag{1}$$

where $Y_1, Y_2, \ldots$ are i.i.d positive random variables with an absolutely continuous distribution function F_Y, representing the claims; $N = (N_t)_{t\geq 0}$ is an independent Poisson process, with intensity $\lambda > 0$, modeling the times at which the claims occur; $x > 0$ denotes the initial surplus; and μ is the premium intensity. We further consider the dividend payments, defined via an adapted and nondecreasing process $D = (D_t)_{t\geq 0}$, representing all the accumulated dividend payments up to time t. Then, the *regulated* process $X = (X_t)_{t\geq 0}$ is given by

$$X_t = R_t - D_t. \tag{2}$$

We observe this regulated process X_t until the time of ruin

$$\tau = \inf\{t \geq 0\colon X_t < 0\}.$$

The time of ruin of an insurance company depends on the chosen dividend strategy. We assume that the usual net profit condition, $\mu > \lambda E(Y_1)$, for the underlying Cramér-Lundberg risk process, is fulfilled. Another natural assumption is that no dividends are paid after the ruin.

Jeanblanc and Shiryaev (1995) and Gerber and Shiu (2004) consider the optimal dividend problem in a Brownian setting. Zhou (2005) study the constant barrier under the Cramér-Lundberg model and Avram et al. (2007) under the Lévy model. For related works considering the dividend problem, we refer to Asmussen and Taksar (1997); Azcue and Muler (2005); Eisenberg and Palmowski (2021); Gao and Yin (2023); Grandis et al. (2007); Noba (2021); Paulsen (2007); Loeffen (2008); Schmidli (2008); Albrecher and Thonhauser (2009); Asmussen and Albrecher (2010); Thonhauser and Albrecher (2011); Eisenberg and Schmidli (2011); Avram et al. (2015) and references therein.

Inspired by Hubalek and Schachermayer (2004), we consider, instead of the classical maximization of the expected value of the discounted dividend payments, the maximization of the expected value of the utility of these payments, for some utility function U. Hubalek and Schachermayer (2004) consider the asymptotic of the expected discounted utility of dividend payments for a Brownian risk process with drift under the assumption that $(D_t)_{t\geq 0}$ is absolutely continuous with respect to the Lebesgue measure. Under the same assumption of $(D_t)_{t\geq 0}$, we perform the asymptotic analysis of the expected utility in a classical compound Poisson risk model, which, due to its jumps, brings an extra level of complexity. As in Hubalek and Schachermayer (2004), we solve some 'peculiar' non-homogenous differential equations.

Assuming that the process D_t admits, almost surely, a density, denoted by $(d_t)_{t\geq 0}$, *namely* for each $t \geq 0$,

$$D_t = \int_0^t d_s ds \quad \text{a.s.},$$

we define the target value function as

$$v(x) = \sup_{(d_t)_{t\geq 0}} \mathbb{E}_x\left(\int_0^\tau e^{-\beta t} U(d_t)\, dt\right), \tag{3}$$

where β is a discount factor, U is a fixed differentiable utility function, which equals 0 on the negative half-line, and $\mathbb{E}_x$ represents the expectation with respect to $\mathbb{P}_x(\cdot) = P(\cdot|X_0 = x)$. Here, the density models the intensity of the dividend payments in continuous time, and thus we will be maximizing the value function $v(x)$ over all admissible dividend *strategies* $(d_t)_{t\geq 0}$. We assume that the dividend density process $(d_t)_{t\geq 0}$ is admissible, whenever it is a nonnegative, adapted and cádlág process, and there are no dividends after the ruin, namely $d_t = 0$, for all $t \geq \tau$. We denote by $\mathfrak{C}$ the set of all admissible strategies $(d_t)_{t\geq 0}$.

Moreover, we restrict ourselves to Markov strategies, meaning that, for every $t \geq 0$, the strategy $(d_t)_{t\geq 0}$ depends only on the amount of the present reserves. We introduce a non-decreasing function c, such that

$$d_t = c(X_t), \quad \text{for any} \quad t \geq 0.$$

The non-decreasing assumption is justified by the fact that the company should be willing to pay more dividends whenever it has larger reserves. Finally, we assume that the ruin cannot be caused by the dividend payment alone and we choose d_t such that the value function given in (3) is well-defined and finite for all $x \geq 0$.

The above dividend problem can be used to monitor the financial state of the company. In particular, it can be considered as a signalling device of future prospects. In this paper, we assume that the company has large reserves and therefore by taking the initial value to infinity we can produce a very transparent optimal strategy and hence a very clear and simple value function, which, we believe, is crucial from a management point of view.

For the above dividend problem, one can formulate the Hamilton-Jacobi-Bellman (HJB) equation of the optimal value function (see Section 2). Although impossible to solve this HJB equation explicitly (see, e.g., Asmussen and Taksar (1997)), one can analyze the asymptotic properties of its solutions for large initial reserves. We focus on the asymptotic analysis of such value functions when the claim sizes are exponentially distributed, with utility functions that are either powers or logarithms (see Section 3). We also introduce a numerical algorithm for identifying such value functions (see Section 4).

2. Hamilton–Jacobi–Bellman Equation

From now on, we assume that $U \in \mathcal{C}^\infty(\mathbb{R}_{>0})$ is increasing and strictly concave, such that $U(0) = 0$, $\lim_{x\to\infty} U_x(x) = 0$ and $\lim_{x\to\infty} U(x) = \infty$, where $f_x(x)$ denotes the derivative of a function f with respect to x. We denote by $\mathfrak{C}_{d_0}$ the set of all admissible strategies $(d_t)_{t\geq 0}$ bounded above by d_0 and let

$$v^{(0)}(x) = \sup_{d_t \in \mathfrak{C}_{d_0}} \mathbb{E}_x\left(\int_0^\tau e^{-\beta t} U(d_t)\, dt\right). \tag{4}$$

Using the verification theorem, one can prove the following theorem.

Theorem 1. *If $d_0 > \mu$ then the value function $v^{(0)}(x)$ is differentiable and fulfills the Hamilton–Jacobi–Bellman equation:*

$$\sup_{0\leq d\leq d_0}\left\{(\mu - d)v_x^{(0)}(x) - (\beta+\lambda)v^{(0)}(x) + U(d) + \lambda\int_0^x v^{(0)}(x-y)dF_Y(y)\right\} = 0. \tag{5}$$

The proof of the above theorem follows the same steps as the proof of Theorem 3.3 of Baran and Palmowski (2017) and therefore we simply refer to them. Since the set of all possible strategies over which we take the supremum in $v^{(0)}(x)$ is smaller than the one for $v(x)$, then one has $v^{(0)}(x) \leq v(x)$. We note also that $v^{(0)}(x)$ depends on d_0. The goal of the next corollary is to prove that $\lim_{d_0\to+\infty} v^{(0)}(x) = v(x)$.

Corollary 1. *The optimal value function $v(x)$ is differentiable and fulfills the Hamilton–Jacobi–Bellman equation:*

$$\sup_{d\geq 0}\left\{(\mu - d)v_x(x) - (\beta+\lambda)v(x) + U(d) + \lambda\int_0^x v(x-y)dF_Y(y)\right\} = 0. \tag{6}$$

Proof. Note that $v^{(0)}(x)$ increases monotonically to $v(x)$, for any fixed $x > 0$, as $d_0 \to \infty$, unless

$$v^{(0)}(x) = \hat{v}^{(0)}(x) := \mathbb{E}_x\left(\int_0^\tau e^{-\beta t} U(d_t^{(0)})\, dt\right), \tag{7}$$

where $d_t^{(0)} = d_0 > \mu$, when $X_t > 0$, and $d_t^{(0)} = \mu$, when $X_t = 0$ (in this way the ruin is not caused by the dividend payments). The reason for that is that the supremum on the left hand side of (5) is a monotone function and thus converges to the supremum given in (6). To exclude (7), it is sufficient to demonstrate that for sufficiently large d_0, for a fixed $x > 0$, the function $\hat{v}^{(0)}(x)$ tends to zero. Observe that the regulated risk process equals either $x + \mu t - d_0 t$, or equals 0 until the nearest jump moment, otherwise, at the time of the first jump, after t. Further, if the first jump happens before t, then either the company becomes ruined by this jump/loss or, it continues, but from an initial position/reserve smaller than x, hence collecting a smaller amount of dividends than $v^{(0)}(x)$. We recall that $U(0) = 0$. Thus

$$\hat{v}^{(0)}(x) \leq e^{-\lambda t}\mathbb{E}_x\left(\int_0^{\tau_0} e^{-\beta s} U(d_0)\, ds\right) + (1 - e^{-\lambda t})v^{(0)}(x),$$

where $\tau_0 = \frac{x+\mu t}{d_0}$. Therefore, for any $\epsilon > 0$, we can find a sufficiently small $t > 0$, such that $\hat{v}^{(0)}(x) \leq \frac{x+\mu t}{d_0} U(d_0) \leq \frac{x(1+\epsilon)}{d_0} U(d_0)$ which tends to zero as $d_0 \to +\infty$, since we assumed that $\lim_{x\to\infty} U'(x) = 0$ and that $\lim_{x\to\infty} U(x) = \infty$. This completes the proof. □

Note that the supremum in (6) is attained for the function

$$c^*(x) = (U')^{-1}(v_x(x)). \tag{8}$$

We end this section by adding two crucial observations. By considering the fix strategy $\bar{c}(y) = y$ and the first jump epoch T we have

$$v(x) \geq \mathbb{E}_x\left(\int_0^T e^{-\beta t} U(\bar{c}(X_t))dt\right) \geq \mathbb{E}_x\left(\int_0^1 e^{-\beta t} U(g_{(x)}(t))dt\right)\mathbb{P}(T > 1)$$
$$\geq \mathbb{E}_x\left(\int_0^1 e^{-\beta t} U(x)dt\right)\mathbb{P}(T > 1) = U(x)\mathbb{P}(T > 1)\frac{1 - e^{-\beta}}{\beta},$$

where the function $g_{(x)}(t)$ describes the deterministic trajectory of the risk process (1) up to the first jump time T, that is, $g_{(x)}(t) = x + \mu t$, $t \leq T$. From the assumption that $\lim_{x\to\infty} U(x) = \infty$, it follows that

$$\lim_{x\to\infty} v(x) = \infty. \tag{9}$$

Moreover, we have the following lemma.

Lemma 1. $\lim_{x\to\infty} v_x(x) = 0.$

Proof. Firstly, we demonstrate that $\lim_{x\to\infty} c^*(x) = \infty$. Recall that, from the definition of an admissible strategy, c^* is a nondecreasing function and hence it is enough to prove that c^* is unbounded. Assuming the contrary, that there exists $L > 0$, such that, for all $x \geq 0$, we have $|c^*(x)| \leq L$, it implies that $d_t^* \leq L$ for all $t \geq 0$. Hence

$$v(x) = \sup_{(d_t)_{t\geq 0}} \mathbb{E}_x\left(\int_0^\tau e^{-\beta t} U(d_t)dt\right) = \mathbb{E}_x\left(\int_0^\tau e^{-\beta t} U(d_t^*)dt\right) \leq$$
$$\leq \mathbb{E}_x\left(\int_0^\tau e^{-\beta t} U(L)dt\right) \leq \mathbb{E}_x\left(\int_0^\infty e^{-\beta t} U(L)dt\right) = \frac{U(L)}{\beta} < \infty.$$

However, this means that $v(x)$ is bounded, which contradicts (9). Thus, indeed $c^*(x) \to \infty$ as $x \to \infty$. Then

$$\lim_{x\to\infty} v_x(x) = \lim_{x\to\infty} U'(c^*(x)) = 0,$$

where the last equality in this equation comes from the Inada condition $\lim_{x\to\infty} U'(x) = 0$ required for the utility function. □

3. Asymptotic Analysis

From now on, we assume that the claims follow an exponential distribution with parameter ξ, that is $Y_i \stackrel{\mathcal{D}}{=} \mathrm{Exp}(\xi)$ for all i. This section is dedicated to the asymptotic analysis of the expected utility of dividend payments, for large initial reserves u.

3.1. Classical Risk Process (1) *and Power Utility Function*

In this subsection, we consider the classical risk process (1) paired with the power utility function

$$U(x) = \frac{x^\alpha}{\alpha}, \qquad \alpha \in (0,1). \tag{10}$$

The supremum in (6) is attained at

$$c^* = (U')^{-1}(v_x) = v_x^{-\frac{1}{1-\alpha}} \tag{11}$$

and thus, after an integration by parts, the Equation (6) simplifies to

$$\mu v_{xx} + (\xi\mu - \beta - \lambda)v_x - \xi\beta v + \xi\frac{1-\alpha}{\alpha}v_x^{-\frac{\alpha}{1-\alpha}} - v_x^{-\frac{1}{1-\alpha}}v_{xx} = 0 \tag{12}$$

where $v_{xx}(x) = v''(x)$ is the second derivative of v. This is a nonlinear second order ODE. Peano Theorem, see (Coddington and Levinson 1987, chp. 1) guarantees the existence of a solution. For uniqueness, we need two boundary conditions. Evaluating $x = 0$ in Equation (6), we have a first initial condition,

$$v(0) = -\frac{\mu}{\beta+\lambda}v_x(0) + \frac{1-\alpha}{\alpha(\beta+\lambda)}v_x(0)^{-\frac{\alpha}{1-\alpha}}. \tag{13}$$

The derivation of the second condition is described later, in Remark 2. In order to asymptotically analyze the solutions of Equation (12), we transform it into a nonlinear first order ODE, via a Riccati type substitution, namely $v_x(x) =: y(v(x))$.

Lemma 2. *As $v \to \infty$, $y(v) \to 0$.*

Proof. Let $v_x(x) = y(v(x))$, then using Lemma 1 concludes the proof. □

From $v_x = y(v)$, we have that $v_{xx} = y_x(v) = y_v v_x = y_v y$. Substituting into Equation (12), it produces the following equation

$$\mu y_v y + (\xi\mu - \beta - \lambda)y - \xi\beta v + \xi\frac{1-\alpha}{\alpha}y^{-\frac{\alpha}{1-\alpha}} - y^{-\frac{\alpha}{1-\alpha}}y_v = 0 \tag{14}$$

which is equivalent with

$$y_v = \frac{(\xi\mu - \beta - \lambda)y - \xi\beta v + \xi\frac{1-\alpha}{\alpha}y^{-\frac{\alpha}{1-\alpha}}}{\mu y - y^{-\frac{\alpha}{1-\alpha}}}. \tag{15}$$

This is a nonlinear first order ODE without known explicit solutions. We focus on the asymptotic behaviour of the solutions and derive the asymptotic optimal strategy of paying dividends $d_t^* = c^*(X_t)$, for $c^*(x)$ a function of the initial reserve.

Note that throughout the paper, $f(x) \sim g(x) \iff \lim_{x\to\infty}\frac{f(x)}{g(x)} = 1$.

Theorem 2. *Let $\alpha = \dfrac{p}{q} \in (0,1)$, where $p, q \in \mathbb{N},\ p < q$. Then, as $x \to \infty$,*

$$v(x) \sim \left(\frac{1-\alpha}{\beta}\right)^{1-\alpha}\frac{x^\alpha}{\alpha}, \tag{16}$$

$$v_x(x) \sim \left(\frac{1-\alpha}{\beta}\right)^{1-\alpha}x^{\alpha-1}, \tag{17}$$

$$c^*(x) \sim \frac{\beta}{1-\alpha}x. \tag{18}$$

Remark 1. *The assumption that α is rational is not restrictive, because the set of all rational numbers is sufficiently large to model various shapes of the power utility function.*

The proof of Theorem 2 is given in Appendix A.

3.2. Classical Risk Process (1) *and Logarithmic Utility Function*

We consider the classical risk process (1) and the logarithmic utility function

$$U(x) = \ln(x+1). \tag{19}$$

The supremum in the Equation (6) is attained for

$$c^* = (U')^{-1}(v_x) = \frac{1}{v_x} - 1 \tag{20}$$

and this equation simplifies to

$$(\mu+1)v_{xx} + (\xi\mu + \xi - \beta - \lambda)v_x - \xi\beta v - \xi \ln v_x - \frac{1}{v_x}v_{xx} - \xi = 0. \tag{21}$$

This is a nonlinear second order ODE with the initial condition

$$v(0) = \frac{\mu+1}{\beta+\lambda}v_x(0) - \frac{\ln v_x(0) + 1}{\beta+\lambda}. \tag{22}$$

For the existence of solutions, see (Coddington and Levinson 1987, chp. 1). Apart from the initial condition above, one more initial condition is required to ensure the uniqueness of solutions. Similarly to the case of the power utility function, the choice of this condition is postponed to Section 4. By a Riccati substitution, $v_x(x) = y(v)$, we transform Equation (21) into the following nonlinear first order ODE

$$(\xi\mu + \xi - \beta - \lambda)y - \xi \ln y - \xi\beta v - \xi + (\mu+1)yy_v - y_v = 0. \tag{23}$$

Theorem 3. *As $x \to \infty$, we have,*

$$v(x) \sim \frac{1}{\beta}(\ln(\beta(x+1)) - 1); \tag{24}$$

$$v_x(x) \sim \frac{1}{\beta(x+1)}; \tag{25}$$

$$c^*(x) \sim \beta x + \beta - 1. \tag{26}$$

The proof of Theorem 3 is given in Appendix B.

4. Numerical Analysis

In this section, we provide a numerical algorithm for calculating the value function for the classical risk process (1) with exponentially distributed claims and power utility function (10). To do this, we first find $v_x(0)$. Then, based on the boundary condition (13), we determine $v(0)$ and numerically solve Equation (12). Obviously, we could propose a similar algorithm for the logarithmic utility function. The considerations regarding the second boundary condition which we formulate in Remark 2 remain true when considering the logarithmic utility functions.

Note that a similar analysis is presented in Baran and Palmowski (2013), from which we retrieve some numerical considerations in the case of the power utility, see Table 1 and Figures 1 and 2. Note that Baran and Palmowski (2013) does not present the derivation of the HJB equation nor the analysis of the logarithmic utility function.

Table 1. Functions $v(x)$ and $v_x(x)$ for $\alpha = 0.5$, $\beta = 0.05$, $\mu = 0.26$, $\xi = 0.4$, $\lambda = 0.1$ and $v_x(0) = 1.9$, $v(0) = 6.8021$.

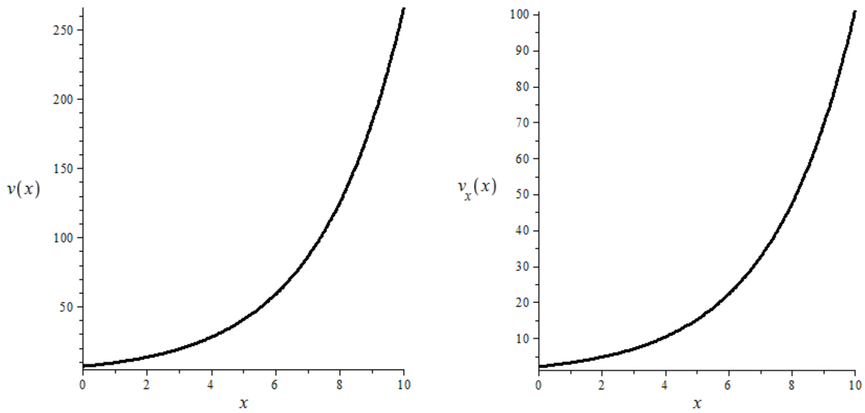
x	$v(x)$	$v_x(x)$	$c(x)$
0	6.8021	1.9000	0.2770
1	8.5790	1.6929	0.3489
2	10.2022	1.5575	0.4122
3	11.7010	1.4431	0.4802
4	13.0940	1.3454	0.5525
5	14.3963	1.2613	0.6286
6	15.6203	1.1884	0.7081
7	16.7762	1.1247	0.7905

Table 1. *Cont.*

x	$v(x)$	$v_x(x)$	$c(x)$
8	17.8723	1.0687	0.8755
9	18.9158	1.0192	0.9626
10	19.9126	0.9752	1.0515

Remark 2. *The choice of $v_x(0)$ is crucial in the context of the optimality of the solution of the HJB equation. Indeed, if we choose $v_x(0)$ and it is too big, then $v(x)$ and $v_x(x)$ go to infinity as $x \to \infty$. In fact, by (11) the discounted cumulative dividends go to 0 (see Table 2). This situation corresponds to a bubble, meaning that the value of the company is not increased by the dividend payments and we cannot derive an optimal solution.*

When $v_x(0)$ is sufficiently large like Figure 2 shows, the function $v(x)$ is concave and $v_x(x)$ tends to 0 as $x \to \infty$, allowing the cumulative discounted dividend payments to increase (see Table 1).

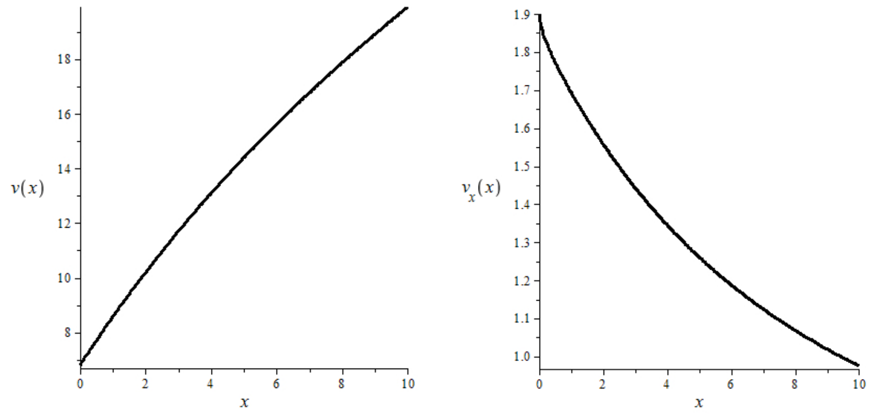


(**a**) Function $v(x)$ (**b**) Function $v_x(x)$

Figure 1. Functions $v(x)$ and $v_x(x)$ for $\alpha = 0.5$, $\beta = 0.05$, $\mu = 0.26$, $\xi = 0.4$, $\lambda = 0.1$ and $v_x(0) = 2$, $v(0) = 6.8$.

Table 2. Functions $v(x)$ and $v_x(x)$ for $\alpha = 0.5$, $\beta = 0.05$, $\mu = 0.26$, $\xi = 0.4$, $\lambda = 0.1$ and $v_x(0) = 2$, $v(0) = 6.8$.

x	$v(x)$	$v_x(x)$	$c(x)$
0	6.8000	2.0000	0.2500
1	9.4022	3.1941	0.0980
2	13.3275	4.7502	0.0443
3	19.1343	7.0039	0.0204
4	27.6771	10.2878	0.0094
5	40.2103	15.0801	0.0044
6	58.5692	22.0787	0.0021
7	85.4378	32.3029	0.0010
8	124.7394	47.2425	0.0004
9	182.2094	69.0750	0.0002
10	266.2320	100.9833	0.0001



(**a**) Function $v(x)$ (**b**) Function $v_x(x)$

Figure 2. Functions $v(x)$ and $v_x(x)$ for $\alpha = 0.5$, $\beta = 0.05$, $\mu = 0.26$, $\xi = 0.4$, $\lambda = 0.1$ and $v_x(0) = 1.9$, $v(0) = 6.8021$.

To find $v_x(0)$ we propose the following algorithm.

- Set initial value $v_x(0) =: b$,
- From the equality (13) derive initial value $v(0) = a$;
- Solve numerically the differential Equation (12) with the initial condition $v(0) = a$;
- Calculate $c(x)$ using $c(x) = v_x(x)^{-\frac{1}{1-\alpha}}$;
- Using the least squares method, approximate $c(x)$ be the linear function $\hat{c}(x) = a_1 x + b_1$. Because of our results from Theorem 2, we assume that $\hat{c}(x)$ is a linear function;
- Let $x(t)$ be a trajectory of the regulated process starting from 0 until the first time claim arrival T. Hence

$$\mu - \hat{c}(x(t)) = x'(t), \quad x(0) = 0, \tag{27}$$

i.e.,

$$x(t) = \frac{\mu - b_1}{a_1} - \frac{\mu - b_1}{a_1} e^{-a_1 t};$$

- Using the least squares method, approximate $v(x)$ by a function of the form $\hat{v}(x) = a_2 x^{\alpha} + b_2$. Because of our results from Theorem 2, we assume that $\hat{v}(x)$ is a power function;
- Calculate

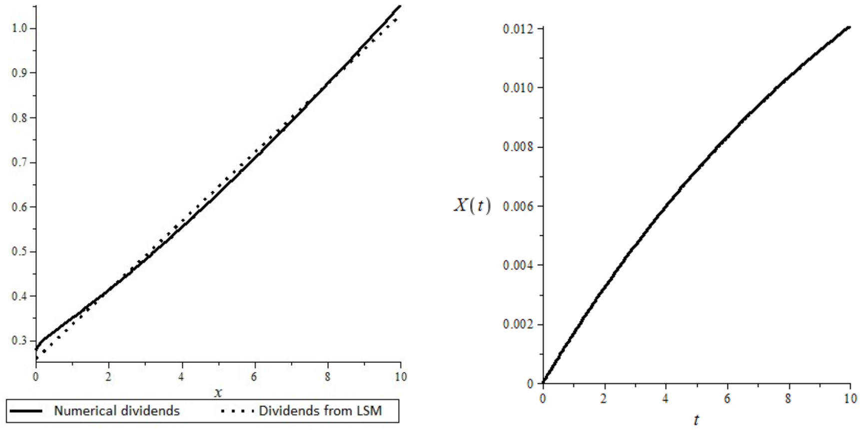
$$A = \mathbb{E}\Big[e^{-\beta T} \hat{v}(X(T) - S)\Big] + \mathbb{E}\Big[\int_0^T e^{-\beta t} U(\hat{c}(X(t)) dt\Big], \tag{28}$$

where $T \overset{\mathcal{D}}{=} \mathrm{Exp}(\lambda)$, $S \overset{\mathcal{D}}{=} \mathrm{Exp}(\xi)$.
- Calculate the value $a - A$;
- Repeat until $|a - A| < \epsilon$ for fixed $\epsilon > 0$.

If we choose $v_x(0) =: b$ hence also $v(0) = a$ correctly, then observing the regulated process right after the first jump occurs, the left hand side A of (28) gives the true estimator of $v(0)$. Hence, A will approximate a. In practice, we should look for the correct a changing $v_x(0)$ by some small fixed value $d > 0$ until $|a - A| < \epsilon$ for a prescribed precision ϵ.

We apply the above procedure in a ten points least square algorithm to the data given in Figure 2. The results are described in the Figures 3 and 4 and the Table 3. At the beginning, we chose $d := 0.01$. We notice that for $b \geq 1.97$, we have a bubble. As per Remark 2, we cannot derive an optimal solution. Thus, the values of b are not greater than 1.97.

We start from the value 1.96 for b and observe the difference $a - A$. Then, we reduce b by d. We noticed that the difference $a - A$ is getting smaller as we are decreasing b. We stop the above procedure when $b = 1.88$ because then $x(t) < 0$ for $t > 0$. Similarly, we can check that all the values of b less that 1.88 are too small. Then, successively, we decrease d to 0.001 and then to $d := 0.0001$. By repeating the above procedure, we can find the "correct" a. For example, if we choose $\epsilon = 0.01$ then $a = 6.800222221$. Table 3 explains how the algorithm works. It contains the results of each step of the loop of this algorithm until $\epsilon = 0.005793808$. Thus, the "correct" value of a is 6.804486491.



(**a**) Functions $c(x)$ and $\hat{c}(x)$ (**b**) Trajectory $x(t)$

Figure 3. Functions $c(x)$, $\hat{c}(x)$ and trajectory $x(t)$ for $\alpha = 0.5$, $\beta = 0.05$, $\mu = 0.26$, $\xi = 0.4$, $\lambda = 0.1$ and $v_x(0) = 1.9$, $v(0) = 6.8021$.

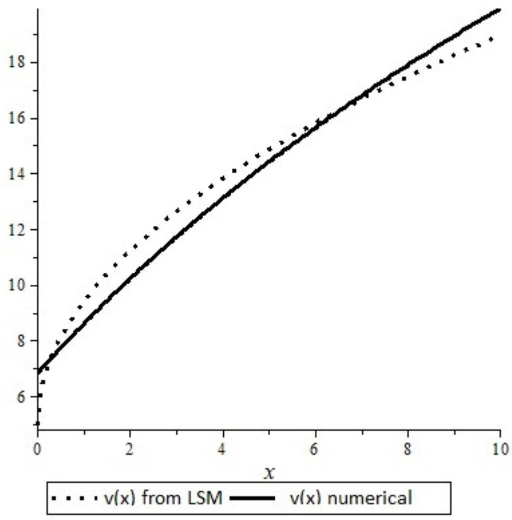


Figure 4. Functions $v(x)$ and $\hat{v}(x)$ for $\alpha = 0.5$, $\beta = 0.05$, $\mu = 0.26$, $\xi = 0.4$, $\lambda = 0.1$ and $v_x(0) = 1.9$, $v(0) = 6.8021$.

Let us recall that our main goal was to derive the asymptotic behavious of the value function for large initial reserves x and to identify its corresponding optimal strategy. The methodology was based on comparing the asymptotic behaviours of components of the HJB equation. This approach produces a very simple solution that can be used instead of numerically solving the HJB equation.

Table 3. The values of initial conditions obtained from procedure of finding $v_x(0)$ (c. = correct, t.b. = too big, t.s. = too small).

b				
Correctness	**Value**	a	A	$a-A$
t.b.	≥1.97	-	-	-
c.	1.96	6.798693877	6.783185889	0.015507988
c.	1.95	6.798803418	6.784849201	0.013954217
c.	1.94	6.799092783	6.786580941	0.012511842
c.	1.93	6.799564767	6.788388955	0.011175812
c.	1.92	6.800222221	6.790283409	0.009938812
c.	1.91	6.801068062	6.792277924	0.008790138
c.	1.90	6.802105263	6.794392618	0.007712645
c.	1.89	6.803336861	6.796662198	0.006674663
t.s.	1.88	-	-	-
t.s.	1.881	-	-	-
c.	1.882	6.804464186	6.798652236	0.005811950
c.	1.8819	6.804479085	6.798679195	0.005799890
t.s.	1.8818	-	-	-
t.s.	⋮	-	-	-
t.s.	1.88185	-	-	-
c.	1.88186	6.804485051	6.798690050	0.005795001
c.	1.881859	6.804485199	6.798690322	0.005794877
c.	1.881858	6.804485348	6.798690594	0.005794754
c.	1.881857	6.804485498	6.798690867	0.005794631
c.	1.881856	6.804485647	6.798691139	0.005794508
c.	1.881855	6.804485795	6.798691412	0.005794383
c.	1.881854	6.804485945	6.798691685	0.005794260
c.	1.881853	6.804486095	6.798691958	0.005794137
c.	1.881852	6.804486243	6.798692231	0.005794012
c.	1.881851	6.804486392	6.798692504	0.005793888
t.s.	1.881850	-	-	-
t.s.	⋮	-	-	-
t.s.	1.8818503	-	-	-
c.	1.8818504	6.804486482	6.798692667	0.005793815
c.	1.88185039	6.804486484	6.798692671	0.005793813
c.	1.88185038	6.804486485	6.798692673	0.005793812
c.	1.88185037	6.804486486	6.798692675	0.005793811
c.	1.88185036	6.804486488	6.798692679	0.005793809
c.	1.88185035	6.804486489	6.798692681	0.005793808
t.s.	1.88185034	-	-	-
t.s.	1.881850341	-	-	-
c.	1.881850342	6.804486491	6.798692684	0.005793807

Still, to compare the asymptotics with the exact values of the value function, we propose a numerical algorithm for solving HJB equations. Using it, we can observe that the asymptotic values are very close to the true ones. In particular, Figure 3 shows that the optimal strategy of paying dividends with intensity $d_t = c(X_t)$ in the case of the power-type utility function is asymptotically linear as (18) suggests. What is interesting, it that this is true even for small values of reserves (starting from $x = 4$). We have observed that this is true for other sets of parameters, which is very promising.

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Appendix A. Proof of Theorem 2

Proof. When $\alpha = \dfrac{p}{q}$, the Equation (14) has the following form:

$$\mu y_v y + (\xi\mu - \beta - \lambda)y - \xi\beta v + \xi\frac{q-p}{p}y^{\frac{p}{p-q}} - y^{\frac{p}{p-q}}y_v = 0.$$

If we make the substitution $z = y^{\frac{1}{p-q}}$, then $z_v = \frac{1}{p-q}y_v z z^{q-p}$, and furthermore

$$(\xi\mu - \beta - \lambda)z - \xi\beta v z^{q-p+1} + \xi\frac{q-p}{p}z^{q+1} - z_v(q-p)\big(\mu z^{p-q} - z^p\big) = 0.$$

If we multiply both sides of the equation by z^{q-p}, we obtain

$$(\xi\mu - \beta - \lambda)z^{q-p+1} - \xi\beta v z^{2q-2p+1} + \xi\frac{q-p}{p}z^{2q-p+1} - (q-p)\mu z_v - (p-q)z^q z_v = 0, \tag{A1}$$

specifically, an equation of the form

$$P(v,z) - z_v Q(v,z) = 0, \tag{A2}$$

where P, Q are polynomials in v and z. Recall that, from (9), $v \to \infty$. Any term on the left-hand side of (A2) is of the form $z^m a_m(v)$ or $z_v z^n a_n(v)$. Marić (1972) proved that if two functions $a_m, a_n \in \mathcal{H}$ (where $\mathcal{H}$ denote the class of Hardy functions) then the set of all terms on the left-hand side of Equation (A2) is totally ordered with respect to the relation $\succeq$, where $a \succeq b$, for $v \to \infty$ means that either $\frac{a}{b} \to \infty$ or $\frac{a}{b} \to l(\neq 0)$ as $v \to \infty$. In other words, heuristically, we can order all terms (which are functions of v) according to the speed that they tends to infinity as $v \to \infty$. (Marić 1972, p. 195) shows that in this set exist two terms of the same order; namely, their quotient tends to a finite limit $l \neq 0$ for $v \to \infty$. Using this result, we can derive the asymptotic behaviour of the solutions of Equation (A2).

Firstly, note that $z \to \infty$. Because of that, we note that in the Equation (A1), the term $(q-p)\mu z_v$ is of a smaller order than the other terms, which contain z_v. Similarly, the term $(\xi\mu - \beta - \lambda)z^{q-p+1}$ has a smaller order than the other terms of Equation (A1), which do not contain z_v. Since we know that there exists two terms of the Equation (A1) of the same order, we have three possibilities to produce the asymptotic behaviour of a solution v of the Equation (A1):

(a) $\xi\beta v z^{2q-2p+1}$ and $\xi\frac{q-p}{p}z^{2q-p+1}$;

(b) $\xi\frac{q-p}{p}z^{2q-p+1}$ and $(p-q)z^q z_v$;
(c) $\xi\beta v z^{2q-2p+1}$ and $(q-p)z^q z_v$.

Lemma A1. *Only the case (a) above produces a feasible asymptotic behaviour.*

Proof. Note that in case (a), both terms have the same order. Indeed, let

$$\lim_{v\to\infty}\frac{\xi\beta v z^{2q-2p+1}}{\xi\frac{q-p}{p}z^{2q-p+1}} = l(\neq 0).$$

Denote

$$\frac{\xi\beta v z^{2q-2p+1}}{\xi\frac{q-p}{p}z^{2q-p+1}} = g(v),$$

where $\lim_{v\to\infty} g(v) = l$, which reduces to

$$z^{-p}(v) = \frac{q-p}{\beta p}v^{-1}g(v).$$

Since $g(v) \sim l$, this becomes

$$z(v) \sim \left(\frac{\beta p}{l(q-p)}\right)^{\frac{1}{p}} v^{\frac{1}{p}}, \quad v\to\infty.$$

Placing the above asymptotics into Equation (A1) and dividing by $v^{\frac{2q-p+1}{p}}$ gives $l=1$. Finally, we obtain the following asymptotics of $z(v)$:

$$z(v) \sim \left(\frac{\beta p}{q-p}\right)^{\frac{1}{p}} v^{\frac{1}{p}}, \quad v\to\infty. \tag{A3}$$

Obviously, in this case $z\to\infty$ for $v\to\infty$, as required.

Similarly, in case (b), we have

$$\lim_{v\to\infty}\frac{(p-q)z^q z_v}{\xi\frac{q-p}{p}z^{2q-p+1}} = l(\neq 0).$$

Following the same steps as in case (a), let

$$\frac{(p-q)z^q z_v}{\xi\frac{q-p}{p}z^{2q-p+1}} = g(v),$$

where $\lim_{v\to\infty} g(v) = l(\neq 0)$. This reduces to

$$z^{-q+p-1}z_v = -\frac{\xi}{p}g(v),$$

which after integration becomes

$$z^{-q+p}(v) = \frac{\xi(q-p)}{p}\int_0^v g(s)ds.$$

From Karamata Theorem (see (Goldie et al. 1989, Prop. 1.5.8)) $\int_0^v g(s)ds \sim lv$, leading to

$$z(v) \sim \left(\frac{l\xi(q-p)}{p}\right)^{\frac{1}{p-q}} v^{\frac{1}{p-q}},$$

for $v \to \infty$. However, for $p - q < 0$, we have $z \to 0$ as $v \to \infty$, which contradicts the assumption that $z \to \infty$ for $v \to \infty$. Thus, this is not acceptable.

In case (c), we have

$$\lim_{v \to \infty} \frac{(q-p)z^q z_v}{\xi \beta v z^{2q-2p+1}} = l(\neq 0).$$

Introducing

$$g(v) = \frac{(q-p)z^q z_v}{\xi \beta v z^{2q-2p+1}} \sim l(\neq 0),$$

as $v \to \infty$, this simplifies into

$$z^{2p-q-1} z_v = \frac{g(v)\xi\beta}{q-p} v. \tag{A4}$$

We will distinguish two cases. Using the same arguments as before, with respect to Karamata arguments given in Goldie et al. (1989), for $v \to \infty$, we encounter two possible asymptotics:

I. If $q \neq 2p$, then, via the separation of variables, we have

$$z(v) \sim \left(\frac{l\xi\beta(2p-q)}{q-p} \right)^{\frac{1}{2p-q}} \left(\frac{v^2}{2} + c \right)^{\frac{1}{2p-q}}.$$

II. If $q = 2p$, then a simple integration leads to

$$z(v) \sim e^{\frac{l\xi\beta}{p}\left(\frac{v^2}{2}+c\right)}.$$

In both of the above cases, c is a constant and its appearance is a consequence of the lack of uniqueness of the solutions of Equation (A4) due to the lack of sufficient boundary conditions for z.

Note that in the first case, the asymptotics of z makes sense only if $q < 2p$ because otherwise $z \to 0$ for $v \to \infty$, leading to a contradiction. In both cases, after substituting the above asymptotics into Equation (A1), the term including $\xi \frac{q-p}{p} z^{2q-p+1}$ dominates any other term. Dividing both sides of Equation (A1) by this asymptotically dominant element leads to the false identity $1 \sim 0$. $\square$

We continue the proof of Theorem 2. From Lemma A1, the asymptotic solution of z is given by (A3). When substituting $y = z^{p-q}$, the asymptotic behavior of $y(v)$ is given by

$$y(v) \sim \left(\frac{\beta p}{q-p} \right)^{\frac{p-q}{p}} v^{\frac{p-q}{p}},$$

which, for $\alpha = \frac{p}{q}$, is equivalent to

$$y(v) \sim \left(\frac{1-\alpha}{\alpha\beta} \right)^{\frac{1-\alpha}{\alpha}} v^{\frac{-(1-\alpha)}{\alpha}}.$$

Recall that $y(v(x)) = v_x(x)$. Hence

$$v_x(x) \sim \left(\frac{1-\alpha}{\alpha\beta} \right)^{\frac{1-\alpha}{\alpha}} v(x)^{\frac{-(1-\alpha)}{\alpha}},$$

We can now solve (via a separation of variables) the equation

$$f_x(x) = \left(\frac{1-\alpha}{\alpha\beta} \right)^{\frac{1-\alpha}{\alpha}} f(x)^{\frac{-(1-\alpha)}{\alpha}}$$

deriving $f(x) = \left(\frac{1-\alpha}{\beta}\right)^{1-\alpha} \frac{(x+c)^\alpha}{\alpha}$ for any constant c. Applying classical Karamata's arguments leads to $f(x) \sim v(x)$, as $x \to \infty$. This produces (16). Using (11) completes the proof. □

Appendix B. Proof of Theorem 3

Proof. We use similar arguments as in the proof of Theorem 2. In fact, one can derive (A2), with the main difference that terms of the form $v^m \ln v$ and v^n will appear in the expressions of P and Q. To satisfy the eliminating procedure given by (Marić 1972, Eq. (3.3)), we mimic all the arguments from Marić (1972) . Thus, one can conclude that also in the case of the logarithmic utility function there exists two of the terms of the Equation (23) of the same order. Now, note that in the Equation (23), the term $(\mu+1)yy_v$ is of a smaller order than y_v. Similarly, the term $(\xi\mu+\xi-\beta-\lambda)y$ is of a smaller order than the other elements, which do not contain y_v. We then have three possibilities:

(a) y_v and $\xi\beta v+\xi$;
(b) y_v and $\xi \ln y$;
(c) $\xi\beta v+\xi$ and $\xi \ln y$.

In case (a)

$$\lim_{v\to\infty} \frac{y_v}{\xi\beta v+\xi} = l(\neq 0),$$

gives

$$y(v) \sim l\xi\beta\frac{v^2}{2} + l\xi v + c.$$

When $v \to \infty$, $y \to \pm\infty$, thus contradicting Lemma 1 ($y \to 0$ when $v \to \infty$).

In case (b) we have

$$\lim_{v\to\infty} \frac{y_v}{\xi \ln y} = l(\neq 0).$$

Let

$$g(v) = \frac{y_v}{\xi \ln y},$$

with $\lim_{v\to\infty} g(v) = l(\neq 0)$. This is equivalent to

$$\xi g(v) = \frac{y_v}{\ln y},$$

which after integration from 0 to v, leads to

$$\xi \int_0^v g(s)ds = \int_0^v \frac{y_s}{\ln y} ds,$$

namely

$$\xi \int_0^v g(s)ds = \frac{v}{\ln v} + \frac{v}{(\ln v)^2} + 2\int_0^v \frac{1}{(\ln y)^3} ds.$$

Using the direct half of Karamata Theorem (see (Goldie et al. 1989, Prop. 1.5.8)), we have that as $v \to \infty$,

$$\xi v l \sim \frac{v}{\ln v} + \frac{v}{(\ln v)^2} + 2\int_0^v \frac{1}{(\ln y)^3} ds,$$

equivalent to

$$\xi l \sim \frac{1}{\ln v} + \frac{1}{(\ln v)^2} + \frac{2}{v}\int_0^v \frac{1}{(\ln y)^3} ds. \tag{A5}$$

Thus, we obtain a contradiction, since the right hand side converges to zero as $v \to \infty$, whereas the left hand side converges to $\xi l \neq 0$..

In case (c), we have

$$\lim_{v\to\infty} \frac{\xi\beta v+\xi}{\xi \ln y} = l(\neq 0),$$

leading to

$$y(v) \sim e^{\frac{\beta}{l} v + \frac{1}{l}}, \quad v \to \infty. \tag{A6}$$

The above asymptotic behaviour makes sense only for $l < 0$, because otherwise $y \to \infty$ when $v \to \infty$. Substituting (A6) into (23) gives $l = -1$. Hence

$$y(v) \sim e^{-\beta v - 1}, \quad v \to \infty.$$

Recall that $y(v) = v_x(x)$. Thus, $v \sim a$ with a solving the equation

$$a_x(x) = e^{-\beta a(x) - 1}.$$

This gives

$$v(x) \sim \frac{1}{\beta}(\ln(\beta(x + C)) - 1).$$

Deriving (25) and (26) is thus straightforward. □

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