

# Optimal Investment in a Dual Risk Model

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**Abstract:** Dual risk models are popular for modeling a venture capital or high-tech company, for which the running cost is deterministic and the profits arrive stochastically over time. Most of the existing literature on dual risk models concentrates on the optimal dividend strategies. In this paper, we propose to study the optimal investment strategy on research and development for the dual risk models to minimize the ruin probability of the underlying company. We will also study the optimization problem when, in addition, the investment in a risky asset is allowed.

**Keywords:** dual risk model; minimizing ruin probability; optimal investment

## 1. Introduction

The classical Cramér–Lundberg model, or the classical compound Poisson risk model, assumes that the surplus process of an insurance company follows the dynamics:

$$dX_t = \rho dt - dJ_t, \quad X_0 = x > 0, \quad (1)$$

where  $\rho > 0$  is the premium rate and  $J_t = \sum_{i=1}^{N_t} Y_i$  is a compound Poisson process, where  $N_t$  is a Poisson process with intensity  $\lambda > 0$  and claim sizes  $Y_i$  are i.i.d. positive random variables independent of the Poisson process with  $\mathbb{E}[Y_1] < \infty$ . One central question in the ruin theory is to study the ruin probability  $\mathbb{P}(\tau < \infty)$ , where  $\tau := \inf\{t > 0 : X_t < 0\}$ .

In recent years, there have been a lot of studies in the insurance and finance literature on the so-called dual risk model, see, e.g., (Afonso et al. 2013; Avanzi et al. 2013; Avanzi et al. 2007; Bayraktar and Egami 2008; Cheung 2012; Cheung and Drekcic 2008; Ng 2009, 2010; Rodríguez-Martínez et al. 2015; Yang and Sendova 2014),

with wealth process following the dynamics:

$$dX_t = -\rho dt + dJ_t, \quad X_0 = x > 0, \quad (2)$$

where  $\rho > 0$  is the cost of running the company and  $J_t = \sum_{i=1}^{N_t} Y_i$ , is the stream of profits, where  $N_t$  is a Poisson process with intensity  $\lambda > 0$  and  $Y_i$  are i.i.d.  $\mathbb{R}^+$  valued random variables with common probability density function  $p(y)$ ,  $y > 0$ , independent of the Poisson process. The dual risk model is used to model the wealth of a venture capital, whose profits depend on the research and development. The classical risk model (1) is most often interpreted as the surplus of an insurance company. On the other hand, the dual risk model (2) can be understood as the wealth of a venture capital or high-tech company. The analogue of the premium in the classical model is the running cost in the dual model, and the claims become the future profits of the company. The ruin probability and the Laplace transform of the ruin time have been well studied for the dual risk model; see, e.g., Afonso et al. (2013). When there is a random delay for the innovations turned to profits, the dual risk model becomes time-inhomogeneous and the ruin probabilities and the distribution of the ruin times are studied in Zhu (2017).

One of the most fundamental questions in the dual risk model is the optimal dividend strategy. Avanzi et al. (2007) worked on optimal dividends in the dual risk model where



**Citation:** Fahim, Arash, and Lingjiong Zhu. 2023. Optimal Investment in a Dual Risk Model. *Risks* 11: 41. <https://doi.org/10.3390/risks11020041>

Academic Editor: Mogens Steffensen

Received: 29 December 2022

Revised: 2 February 2023

Accepted: 6 February 2023

Published: 9 February 2023



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the optimal strategy is a barrier strategy. [Avanzi et al. \(2013\)](#) studied a dividend barrier strategy for the dual risk model, whereby dividend decisions are made only periodically, but still allow ruin to occur at any time. A dual model with a threshold dividend strategy with exponential interclaim times was studied in [Ng \(2009\)](#). [Afonso et al. \(2013\)](#) also worked on dividend problem in the dual risk model, assuming exponential interclaim times. A new approach for the calculation of expected discounted dividends was presented and ruin and dividend probabilities, number of dividends, time to a dividend, and the distribution for the amount of single dividends were studied. Dividend moments in the dual risk model were considered in [Cheung and Drekić \(2008\)](#). They derived integro-differential equations for the moments of the total discounted dividends which can be solved explicitly assuming the jump size distribution has a rational Laplace transform. The expected discounted dividends assuming the profits follow a Phase Type distribution were studied in [Rodríguez-Martínez et al. \(2015\)](#). The Laplace transform of the ruin time, expected discounted dividends for the Sparre–Andersen dual model were derived in [Yang and Sendova \(2014\)](#). More recently, [Yang et al. \(2020\)](#) obtained an explicit expression of the expected discounted dividends in a dual risk model with the threshold dividend strategy and the optimal threshold level were derived. [Avanzi et al. \(2020\)](#) considered the optimal periodic dividend strategies for a general class of dual risk models with fixed transaction costs. In [Fahim and Zhu \(2022\)](#), they obtained the asymptotic analysis for optimal dividends in the dual risk model. [Liu et al. \(2023\)](#) studied the optimal dividend strategy for the dual model with surplus-dependent expense.

So far, the optimization problems studied in the literature on dual risk models have been almost exclusively devoted to the optimal dividend strategy. In this paper, we consider a different type of optimization problem. For a venture capital, or a high-tech company, the investment strategy on research and development (R&D) is crucial. A decision to increase the investment on research and development will increase the running cost of the company, but that will also boost the possibility of the future profits. Therefore, we believe that it is of fundamental interest to understand the optimal investment strategy to strengthen the position of the company.

It is well known that research and development is a basic engine of economic and social growth. It is a considerable amount of spending among many leading corporations in the world. A 2014 FORTUNE article listed the top ten biggest R&D spenders worldwide in the year 2013, including Volkswagen, Samsung, Intel, Microsoft, Roche, Novartis, Toyota, Johnson & Johnson, Google and Merck, with Intel spending as much as 20.1% of their revenue on R&D, see [Casey and Hackett \(2014\)](#). Many technology giants increase their R&D spending consistently, year over year, see, e.g., Table 1 for the R&D and percentage of the revenues of Alphabet, Amazon, Tesla in the years 2018–2021<sup>1</sup>. Notice that in the case of Alphabet, even though the R&D expenditure increases year by year, it increases in line with the increase of the total revenues so that as the percentage of revenues, the number does not change much. The same can be said about Amazon. For some companies, both the absolute R&D expenditure amount and the percentage as the revenues remain reasonably stable, see, e.g., Table 1 for Merck in the years 2018–2021, with the year of 2020 being the only exception which witnessed an unusually high R&D expenditure. For some companies, both the absolute R&D expenditure amount and the revenues can change dramatically, see, e.g., Table 1 for Alphabet, Amazon, Tesla in the years 2018–2021. The case of Tesla is exceptional but not unusual for a new high-tech company in the sense that the total revenues has astronomical growth and the R&D expenditure as the percentage of revenues actually declines during this period even though it had a spectacular increase in R&D expenditure in the year of 2021. Another company that has enjoyed similar phenomenal growth as Tesla is Amazon, see Table 1. However, Amazon's overall growth is not as fast as Tesla.

Since it is expensed rather than capitalized, cuts on research and development increase profit in the short term, but they can hurt the strength of a company in the long run, even if the detrimental impact of the cuts may not be felt for a few years. In the most

recent recession, firms with revenues greater than 100 million USD reduced their research and development intensity (divided by revenue) by 5.6%, even though the advertising intensity actually increased 3.4%, see [Marie Knott \(2012\)](#). In the long run, the research and development does help the company grow and increase the value of a company. Using a measure of the so-called research quotient, a study over all publicly traded US companies from 1981 through 2006 suggested that a 10% increase in research quotient results in an increase in market value of 1.1%, see [Marie Knott \(2012\)](#). Indeed, the US government also encourages the research and development activities. The Research & Experimentation Tax Credit is a general business tax credit passed by the Congress in 1981, as a response to the concerns that research spending declines had adversely affected the country's economic growth, productivity gains, and competitiveness within the global marketplace. According to a study by Ernst & Young, in the year 2005, 17,700 US corporations claimed 6.6 billion USD R&D tax credits on their tax returns<sup>2</sup>.

**Table 1.** R&D spending by Alphabet, Amazon, Tesla and Merck during 2018–2021.

<b>Alphabet</b>	<b>2018</b>	<b>2019</b>	<b>2020</b>	<b>2021</b>
R&D (millions)	\$21,419	\$26,018	\$27,573	\$31,562
Revenues (millions)	\$136,819	\$161,857	\$182,527	\$257,637
As % of Revenues	15.7%	16.1%	15.1%	12.3%
<b>Amazon</b>	<b>2018</b>	<b>2019</b>	<b>2020</b>	<b>2021</b>
R&D (millions)	\$28,837	\$35,931	\$42,740	\$56,052
Revenues (millions)	\$232,887	\$280,522	\$386,064	\$469,822
As % of Revenues	12.4%	12.8%	11.1%	11.9%
<b>Tesla</b>	<b>2018</b>	<b>2019</b>	<b>2020</b>	<b>2021</b>
R&D (millions)	\$1460	\$1343	\$1491	\$2593
Revenues (millions)	\$21,461	\$24,578	\$31,536	\$58,823
As % of Revenues	6.8%	5.5%	4.7%	4.4%
<b>Merck</b>	<b>2018</b>	<b>2019</b>	<b>2020</b>	<b>2021</b>
R&D (millions)	\$9,752	\$9,724	\$13,397	\$12,245
Revenues (millions)	\$42,294	\$39,121	\$41,518	\$48,704
As % of Revenues	23.1%	24.9%	32.3%	25.1%

Optimal investment problems have a long history in finance and related fields. For example, [Merton \(1969, 1971\)](#) formulated and studied the problem of optimal allocation between risky assets and a risk-free asset to maximize expected utility; [Fleming and Zariphopoulou \(1991\)](#) considered the optimal investment and consumption problem where short-selling is not allowed but borrowing is allowed. [Davis \(1990\)](#), and [Shreve and Soner \(1994\)](#) studied optimal investment and consumption with proportional transaction costs and [Morton and Pliska \(1990\)](#) considered optimal portfolio management with fixed transaction costs. [Grossman and Zhou \(1993\)](#) studied optimal investment strategies for controlling drawdowns. [Fleming and Sheu \(2000\)](#) studied the optimal investment problem to maximize the long-term growth rate of expected utility of wealth. [Hipp and Plum \(2000\)](#) studied the optimal investment for insurers. [Carr et al. \(2001\)](#) considered the problem of optimal investment in a risky asset, and in derivatives written on the price process of this asset. Finally, there are also a limited number of works on the optimal venture capital investments, see, e.g., [Bayraktar and Egami \(2008\)](#). However, to the best of our knowledge, the optimal investment in research and development for the dual risk model has never been studied in the previous literature, and our paper is the first one that considers this problem.

We propose to study the optimal investment strategy on research and development for the dual risk models to minimize the ruin probability of the underlying company. In addition to the investment in research and development, we will also allow the investment in a risky asset, e.g., a market index. The possibility that an insurer can invest part of the

surplus into a risky asset to minimize the ruin probability was studied by [Browne \(1995\)](#) for the case that the insurance business is modeled by a Brownian motion with constant drift and the risky asset is modeled as a geometric Brownian motion. Later, [Hipp and Plum \(2000\)](#) studied the optimal investment in a market index for insurers in the classical compound Poisson risk model. We will study the optimal investment problem when both investment in research and development and investment in a risky asset are allowed. Unlike the problem of minimizing the ruin probability for an insurer in the classical risk model [Hipp and Plum \(2000\)](#), we will obtain closed-form formulas in the dual risk model.

Since the works of [Browne \(1995\)](#) and [Hipp and Plum \(2000\)](#), the optimal investment in the market for the classical risk model and related models have been extensively studied. In [Liu and Yang \(2004\)](#), they generalized the works by [Hipp and Plum \(2000\)](#) by including a risk-free asset. In [Schmidli \(2002\)](#), the optimization problem of minimizing the ruin probability for the classical risk model is studied when investment in a risky asset and proportional reinsurance are both allowed. The asymptotic ruin probability for the classical risk model under the optimal investment in a risky asset is obtained by [Gaier et al. \(2003\)](#) for large initial wealth. The asymptotics for small claim sizes were obtained in [Hipp and Schmidli \(2004\)](#). In [Yang and Zhang \(2005\)](#), they studied the optimal investment for an insurer when the risk process is compound Poisson process perturbed by a standard Brownian motion and the insurer can invest in the money market and in a risky asset. In [Gaier and Grandits \(2002\)](#), the case when the claim sizes are of regularly varying tails were studied. The results were then extended to include interest rates in [Gaier and Grandits \(2004\)](#). The case for subexponential claims was investigated in [Schmidli \(2005\)](#). In [Promislow and Young \(2005\)](#), they studied the problem of minimizing the probability of ruin of an insurer when the claim process is modeled by a Brownian motion with drift optimizing over the investment in a risky asset and purchasing quota-share reinsurance. In [Wang et al. \(2007\)](#), they adopted the martingale approach to study the optimal investment problem for an insurer when the insurer's risk process is modeled by a Lévy process with possible investment in a security market described by the standard Black–Scholes model. When the underlying investor is an individual rather than an insurance company, the optimal investment problem of minimizing the ruin probability was studied in, e.g., [Bayraktar and Young \(2007\)](#). In [Azcue and Muler \(2009\)](#), they studied the minimization of the ruin probability for the classical risk model with possible investment in a risky asset that follows a geometric Brownian motion under the borrowing constraints. There have been many other works in this area. For a survey, we refer to [Paulsen \(2008\)](#) and the references therein.

This paper is organized as follows. We first introduce a state-dependent dual risk model that generalizes the classical dual risk model (Section 2). When the size of a company increases, the cost usually also increases, while the resource of income will also increase in general, which makes it natural to study a state-dependent dual risk model. Then, we study the optimal investment strategy on research and development to minimize the ruin probability of the company (Section 3), with a further discussion of a state-dependent example in Section 3.1. As a special case, the state-independent model is discussed in Section 3.2, with a further discussion of a state-independent example in Section 3.3. Next, we study the joint investment in research and development and a market index to minimize the ruin probability in Section 4. Finally, we provide some numerical studies in Section 5 to better understand how the minimized ruin probability and the optimal strategy depend on the parameters in the model.

## 2. A State-Dependent Dual Risk Model

We introduce a state-dependent dual risk model with the wealth process being defined as follows:

$$dX_t = -\rho(X_t)dt + dJ_t, \quad X_0 > 0, \quad (3)$$

where  $J_t = \sum_{i=1}^{N_t} Y_i$ , where  $N_t$  is a simple point process with intensity  $\lambda(X_{t-})$  at time  $t$ , and  $Y_i$  are i.i.d. positive random variables with finite mean and independent of  $\mathcal{F}_{\tau_i-}$ , where

$\mathcal{F}_t$  is the natural filtration generated by  $X_t$  process,  $\tau_i$  is the  $i$ -th arrival time of  $N_t$  and we further assume that  $\rho(\cdot), \lambda(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  are increasing functions. The state-dependent dual risk model (3) was first introduced in [Zhu \(2015b\)](#), in which ruin probability and the Laplace transform of the ruin time were studied.

The motivation of introducing state dependence for the dual risk model is the following. First, the cost of a company usually increases as the size of the company increases. For example, the running cost of a small business and a Fortune 500 company are vastly different. Second, as the size of a company increases, the arrival intensity of the future profits might increase. It may be due to the fact that the larger a company gets, the more resources for income it will obtain. It is also well known in the finance literature that as a company gets larger and stronger, it can enjoy more benefits, e.g., net present value (NPV), which for example might be due to the opportunities brought by franchising. As we can see from [Table 1](#), the R&D expenditure may be far from being constant as the size of the company and the revenue of the company change. More realistically, the R&D expenditure and other costs of running the company should be state-dependent.

Let  $\tau := \inf\{t > 0 : X_t \leq 0\}$  be the ruin time of  $X_t$  process. The eventual ruin probability is defined as the function  $\psi(x) := \mathbb{P}(\tau < \infty | X_0 = x)$  to emphasize the dependence on the initial wealth  $x$ . Note that for the state-independent dual risk model,  $\lambda(\cdot) \equiv \lambda$  and  $\rho(\cdot) \equiv \rho$ , under the assumption  $\lambda \mathbb{E}[Y_1] > \rho$ , the ruin probability  $\psi(x)$  is less than 1. Indeed,  $\psi(x) = e^{-\alpha x}$ , where  $\alpha > 0$  is the unique solution to the equation; see, e.g., [Afonso et al. \(2013\)](#):

$$\rho\alpha + \lambda \int_0^\infty [e^{-\alpha y} - 1]p(y)dy = 0. \quad (4)$$

For the state-dependent dual risk model, there is no simple closed-form formula for the ruin probability. Nevertheless, for the special case when the jump sizes  $Y_i$  are i.i.d. exponentially distributed, there is a closed-form expression for the ruin probability; see [Theorem 1](#) in [Zhu \(2015b\)](#).

Finally, we notice that the  $X_t$  process in (3) is an extension of the (nonlinear) marked Hawkes process with exponential kernel (see, e.g., [Brémaud and Massoulié 1996](#); [Gao and Zhu 2018a, 2018b](#); [Hawkes 1971](#); [Zhu 2015a](#)),

that is,  $N_t$  is a simple point process with intensity  $\lambda(X_t)$ , where

$$X_t := X_0 e^{-\beta t} + \sum_{i:\tau_i < t} Y_i e^{-\beta(t-\tau_i)}, \quad (5)$$

where  $\tau_i$  is the  $i$ -th arrival time of  $N_t$ , and  $Y_i$  are i.i.d. positive random variables independent of  $\mathcal{F}_{\tau_i^-}$  with finite mean and  $X_0, \beta > 0$  are given constants, where  $X_t$  in (5) satisfies the dynamics (3) with  $\rho(x) := \beta x$ . When  $\lambda(\cdot)$  is linear, it is called linear Hawkes process, named after [Hawkes \(1971\)](#). When  $\lambda(\cdot)$  is nonlinear, the Hawkes process is said to be nonlinear which was first introduced by [Brémaud and Massoulié \(1996\)](#). Hawkes processes have wide applications in finance, neuroscience, social networks, criminology, seismology, and many other fields; see [Gao and Zhu \(2021\)](#) and the references therein. Since the  $X_t$  process in (3) is an extension of the (nonlinear) marked Hawkes process with exponential kernel, our paper also contributes to the literature on the Hawkes process.

### 3. Minimizing the Ruin Probability

In this section, we study the optimization control problem of minimizing the ruin probability for the dual risk model. The management of the underlying company can decide whether or not to increase the capital spending on research and development to boost the future profits. Our goal is to find the optimal expenditure on research and development to minimize the probability that the company is eventually ruined.

Before we proceed, we introduce the investment on research and development  $C \in \mathcal{C}$ , where  $\mathcal{C}$  is the set of all admissible strategies, defined as

$$\mathcal{C} := \{C : [0, \infty) \times \Omega \rightarrow \mathbb{R}_{\geq 0} : C \text{ is progressively measurable, bounded and predictable}\}. \quad (6)$$

Given the control  $C \in \mathcal{C}$ , the wealth process has the dynamics

$$dX_t^C = -(\rho(X_t) + C_t)dt + dJ_t^C, \tag{7}$$

where  $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is increasing and  $J_t = \sum_{i=1}^{N_t} Y_i$ , where  $Y_i$  are defined same as before and  $N_t$  is a simple point process with intensity  $F(X_{t-}, C_{t-})$  at time  $t$ , where  $F(x, c) : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is measurable in  $(x, c)$  and increasing in both  $x$  and  $c$  and  $F(x, 0) = \lambda(x)$  for every  $x \in \mathbb{R}_+$ , where  $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is increasing.

We define  $\tau^C$  as the ruin time of the  $X^C$  process under the control  $C \in \mathcal{C}$  by  $\tau^C := \inf\{t \geq 0 : X_t^C \leq 0\}$ . We are interested in studying the optimization problem:

$$V(x) := \min_{C \in \mathcal{C}} \mathbb{P}(\tau^C < \infty | X_0^C = x). \tag{8}$$

From the optimal control point of view, it is also interesting to study the state-dependent case, which adds a technical contribution to the literature of stochastic optimal control theory. We will show that the optimal strategy is in general state-dependent when the underlying dual risk model is state-dependent, and it exhibits a closed-form expression.

**Theorem 1.** *The optimal strategy  $C^*$  is given by*

$$C_t^* = C^*(X_t) \in \arg \min_{C \geq 0} \frac{\rho(X_t) + C}{F(X_t, C)}, \tag{9}$$

*provided that the minimum exists.*

**Proof of Theorem 1.** For any control  $C \in \mathcal{C}$ , we have

$$dX_t^C = -(\rho(X_t) + C_t)dt + dJ_t^C, \tag{10}$$

where  $J_t^C = \sum_{i=1}^{N_t^C} Y_i$ , where  $N_t^C$  is a simple point process with intensity  $F(X_{t-}, C_{t-})$  at time  $t$  and  $Y_i$  are i.i.d. with probability density function  $p(y)$  defined as before.

Let us introduce a random time change and define the random time  $T(t)$  via:

$$\int_0^{T(t)} F(X_{s-}, C_{s-}) ds = t. \tag{11}$$

Then, it is easy to see that  $T(0) = 0$  and  $T(t) \rightarrow \infty$  as  $t \rightarrow \infty$  since  $C \in \mathcal{C}$  is bounded. It follows from (10) that

$$dX_{T(t)} = -(\rho(X_{T(t)}) + C_{T(t)})dT(t) + dJ_{T(t)}^C. \tag{12}$$

Under the random time change (11), we have

$$\frac{dT(t)}{dt} = \frac{1}{F(X_t, C_t)},$$

and  $J_{T(t)}^C$  is distributed as  $\bar{J}_t := \sum_{i=1}^{\bar{N}_t} Y_i$ , where  $\bar{N}_t$  is a standard Poisson process with intensity 1; see, e.g., Meyer (1971) for the random time change for simple point processes. Therefore, we obtain

$$dX_{T(t)} = -\frac{\rho(X_{T(t)}) + C_{T(t)}}{F(X_t, C_t)} dt + d\bar{J}_t. \tag{13}$$

Let us also notice that  $\mathbb{P}(X_t \text{ ever gets ruined}) = \mathbb{P}(X_{T(t)} \text{ ever gets ruined})$ . Therefore, the optimal strategy is given by (9) provided that the minimum exists. This completes the proof.  $\square$

In Theorem 1, we obtain the closed-form expression of the optimal strategy  $C^*$ . However, we do not have a closed form for the minimized ruin probability  $\mathbb{P}(\tau^{C^*} < \infty | X_0^{C^*} = x)$ . Next, we will show that we can obtain a closed form for the ruin probability in the special case when the jump sizes  $Y_i$  follow exponential distributions. We first recall the following result from Zhu (2015b), which states that the ruin probability for a state-dependent dual risk model with the exponentially distributed  $Y_i$  has a closed-form expression.

**Theorem 2** (Theorem 1 in Zhu (2015b)). *Consider the dual risk model:  $dX_t = -\rho(X_t)dt + dJ_t$ , where  $X_0 = x > 0$ ,  $J_t = \sum_{i=1}^{N_t} Y_i$ , where  $Y_i$  are exponential random variables with the probability density function  $p(y) = \nu e^{-\nu y}$ ,  $\nu > 0$ , and  $N_t$  is a simple point process with intensity  $\lambda(X_{t-})$  at time  $t$ , where  $\rho(\cdot), \lambda(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  are increasing functions. Then,*

$$\mathbb{P}(\tau < \infty | X_0 = x) = \frac{\int_x^\infty \frac{\lambda(y)}{\rho(y)} e^{\nu y - \int_0^y \frac{\lambda(w)}{\rho(w)} dw} dy}{\int_0^\infty \frac{\lambda(y)}{\rho(y)} e^{\nu y - \int_0^y \frac{\lambda(w)}{\rho(w)} dw} dy}. \tag{14}$$

As a corollary of Theorems 1 and 2, we obtain the closed form for the minimized ruin probability when the jump sizes  $Y_i$  are i.i.d. exponentially distributed.

**Proposition 1.** *Assume  $p(y) = \nu e^{-\nu y}$ , where  $\nu > 0$ . Assume also the integral*

$$\int_0^\infty \frac{F(y, C^*(y))}{\rho(y) + C^*(y)} e^{\nu y - \int_0^y \frac{F(w, C^*(w))}{\rho(w) + C^*(w)} dw} dy$$

*exists and is finite. Then,*

$$\min_{C \in \mathcal{C}} \mathbb{P}(\tau^C < \infty | X_0^C = x) = \frac{\int_x^\infty \frac{F(y, C^*(y))}{\rho(y) + C^*(y)} e^{\nu y - \int_0^y \frac{F(w, C^*(w))}{\rho(w) + C^*(w)} dw} dy}{\int_0^\infty \frac{F(y, C^*(y))}{\rho(y) + C^*(y)} e^{\nu y - \int_0^y \frac{F(w, C^*(w))}{\rho(w) + C^*(w)} dw} dy}. \tag{15}$$

**Proof of Proposition 1.** The proposition follows immediately from Theorems 1 and 2.  $\square$

### 3.1. A State-Dependent Example

In this section, we study a state-dependent example in detail. We assume that

$$F(x, c) = \lambda(x) + \delta(x)c^\gamma, \tag{16}$$

where  $\delta(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is increasing, and  $\gamma > 0$ . We recall that  $\lambda(\cdot)$  is increasing and thus,  $\lambda(\cdot) \geq \lambda(0) > 0$ . Let us also assume that  $\rho(\cdot) \leq \rho(\infty) < \infty$ . Under our assumptions,  $F(x, c)$  is increasing in both  $x$  and  $c$ , and  $F(x, 0) = \lambda(x)$ .

Notice when  $\gamma > 1$ , for any constant strategy  $C_t \equiv C$ , where  $C > 0$  is sufficiently large, the ruin probability is bounded above by the ruin probability of the following process:

$$dX_t = -(\rho(\infty) + C)dt + dJ_t, \tag{17}$$

where  $J_t = \sum_{i=1}^{N_t} Y_i$  is compound Poisson with  $N_t$  being the Poisson process with intensity  $\lambda(0) + \delta(0)C^\gamma$ .

By the ruin probability for state-independent dual risk model (see, e.g., Afonso et al. 2013), the ruin probability of the  $X_t$  process defined in (17) is given by  $e^{-\alpha_C x}$ , where  $\alpha_C$  is the unique positive solution to the equation:

$$(\rho(\infty) + C)\alpha_C + (\lambda(0) + \delta(0)C^\gamma) \int_0^\infty [e^{-\alpha_C y} - 1]p(y)dy = 0. \tag{18}$$

We can rewrite this equation as:

$$\frac{\rho(\infty) + C}{\lambda(0) + \delta(0)C^\gamma} \alpha_C = \int_0^\infty [1 - e^{-\alpha_C y}] p(y) dy. \tag{19}$$

The right-hand side of the above equation is bounded between 0 and 1. In the left-hand side of the above equation,  $\lim_{C \rightarrow \infty} \frac{\rho(\infty) + C}{\delta(0)C^\gamma} = 0$ , which implies that  $\alpha_C \rightarrow \infty$  as  $C \rightarrow \infty$ . Hence,  $V(x) \leq \inf_{C > 0} e^{-\alpha_C x} = 0$  and the minimized ruin probability is trivially zero.

Therefore, in the rest of this section, we only consider two cases: (i)  $0 < \gamma < 1$ ; (ii)  $\gamma = 1$ .

### 3.1.1. The $0 < \gamma < 1$ Case

Under the assumption that  $0 < \gamma < 1$ , it is easy to see from Theorem 1 that the optimal strategy  $C_{T(t)}$  is the strategy that minimizes the drift:

$$\frac{\rho(X_{T(t)}) + C_{T(t)}}{\lambda(X_{T(t)}) + \delta(X_{T(t)})C_{T(t)}^\gamma}. \tag{20}$$

It is easy to compute from (20) that the optimal strategy satisfies

$$\lambda(X_{T(t)}) + \delta(X_{T(t)})(1 - \gamma)C_{T(t)}^\gamma = \rho(X_{T(t)})\delta(X_{T(t)})\gamma C_{T(t)}^{\gamma-1}. \tag{21}$$

Therefore, for any  $t > 0$ , the optimal strategy  $C_t$  satisfies

$$\lambda(X_t) + \delta(X_t)(1 - \gamma)C_t^\gamma = \rho(X_t)\delta(X_t)\gamma C_t^{\gamma-1}. \tag{22}$$

It is clear that the optimal strategy  $C_t$  is a function of  $X_t$  and we denote it as  $C^*(X_t)$ . Then, under the optimal strategy,

$$dX_t = -(\rho(X_t) + C^*(X_t))dt + dJ_t, \tag{23}$$

where  $J_t = \sum_{i=1}^{N_t} Y_i$ , where  $N_t$  has intensity  $\lambda(X_{t-}) + \delta(X_{t-})C^*(X_{t-})^\gamma$  at time  $t$ .

When the probability density function  $p(y) = \nu e^{-\nu y}$  of jump sizes  $Y_i$  is exponential, it follows from Proposition 1 that we have the following result:

**Proposition 2.** Assume  $p(y) = \nu e^{-\nu y}$ , where  $\nu > 0$ . Assume also the integral

$$\int_0^\infty \frac{\lambda(y) + \delta(y)C^*(y)^\gamma}{\rho(y) + C^*(y)} e^{\nu y - \int_0^y \frac{\lambda(w) + \delta(w)C^*(w)^\gamma}{\rho(w) + C^*(w)} dw} dy$$

exists and is finite. Then,

$$V(x) = \frac{\int_x^\infty \frac{\lambda(y) + \delta(y)C^*(y)^\gamma}{\rho(y) + C^*(y)} e^{\nu y - \int_0^y \frac{\lambda(w) + \delta(w)C^*(w)^\gamma}{\rho(w) + C^*(w)} dw} dy}{\int_0^\infty \frac{\lambda(y) + \delta(y)C^*(y)^\gamma}{\rho(y) + C^*(y)} e^{\nu y - \int_0^y \frac{\lambda(w) + \delta(w)C^*(w)^\gamma}{\rho(w) + C^*(w)} dw} dy}. \tag{24}$$

**Proof of Proposition 2.** The proposition follows immediately from Proposition 1. □

Next, in the following example, we show that with particular model specifications, the optimal  $C^*$  and the minimized ruin probability  $V(x)$  in (24) admit a simpler closed-form formula.

**Example 1.** Let  $\rho(x) = \rho_0$ ,  $\lambda(x) = \lambda_0(c_1x + c_2)$ , and  $\delta(x) = \delta_0(c_1x + c_2)$ , where  $\rho_0, \lambda_0, \delta_0, c_1, c_2$  are positive constants. Then, the optimal investment rate  $C^*(x)$  is a constant  $C^*(x) \equiv C_0$ , where  $C_0$  is the unique positive solution to the equation:

$$\lambda_0 + \delta_0(1 - \gamma)C_0^\gamma = \rho_0\delta_0\gamma C_0^{\gamma-1}. \tag{25}$$

Hence, the minimized ruin probability in (24) can be computed as:

$$\begin{aligned} V(x) &= \frac{\int_x^\infty \frac{\lambda_0 + \delta_0 C_0^\gamma}{\rho_0 + C_0} (c_1y + c_2) e^{\nu y - \int_0^y \frac{\lambda_0 + \delta_0 C_0^\gamma}{\rho_0 + C_0} (c_1w + c_2) dw} dy}{\int_0^\infty \frac{\lambda_0 + \delta_0 C_0^\gamma}{\rho_0 + C_0} (c_1y + c_2) e^{\nu y - \int_0^y \frac{\lambda_0 + \delta_0 C_0^\gamma}{\rho_0 + C_0} (c_1w + c_2) dw} dy} \\ &= \frac{\int_x^\infty (c_1y + c_2) e^{\left(\nu - \frac{\lambda_0 + \delta_0 C_0^\gamma}{\rho_0 + C_0} c_2\right) y - \frac{\lambda_0 + \delta_0 C_0^\gamma}{\rho_0 + C_0} \frac{c_1}{2} y^2} dy}{\int_0^\infty (c_1y + c_2) e^{\left(\nu - \frac{\lambda_0 + \delta_0 C_0^\gamma}{\rho_0 + C_0} c_2\right) y - \frac{\lambda_0 + \delta_0 C_0^\gamma}{\rho_0 + C_0} \frac{c_1}{2} y^2} dy} \\ &= \frac{\frac{1}{4d^{3/2}} e^{-dy^2} \left[ \sqrt{\pi} e^{\frac{c^2}{4d} + dy^2} (ac + 2bd) \operatorname{erf}\left(\frac{2dy - c}{2\sqrt{d}}\right) - 2a\sqrt{d}e^{cy} \right] \Big|_{y=x}^\infty}{\frac{1}{4d^{3/2}} e^{-dy^2} \left[ \sqrt{\pi} e^{\frac{c^2}{4d} + dy^2} (ac + 2bd) \operatorname{erf}\left(\frac{2dy - c}{2\sqrt{d}}\right) - 2a\sqrt{d}e^{cy} \right] \Big|_{y=0}^\infty} \\ &= \frac{2a\sqrt{d}e^{cx - dx^2} + \sqrt{\pi} e^{\frac{c^2}{4d}} (ac + 2bd) \operatorname{erfc}\left(\frac{2dx - c}{2\sqrt{d}}\right)}{2a\sqrt{d} + \sqrt{\pi} e^{\frac{c^2}{4d}} (ac + 2bd) \operatorname{erfc}\left(\frac{-c}{2\sqrt{d}}\right)}, \end{aligned} \tag{26}$$

where  $\operatorname{erf}(x) := \frac{2}{\sqrt{2\pi}} \int_0^x e^{-t^2} dt$  is the error function and  $\operatorname{erfc}(x) := 1 - \operatorname{erf}(x)$  is the complementary error function and  $a := c_1, b := c_2$ , and

$$c := \nu - \frac{\lambda_0 + \delta_0 C_0^\gamma}{\rho_0 + C_0} c_2, \quad d := \frac{\lambda_0 + \delta_0 C_0^\gamma}{\rho_0 + C_0} \frac{c_1}{2}. \tag{27}$$

### 3.1.2. The $\gamma = 1$ Case

When  $\gamma = 1$ , it follows from Theorem 1 that the optimal  $C^*(x)$  satisfies  $C^*(x) = 0$  in the region where  $\delta(x) \leq \frac{\lambda(x)}{\rho(x)}$  and the “optimal”  $C^*(x) = \infty$  in the region where  $\delta(x) > \frac{\lambda(x)}{\rho(x)}$ .

**Remark 1.** If we impose a research and development budget constraint by  $M \in (0, \infty)$ , the maximum capacity, then the admissible set of controls is given by  $\mathcal{C}_M := \{C \in \mathcal{C} : \sup_{t \geq 0} C_t \leq M\}$ . Then, the above analysis implies that  $C^*(x) = 0$  in the region  $\delta(x) \leq \frac{\lambda(x)}{\rho(x)}$  and  $C^*(x) = M$  in the region  $\delta(x) > \frac{\lambda(x)}{\rho(x)}$ .

Next, in the following example, we show that with particular model specifications, the optimal  $C^*$  the minimized ruin probability  $V(x)$  admit simpler closed-form formulas.

**Example 2.** Let  $\rho(x) = \rho_0(c_1x + c_2)$ ,  $\lambda(x) = \left(\nu + \frac{\lambda_0}{1+x}\right)\rho(x)$ , and  $\delta(x) = \delta_0$ , where  $\rho_0, c_1, c_2, \lambda_0, \delta_0$  are positive constants. We further assume that  $\nu < \delta_0 < \nu + \lambda_0$ . Then, the optimal  $C^*$  is given by:

$$C^*(x) = \begin{cases} 0 & \text{if } x \leq \frac{\lambda_0 - \delta_0 + \nu}{\delta_0 - \nu}, \\ +\infty & \text{if } x > \frac{\lambda_0 - \delta_0 + \nu}{\delta_0 - \nu}. \end{cases} \tag{28}$$

Let us define:

$$x^* := \frac{\lambda_0 - \delta_0 + \nu}{\delta_0 - \nu}. \tag{29}$$

Then, we can compute that for any  $y \leq x^*$ ,

$$\int_0^y \frac{\lambda(w) + \delta(w)C^*(w)}{\rho(w) + C^*(w)} dw = \int_0^y \left( \nu + \frac{\lambda_0}{1+w} \right) dw = \nu y + \lambda_0 \log(1+y), \tag{30}$$

and for any  $y > x^*$ ,

$$\int_0^y \frac{\lambda(w) + \delta(w)C^*(w)}{\rho(w) + C^*(w)} dw = \nu x^* + \lambda_0 \log(1+x^*) + \delta_0(y-x^*). \tag{31}$$

Therefore, for  $x > x^*$ , we have

$$\begin{aligned} & \int_x^\infty \frac{\lambda(y) + \delta(y)C^*(y)}{\rho(y) + C^*(y)} e^{\nu y - \int_0^y \frac{\lambda(w) + \delta(w)C^*(w)}{\rho(w) + C^*(w)} dw} dy \\ &= \int_x^\infty \delta_0 e^{\nu y - \nu x^* - \lambda_0 \log(1+x^*) - \delta_0(y-x^*)} dy = \frac{e^{-\nu x^* + \delta_0 x^*}}{(1+x^*)^{\lambda_0}} \frac{\delta_0}{\delta_0 - \nu} e^{-(\delta_0 - \nu)x}, \end{aligned} \tag{32}$$

and for  $x \leq x^*$ , we have

$$\begin{aligned} & \int_x^\infty \frac{\lambda(y) + \delta(y)C^*(y)}{\rho(y) + C^*(y)} e^{\nu y - \int_0^y \frac{\lambda(w) + \delta(w)C^*(w)}{\rho(w) + C^*(w)} dw} dy \\ &= \int_x^{x^*} \left( \nu + \frac{\lambda_0}{1+y} \right) e^{\nu y - \nu y - \lambda_0 \log(1+y)} dy + \frac{1}{(1+x^*)^{\lambda_0}} \frac{\delta_0}{\delta_0 - \nu} \\ &= \frac{\nu}{1-\lambda_0} \left[ (1+x^*)^{-\lambda_0+1} - (1+x)^{-\lambda_0+1} \right] + (1+x)^{-\lambda_0} - (1+x^*)^{-\lambda_0} + \frac{1}{(1+x^*)^{\lambda_0}} \frac{\delta_0}{\delta_0 - \nu}. \end{aligned} \tag{33}$$

Hence, we conclude that for  $x > x^*$ , we have

$$V(x) = \frac{\frac{e^{-\nu x^* + \delta_0 x^*}}{(1+x^*)^{\lambda_0}} \frac{\delta_0}{\delta_0 - \nu} e^{-(\delta_0 - \nu)x}}{\frac{\nu}{1-\lambda_0} \left[ (1+x^*)^{-\lambda_0+1} - 1 \right] + 1 - (1+x^*)^{-\lambda_0} + \frac{1}{(1+x^*)^{\lambda_0}} \frac{\delta_0}{\delta_0 - \nu}}, \tag{34}$$

and for  $x \leq x^*$ , we have

$$V(x) = \frac{\frac{\nu}{1-\lambda_0} \left[ (1+x^*)^{-\lambda_0+1} - (1+x)^{-\lambda_0+1} \right] + (1+x)^{-\lambda_0} - (1+x^*)^{-\lambda_0} + \frac{1}{(1+x^*)^{\lambda_0}} \frac{\delta_0}{\delta_0 - \nu}}{\frac{\nu}{1-\lambda_0} \left[ (1+x^*)^{-\lambda_0+1} - 1 \right] + 1 - (1+x^*)^{-\lambda_0} + \frac{1}{(1+x^*)^{\lambda_0}} \frac{\delta_0}{\delta_0 - \nu}}. \tag{35}$$

### 3.2. The State-Independent Case

In this section, we consider the state-independent case, that is,

$$\rho(\cdot) \equiv \rho, \quad \lambda(\cdot) \equiv \lambda, \tag{36}$$

and

$$F(\cdot, c) \equiv F(c), \tag{37}$$

where  $\rho, \lambda > 0$  and  $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is increasing. Under the assumptions (36) and (37), we have the following result, which is a corollary of Theorem 1 and the ruin probability for the state-independent dual risk model (Equation (4)).

**Theorem 3.** *The optimal strategy  $C^*$  is constant, given by*

$$C^* = \arg \min_{C \geq 0} \frac{\rho + C}{F(C)}, \tag{38}$$

provided that the minimum exists and the minimized ruin probability is  $V(x) = e^{-\beta x}$ , where

$$(\rho + C^*)\beta + F(C^*) \int_0^\infty [e^{-\beta y} - 1]p(y)dy = 0. \tag{39}$$

**Proof of Theorem 3.** Under the assumptions (36) and (37), it follows from Theorem 1 that the optimal strategy  $C^*$  is constant, which is given by  $C^* = \arg \min_{C \geq 0} \frac{\rho + C}{F(C)}$ . With the optimal  $C^*$ , we have

$$dX_t = -(\rho + C^*)dt + dJ_t, \tag{40}$$

where  $J_t = \sum_{i=1}^{N_t} Y_i$  is compound Poisson, where  $N_t$  is Poisson with intensity  $F(C^*)$ .

By the formula for the ruin probability for the state-independent dual risk model, see, e.g., Equation (4), we have  $V(x) = e^{-\beta x}$ , where  $\beta$  satisfies the equation (39). This completes the proof.  $\square$

### 3.3. A State-Independent Example

In this section, we consider a state-independent example, that is,

$$\rho(\cdot) \equiv \rho, \quad \lambda(\cdot) \equiv \lambda, \tag{41}$$

and

$$F(x, c) = \lambda + \delta c^\gamma, \quad \delta, \gamma > 0. \tag{42}$$

In this special case, by Theorem 3, the optimal strategy  $C^*$  is constant and given by

$$C^* = \arg \min_{C \geq 0} \frac{\rho + C}{\lambda + \delta C^\gamma}. \tag{43}$$

By following the discussions in the more general state-dependent case in Section 3.1, the case  $\gamma \geq 1$  is trivial and in the rest we only consider the cases  $0 < \gamma < 1$  and  $\gamma = 1$ .

#### 3.3.1. The $0 < \gamma < 1$ Case

We first consider the case that  $0 < \gamma < 1$ . In this case, the intensity  $F(X_t, C_t) = \lambda + \delta C_t^\gamma$  is a concave and increasing function of  $C_t$ . What this indicates is that the initial investment of research and development can boost the prospect of future profits, but the margin decreases with the increase of the investment.

When it is allowed to invest in research and development, we will see later that the condition

$$(\rho - \lambda \mathbb{E}[Y_1]) - (\delta \gamma)^{\frac{1}{1-\gamma}} \left( \frac{1}{\gamma} - 1 \right) (\mathbb{E}[Y_1])^{\frac{1}{1-\gamma}} < 0 \tag{44}$$

is sufficient to guarantee that  $V(x) < 1$ . Note that this is weaker than the usual condition  $\rho - \lambda \mathbb{E}[Y_1] < 0$  for the dual risk model. We have the following result.

**Proposition 3.** Under the assumption (44),

$$V(x) = \min_{C \in \mathcal{C}} \mathbb{P}(\tau^C < \infty | X_0^C = x) = e^{-\beta x}, \tag{45}$$

where  $\beta$  is the unique positive value that satisfies the equation:

$$\begin{aligned} & \beta \left[ \rho + \left( \frac{1}{\delta \gamma} \right)^{\frac{1}{\gamma-1}} \left( \frac{\beta}{1 - \int_0^\infty e^{-\beta y} p(y) dy} \right)^{\frac{1}{\gamma-1}} \right] \\ & - \left[ \lambda + \delta \left( \frac{1}{\delta \gamma} \right)^{\frac{\gamma}{\gamma-1}} \left( \frac{\beta}{1 - \int_0^\infty e^{-\beta y} p(y) dy} \right)^{\frac{\gamma}{\gamma-1}} \right] \left( 1 - \int_0^\infty e^{-\beta y} p(y) dy \right) = 0, \end{aligned} \tag{46}$$

and the optimal strategy is given by

$$C^* = \left(\frac{1}{\delta\gamma}\right)^{\frac{1}{\gamma-1}} \left(\frac{\beta}{1 - \int_0^\infty e^{-\beta y} p(y) dy}\right)^{\frac{1}{\gamma-1}}, \tag{47}$$

which also satisfies the following equation:

$$\lambda + (1 - \gamma)\delta(C^*)^\gamma = \rho\delta\gamma(C^*)^{\gamma-1}. \tag{48}$$

**Proof of Proposition 3.** It follows from Theorem 3 that the optimal strategy is given by

$$C^* = \left(\frac{1}{\delta\gamma}\right)^{\frac{1}{\gamma-1}} \left(\frac{\beta}{1 - \int_0^\infty e^{-\beta y} p(y) dy}\right)^{\frac{1}{\gamma-1}}, \tag{49}$$

and the minimized ruin probability  $V(x)$  satisfies Equation (46).

To show that (46) has a unique positive solution, it is equivalent to show that  $F(\beta) = 0$  has a unique positive solution where

$$F(\beta) := \beta \left[ \rho - (\delta\gamma)^{\frac{1}{1-\gamma}} \left(\frac{1}{\gamma} - 1\right) [g(\beta)]^{\frac{1}{1-\gamma}} - \lambda g(\beta) \right], \tag{50}$$

and

$$g(\beta) := \frac{1 - \int_0^\infty e^{-\beta y} p(y) dy}{\beta}. \tag{51}$$

It is easy to compute that for  $\beta > 0$ ,

$$g'(\beta) = \frac{1}{\beta^2} \int_0^\infty [\beta y e^{-\beta y} - 1 + e^{-\beta y}] p(y) dy. \tag{52}$$

Let  $h(x) := xe^{-x} - 1 + e^{-x}$ ,  $x \geq 0$ . Then,  $h(0) = 0$  and  $h(x) \rightarrow -1$  as  $x \rightarrow \infty$ . Moreover,  $h'(x) = -xe^{-x} < 0$  for  $x > 0$ . Thus,  $h(x) \leq 0$  for any  $x \geq 0$  and therefore,  $g'(\beta) \leq 0$  for any  $\beta > 0$  and  $g(\beta)$  is a decreasing function of  $\beta$ .

Note that  $F(\beta) = 0$  for  $\beta > 0$  if and only if  $G(\beta) = 0$  for  $\beta > 0$ , where

$$G(\beta) := \rho - (\delta\gamma)^{\frac{1}{1-\gamma}} \left(\frac{1}{\gamma} - 1\right) [g(\beta)]^{\frac{1}{1-\gamma}} - \lambda g(\beta). \tag{53}$$

Note that by L'Hôpital's rule,  $\lim_{\beta \rightarrow 0^+} g(\beta) = \mathbb{E}[Y_1]$ . Therefore,

$$\lim_{\beta \rightarrow 0^+} G(\beta) = (\rho - \lambda \mathbb{E}[Y_1]) - (\delta\gamma)^{\frac{1}{1-\gamma}} \left(\frac{1}{\gamma} - 1\right) (\mathbb{E}[Y_1])^{\frac{1}{1-\gamma}} < 0. \tag{54}$$

On the other hand,  $g(\beta) \rightarrow 0$  as  $\beta \rightarrow \infty$ ; therefore,  $G(\beta) \rightarrow \rho > 0$  as  $\beta \rightarrow \infty$ . Since  $g(\beta)$  is a decreasing function in  $\beta$  and  $0 < \gamma < 1$ , it follows that  $G(\beta)$  is increasing in  $\beta$ . Hence, we conclude that  $G(\beta) = 0$  has a unique positive solution. This completes the proof.  $\square$

In the following example, we show that when  $Y_i$  are exponentially distributed, we are able to compute out  $\beta$  and  $C^*$  in simple closed forms.

**Example 3.** When  $p(y) = \nu e^{-\nu y}$ ,  $\nu > 0$ ,  $\beta$  satisfies

$$\beta \left[ \rho + \left(\frac{1}{\delta\gamma}\right)^{\frac{1}{\gamma-1}} (\beta + \nu)^{\frac{1}{\gamma-1}} \right] = \left[ \lambda + \delta \left(\frac{1}{\delta\gamma}\right)^{\frac{\gamma}{\gamma-1}} (\beta + \nu)^{\frac{\gamma}{\gamma-1}} \right] \frac{\beta}{\beta + \nu}, \tag{55}$$

which implies that

$$\rho(\beta + \nu) = \lambda + \left(\frac{1}{\gamma} - 1\right) \left(\frac{1}{\delta\gamma}\right)^{\frac{1}{\gamma-1}} (\beta + \nu)^{\frac{\gamma}{\gamma-1}}. \tag{56}$$

In particular, when  $\gamma = \frac{1}{2}$ , we obtain  $\rho(\beta + \nu)^2 = \lambda(\beta + \nu) + \frac{\delta^2}{4}$ , which implies  $\beta = \frac{\lambda + \sqrt{\lambda^2 + \rho\delta^2}}{2\rho} - \nu$ , and thus, the optimal  $C^*$  is given by

$$C^* = \frac{\delta^2\rho^2}{(\lambda + \sqrt{\lambda^2 + \rho\delta^2})^2}. \tag{57}$$

**Remark 2.** We have already shown in Proposition 3 that  $V(x) = e^{-\beta x}$ , where  $\beta$  is the unique positive solution to Equation (46) and that it is equivalent to

$$\rho - (\delta\gamma)^{\frac{1}{1-\gamma}} \left(\frac{1}{\gamma} - 1\right) [g(\beta)]^{\frac{1}{1-\gamma}} - \lambda g(\beta) = 0, \tag{58}$$

where  $g(\beta)$  is defined in (51). Now, let us discuss how the value  $\beta$  (and hence the value function  $V(x) = e^{-\beta x}$ ) and the optimal investment rate  $C^*$  depend on the parameters  $\rho$ ,  $\lambda$  and  $\delta$ . By (58), we have the following observations:

(i) As  $\rho$  increases,  $g(\beta)$  increases. Since  $g(\beta)$  is decreasing in  $\beta$ , we conclude that  $\beta$  decreases as  $\rho$  increases. Intuitively, this means that as the fixed running cost for research and investment increases, the ruin probability increases. Asymptotically, as  $\rho \rightarrow 0$ ,  $g(\beta) \rightarrow 0$ . When  $g(\beta) \rightarrow 0$ , since  $0 < \gamma < 1$ , we must have  $[g(\beta)]^{\frac{1}{1-\gamma}} \ll g(\beta)$ . Therefore, by (58), as  $\rho \rightarrow 0$ , we have  $g(\beta) \sim \frac{\rho}{\lambda}$ . From the definition of  $g(\beta)$ , we have  $g(\beta) \sim \frac{1}{\beta}$  as  $\beta \rightarrow \infty$ . Hence, we conclude that  $\beta \sim \frac{\lambda}{\rho}$ , as  $\rho \rightarrow 0$ . Therefore, the optimal  $C^*$  satisfies

$$C^* \sim (\delta\gamma)^{\frac{1}{1-\gamma}} \left(\frac{\rho}{\lambda}\right)^{\frac{1}{1-\gamma}}, \quad \text{as } \rho \rightarrow 0. \tag{59}$$

(ii) As  $\delta$  increases,  $g(\beta)$  decreases. Since  $g(\beta)$  is decreasing in  $\beta$ , we conclude that  $\beta$  increases as  $\delta$  increases. Intuitively, this indicates that if the prospect of future profits given the investment in research and development increases, then the ruin probability decreases. Asymptotically, as  $\delta \rightarrow \infty$ , we have  $g(\beta) \rightarrow 0$ , and thus,  $(\delta\gamma)^{\frac{1}{1-\gamma}} \left(\frac{1}{\gamma} - 1\right) [g(\beta)]^{\frac{1}{1-\gamma}} \rightarrow \rho$ , which implies that as  $\delta \rightarrow \infty$ , we have  $g(\beta) \sim \frac{\rho^{1-\gamma}}{\gamma\delta} \left(\frac{1}{\gamma} - 1\right)^{\gamma-1}$ . Since  $g(\beta) \sim \frac{1}{\beta}$  as  $\beta \rightarrow \infty$ , we conclude that  $\beta \sim \frac{\gamma\delta}{\rho^{1-\gamma}} \left(\frac{1}{\gamma} - 1\right)^{1-\gamma}$ , as  $\delta \rightarrow \infty$ . Moreover, the optimal  $C^*$  satisfies:

$$C^* \rightarrow \frac{\rho}{\frac{1}{\gamma} - 1}, \quad \text{as } \delta \rightarrow \infty. \tag{60}$$

Now, if  $\delta \rightarrow 0$ , then  $g(\beta) \rightarrow \frac{\rho}{\lambda}$ . Therefore, as  $\delta \rightarrow 0$ ,  $\beta \rightarrow \alpha$ , where we recall that  $\alpha$  is the unique positive value so that  $1 - \int_0^\infty e^{-\alpha y} p(y) dy = \alpha \frac{\rho}{\lambda}$ , which is the same as defined in (4). Moreover, the optimal  $C^*$  satisfies

$$C^* \sim (\delta\gamma)^{\frac{1}{1-\gamma}} \left(\frac{\rho}{\lambda}\right)^{\frac{1}{1-\gamma}}, \quad \text{as } \delta \rightarrow 0. \tag{61}$$

Intuitively, it says that as  $\delta \rightarrow 0$ , there is no value investing in research and development.

(iii) Similarly, as  $\lambda$  increases,  $\beta$  increases, and the ruin probability decreases. As  $\lambda \rightarrow \infty$ , we have  $g(\beta) \rightarrow 0$ . Thus,  $\lambda g(\beta) \rightarrow \rho$ , and  $g(\beta) \sim \frac{\rho}{\lambda}$ . Since  $g(\beta) \sim \frac{1}{\beta}$  as  $\beta \rightarrow \infty$ , we conclude that  $\beta \sim \frac{\lambda}{\rho}$ , as  $\lambda \rightarrow \infty$ . Moreover, the optimal  $C^*$  satisfies:

$$C^* \sim (\delta\gamma)^{\frac{1}{1-\gamma}} \left(\frac{\rho}{\lambda}\right)^{\frac{1}{1-\gamma}}, \quad \text{as } \lambda \rightarrow \infty. \tag{62}$$

(iv) Assume that the parameters are chosen so that

$$(\rho - \lambda \mathbb{E}[Y_1]) - (\delta\gamma)^{\frac{1}{1-\gamma}} \left(\frac{1}{\gamma} - 1\right) (\mathbb{E}[Y_1])^{\frac{1}{1-\gamma}} \rightarrow 0. \tag{63}$$

Then, it follows that  $g(\beta) \rightarrow \mathbb{E}[Y_1]$  and  $\beta \rightarrow 0$ . More precisely, as  $\beta \rightarrow 0$ ,  $g(\beta) \sim \mathbb{E}[Y_1] - \frac{\beta}{2} \mathbb{E}[Y_1^2]$  if  $\mathbb{E}[Y_1^2] < \infty$ , and (58) becomes

$$\rho - (\delta\gamma)^{\frac{1}{1-\gamma}} \left(\frac{1}{\gamma} - 1\right) \left(\mathbb{E}[Y_1] - \frac{\beta}{2} \mathbb{E}[Y_1^2]\right)^{\frac{1}{1-\gamma}} - \lambda \left(\mathbb{E}[Y_1] - \frac{\beta}{2} \mathbb{E}[Y_1^2]\right) = O(\beta^2), \tag{64}$$

as  $\beta \rightarrow 0$ . Then, it follows that

$$\begin{aligned} \rho - (\delta\gamma)^{\frac{1}{1-\gamma}} \left(\frac{1}{\gamma} - 1\right) \left(\mathbb{E}[Y_1]^{\frac{1}{1-\gamma}} - \frac{1}{2(1-\gamma)} (\mathbb{E}[Y_1])^{\frac{\gamma}{1-\gamma}} \mathbb{E}[Y_1^2] \beta\right) \\ - \lambda \left(\mathbb{E}[Y_1] - \frac{\beta}{2} \mathbb{E}[Y_1^2]\right) = O(\beta^2), \end{aligned} \tag{65}$$

as  $\beta \rightarrow 0$ . Hence, we conclude that

$$\beta \sim \frac{-(\rho - \lambda \mathbb{E}[Y_1]) + (\delta\gamma)^{\frac{1}{1-\gamma}} \left(\frac{1}{\gamma} - 1\right) (\mathbb{E}[Y_1])^{\frac{1}{1-\gamma}}}{(\delta\gamma)^{\frac{1}{1-\gamma}} \frac{1}{2\gamma} (\mathbb{E}[Y_1])^{\frac{\gamma}{1-\gamma}} \mathbb{E}[Y_1^2] + \frac{\lambda}{2} \mathbb{E}[Y_1^2]}. \tag{66}$$

Moreover, the optimal  $C^*$  satisfies:

$$C^* \sim (\delta\gamma)^{\frac{1}{1-\gamma}} (\mathbb{E}[Y_1])^{\frac{1}{1-\gamma}}. \tag{67}$$

**Remark 3.** The value function  $V(x) = e^{-\beta x}$  and the optimal investment rate  $C^*$  also depend on the parameter  $\gamma$ . We will study the  $\gamma = 1$  case in details later. For the moment, let us try to understand the asymptotic behavior of the value function and the optimal investment rate as  $\gamma \rightarrow 1^-$ . We will also obtain the asymptotics as  $\gamma \rightarrow 0^+$ . Let us recall that the optimal  $C^*$  satisfies the equation:

$$\lambda + (1 - \gamma)\delta(C^*)^\gamma = \rho\delta\gamma(C^*)^{\gamma-1}. \tag{68}$$

Thus, we have  $(1 - \gamma)\delta(C^*)^\gamma \leq \rho\delta\gamma(C^*)^{\gamma-1}$  which implies that  $C^* \leq \frac{\rho\gamma}{1-\gamma}$ . Thus,  $C^* \rightarrow 0$  as  $\gamma \rightarrow 0$ . Note that  $\lim_{\gamma \rightarrow 0^+} \gamma^\gamma = 1$ . Therefore, we can check that

$$C^* \sim \frac{\rho\delta}{\lambda + \delta} \gamma, \quad \text{as } \gamma \rightarrow 0^+. \tag{69}$$

Now, let us consider the  $\gamma \rightarrow 1^-$  limit. Let us rewrite that Equation (68) as

$$\frac{\lambda}{(1 - \gamma)^{1-\gamma}} + \delta D^\gamma = \frac{\rho\delta\gamma}{D^{1-\gamma}}, \tag{70}$$

where  $D = (1 - \gamma)C^*$ . Let us first consider the case  $\rho\delta > \lambda$ . Notice first that  $\lim_{\gamma \rightarrow 1^-} (1 - \gamma)^{1-\gamma} = 1$ . First,  $D$  cannot go to 0 as  $\gamma \rightarrow 1^-$ , because otherwise, the left-hand side of (70) goes to  $\lambda$  and as  $D$  goes to 0,  $D < 1$  and  $D^{1-\gamma} \leq 1$ , so the right-hand side of (70) is greater than  $\rho\delta\gamma$ . Then, in the limit as  $\gamma \rightarrow 1^-$ , we obtain  $\lambda \geq \rho\delta$ , which is a contradiction. Second,  $D$  cannot go to  $\infty$  as  $\gamma \rightarrow 1^-$ . To see this, notice that as  $D \rightarrow \infty$ , the left hand side of (70) goes to  $\infty$  and in the right-hand side of (70), for large  $D$ ,  $D > 1$  and  $D^{1-\gamma} \geq 1$  and hence, the right-hand side is less than  $\rho\delta$ , which is a contradiction.

Therefore, if  $\rho\delta > \lambda$ ,  $D$  converges to a positive constant, which from (70) we can see that the limit is  $\frac{\rho\delta - \lambda}{\delta}$ , and we have

$$C^* \sim \frac{\rho\delta - \lambda}{\delta} \frac{1}{1 - \gamma}, \quad \text{as } \gamma \rightarrow 1^- \tag{71}$$

If  $\rho\delta < \lambda$ , then the optimal  $C^* \rightarrow 0$  as  $\gamma \rightarrow 1^-$ . To see this, notice that if  $\limsup_{\gamma \rightarrow 1^-} C^* \in (0, \infty)$ , then in (68), we have  $\limsup_{\gamma \rightarrow 1^-} \rho\delta\gamma(C^*)^{\gamma-1} = \rho\delta$  and  $\limsup_{\gamma \rightarrow 1^-} [\lambda + (1 - \gamma)\delta(C^*)^\gamma] = \lambda$ , which is a contradiction since  $\rho\delta < \lambda$ . If  $\limsup_{\gamma \rightarrow 1^-} C^* = \infty$ , then for  $C^* > 1$ , we have from (68) that  $\lambda < \lambda + (1 - \gamma)\delta(C^*)^\gamma = \rho\delta\gamma(C^*)^{\gamma-1} < \rho\delta$ , which is again a contradiction. Hence, we must have  $C^* \rightarrow 0$ .

Since  $C^* \rightarrow 0$ ,  $(1 - \gamma)\delta(C^*)^\gamma \ll \rho\delta\gamma(C^*)^{\gamma-1}$ , and thus

$$C^* \sim \left(\frac{\lambda}{\rho\delta\gamma}\right)^{\frac{1}{\gamma-1}} \sim \frac{1}{e} \left(\frac{\rho\delta}{\lambda}\right)^{\frac{1}{1-\gamma}}, \quad \text{as } \gamma \rightarrow 1^- \tag{72}$$

If  $\rho\delta = \lambda$ , the optimal  $C^*$  satisfies the equation:

$$\lambda = \frac{(1 - \gamma)\delta(C^*)^\gamma}{\gamma(C^*)^{\gamma-1} - 1} \tag{73}$$

Assume that  $C^* > 0$  is fixed, then by L'Hôpital's rule,

$$\lim_{\gamma \rightarrow 1^-} \frac{(1 - \gamma)\delta(C^*)^\gamma}{\gamma(C^*)^{\gamma-1} - 1} = \lim_{\gamma \rightarrow 1^-} \frac{-\delta(C^*)^\gamma + (1 - \gamma)\delta(C^*)^\gamma \log C^*}{(C^*)^{\gamma-1} + \gamma(C^*)^{\gamma-1} \log C^*} = \frac{-\delta C^*}{1 + \log C^*} \tag{74}$$

Therefore, as  $\gamma \rightarrow 1^-$ ,  $C^*$  converges to the unique positive solution to the equation:  $\delta x + \lambda(1 + \log x) = 0$ .

### 3.3.2. The $\gamma = 1$ Case

When  $\gamma = 1$ , it follows from Theorem 3 that the optimal strategy  $C^*$  is constant and it is given by

$$C^* = \arg \min_{C \geq 0} \frac{\rho + C}{\lambda + \delta C} \tag{75}$$

When  $\frac{\rho}{\lambda} < \frac{1}{\delta}$ , then  $\inf_{C \geq 0} \frac{\rho + C}{\lambda + \delta C} = \frac{\rho}{\lambda}$  and the optimal strategy is  $C_t \equiv 0$ . In this case, the value function  $V(x) = e^{-\beta x}$ , where

$$\rho\beta + \lambda \int_0^\infty [e^{-\beta y} - 1]p(y)dy = 0 \tag{76}$$

When  $\frac{\rho}{\lambda} > \frac{1}{\delta}$ , then  $\inf_{C \geq 0} \frac{\rho + C}{\lambda + \delta C} = \frac{1}{\delta}$ . Additionally, for any  $C \in \mathcal{C}$  and  $\bar{C} := \|C\|_\infty$ , the strategy  $\bar{C}$  is more optimal than  $C$ . The "optimal strategy" is  $C_t \equiv \infty$ . Let us also assume that  $\delta\mathbb{E}[Y_1] > 1$ . In this case, the value function  $V(x) = e^{-\beta x}$ , where

$$\beta + \delta \int_0^\infty [e^{-\beta y} - 1]p(y)dy = 0 \tag{77}$$

When  $\frac{\rho}{\lambda} = \frac{1}{\delta}$ , in terms of ruin probability, it does not make a difference whether the company decides to invest in research and development or not.

**Remark 4.** When  $\frac{\rho}{\lambda} \geq \frac{1}{\delta}$ ,  $V(x) = e^{-\beta x}$ , where  $\beta$  satisfies (77) that is independent of  $\rho$  and  $\lambda$ . Asymptotically, when  $\frac{\rho}{\lambda} \rightarrow 0$ , it is easy to see that  $\beta \sim \frac{\lambda}{\rho}$ .

**Example 4.** In the special case that  $p(y) = ve^{-vy}$ , when  $\frac{\rho}{\lambda} < \frac{1}{\delta}$ , then the optimal  $C \equiv 0$  and  $V(x) = e^{-(\frac{\lambda}{\rho}-v)x}$ , and when  $\frac{\rho}{\lambda} > \frac{1}{\delta}$  and  $\frac{\delta}{v} > 1$ , then the optimal  $C \equiv \infty$  and  $V(x) = e^{-(\delta-v)x}$ .

#### 4. Investing in a Market Index

We have already studied the optimal investment in research and development for a venture capital or high-tech company in the dual risk model in Section 3, and now, let us also add the possibility of the alternative investment in a risky asset in the market, which is a capital market index modeled by a geometric Brownian motion.

For simplicity, we restrict our discussions to the state-independent case as in Section 3.3:

$$\rho(\cdot) \equiv \rho, \quad \lambda(\cdot) \equiv \lambda, \tag{78}$$

where  $\rho, \lambda > 0$  and

$$F(x, c) = \lambda + \delta c^\gamma, \quad \delta, \gamma > 0. \tag{79}$$

Let us assume that the market index  $S_t$  follows a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{80}$$

where  $\mu, \sigma > 0$  and  $W_t$  is a standard Brownian motion.

Assume that at time  $t$ , the company can invest  $\theta_t$  shares of the market index  $S_t$  and  $C_t$  in research and development. Thus, the wealth process of the company satisfies the dynamics:

$$dX_t = -(\rho + C_t)dt + dJ_t^C + \theta_t dS_t, \quad X_0 = x > 0 \tag{81}$$

The invested amount in the market index is  $A_t = \theta_t S_t$  at time  $t$ .

We are interested in finding optimal investment strategies to minimize the probability of ruin:

$$V(x) := \inf_{C \in \mathcal{C}, A \in \mathcal{A}} \mathbb{P}(\tau < \infty | X_0 = x), \tag{82}$$

where  $\mathcal{C}$  is the same as defined before and  $\mathcal{A}$  is the admissible strategies for investment in the market index, defined as:

$$\mathcal{A} := \left\{ A : [0, \infty) \times \Omega \rightarrow \mathbb{R} : A \text{ is progressively measurable} \right. \tag{83}$$

$$\left. \text{and for any } t > 0, \mathbb{E} \left[ \int_0^t A_s^2 ds \right] < \infty. \right\}.$$

For any given  $C \in \mathcal{C}$  and  $A \in \mathcal{A}$ , we write  $X^{C,A} = X$  to emphasize the dependence on  $C$  and  $A$ .

With additional investment in a market index, the random time change argument in the analysis in Section 3 no longer applies. Instead, we rely on the stochastic optimal control theory (see, e.g., Fleming and Soner 1993), which suggests that the Hamilton–Jacobi–Bellman equation for  $V(x)$  is given by

$$\inf_{C \geq 0, A \in \mathbb{R}} \left\{ -(\rho + C)V'(x) + (\lambda + \delta C^\gamma) \int_0^\infty [V(x + y) - V(x)]p(y)dy \right. \tag{84}$$

$$\left. + A\mu V'(x) + \frac{1}{2}A^2\sigma^2 V''(x) \right\} = 0,$$

with boundary condition  $V(0) = 1$ .

Similar to the case in Section 3, the case  $\gamma \geq 1$  leads to triviality and for the rest, we consider two cases:  $0 < \gamma < 1$  and  $\gamma = 1$ .

4.1. The  $0 < \gamma < 1$  Case

In this section, we consider the  $0 < \gamma < 1$  case. We start with the following technical lemma.

**Lemma 1.**  $V(x) = e^{-\beta x}$  is a solution to the Hamilton–Jacobi–Bellman Equation (84), where  $\beta > 0$  is the unique solution to the equation:

$$\beta \left[ \rho + \left( \frac{1}{\delta\gamma} \right)^{\frac{1}{\gamma-1}} \left( \frac{\beta}{1 - \int_0^\infty e^{-\beta y} p(y) dy} \right)^{\frac{1}{\gamma-1}} \right] - \left[ \lambda + \delta \left( \frac{1}{\delta\gamma} \right)^{\frac{\gamma}{\gamma-1}} \left( \frac{\beta}{1 - \int_0^\infty e^{-\beta y} p(y) dy} \right)^{\frac{\gamma}{\gamma-1}} \right] \left( 1 - \int_0^\infty e^{-\beta y} p(y) dy \right) - \frac{1}{2} \frac{\mu^2}{\sigma^2} = 0. \tag{85}$$

Given  $V(x) = e^{-\beta x}$  and let

$$(C^*, A^*) \in \operatorname{argmin} \left\{ -(\rho + C)V'(x) + (\lambda + \delta C^\gamma) \int_0^\infty [V(x+y) - V(x)]p(y)dy + A\mu V'(x) + \frac{1}{2}A^2\sigma^2 V''(x) \right\}. \tag{86}$$

Then, we have

$$C^* = \left( \frac{1}{\delta\gamma} \right)^{\frac{1}{\gamma-1}} \left( \frac{\beta}{1 - \int_0^\infty e^{-\beta y} p(y) dy} \right)^{\frac{1}{\gamma-1}}, \quad A^* = \frac{\mu}{\sigma^2 \beta}. \tag{87}$$

**Proof of Lemma 1.** Assume that  $V'(x) < 0$  and  $V''(x) > 0$ , then the optimal  $C$  and  $A$  are given respectively by

$$C = \left( \frac{1}{\delta\gamma} \right)^{\frac{1}{\gamma-1}} \left( \frac{V'(x)}{\int_0^\infty [V(x+y) - V(x)]p(y)dy} \right)^{\frac{1}{\gamma-1}}, \quad A = -\frac{\mu V'(x)}{\sigma^2 V''(x)}, \tag{88}$$

and the Hamilton–Jacobi–Bellman equation becomes

$$\begin{aligned} & - \left[ \rho + \left( \frac{1}{\delta\gamma} \right)^{\frac{1}{\gamma-1}} \left( \frac{V'(x)}{\int_0^\infty [V(x+y) - V(x)]p(y)dy} \right)^{\frac{1}{\gamma-1}} \right] V'(x) \\ & + \left[ \lambda + \delta \left( \frac{1}{\delta\gamma} \right)^{\frac{\gamma}{\gamma-1}} \left( \frac{V'(x)}{\int_0^\infty [V(x+y) - V(x)]p(y)dy} \right)^{\frac{\gamma}{\gamma-1}} \right] \\ & \cdot \int_0^\infty [V(x+y) - V(x)]p(y)dy - \frac{1}{2} \frac{\mu^2}{\sigma^2} \frac{(V'(x))^2}{V''(x)} = 0. \end{aligned} \tag{89}$$

We can see that  $V(x) = e^{-\beta x}$ , where  $\beta > 0$  is the unique solution to the equation:

$$\beta \left[ \rho + \left( \frac{1}{\delta\gamma} \right)^{\frac{1}{\gamma-1}} \left( \frac{\beta}{1 - \int_0^\infty e^{-\beta y} p(y) dy} \right)^{\frac{1}{\gamma-1}} \right] - \left[ \lambda + \delta \left( \frac{1}{\delta\gamma} \right)^{\frac{\gamma}{\gamma-1}} \left( \frac{\beta}{1 - \int_0^\infty e^{-\beta y} p(y) dy} \right)^{\frac{\gamma}{\gamma-1}} \right] \left( 1 - \int_0^\infty e^{-\beta y} p(y) dy \right) - \frac{1}{2} \frac{\mu^2}{\sigma^2} = 0. \tag{90}$$

Recall the definition  $g(\beta) = \frac{1}{\beta} [1 - \int_0^\infty e^{-\beta y} p(y) dy]$  and we want to show that the equation

$$H(\beta) := \rho - (\delta\gamma)^{\frac{1}{1-\gamma}} \left( \frac{1}{\gamma-1} \right) [g(\beta)]^{\frac{1}{1-\gamma}} - \lambda g(\beta) - \frac{1}{2} \frac{\mu^2}{\sigma^2} \frac{1}{\beta} = 0 \tag{91}$$

has a unique positive solution. It is easy to see that  $\lim_{\beta \rightarrow 0^+} g(\beta) = \mathbb{E}[Y_1]$  and  $\lim_{\beta \rightarrow \infty} g(\beta) = 0$ . Thus,  $H(\beta) \sim -\frac{1}{2} \frac{\mu^2}{\sigma^2 \beta} < 0$  as  $\beta \rightarrow 0^+$  and  $H(\beta) \rightarrow \rho$  as  $\beta \rightarrow \infty$ . We have already proved that  $g(\beta)$  is decreasing in  $\beta$ . Moreover,  $\frac{1}{\beta}$  is also decreasing in  $\beta$ . Therefore,  $H(\beta)$  is increasing in  $\beta$  and hence, there exists a unique positive value  $\beta$  so that  $H(\beta) = 0$ .

Finally, we can compute that the optimal  $C^*$  and  $A^*$  are given by (87). This completes the proof.  $\square$

**A Verification Theorem**

Let us recall from (84) that the Hamilton–Jacobi–Bellman equation is given by

$$0 = \inf_{C>0, A \in \mathbb{R}} \left\{ -(\rho + C)V'(x) + (\lambda + \delta C^\gamma) \int_0^\infty [V(x+y) - V(x)]p(y)dy + A\mu V'(x) + \frac{1}{2} A^2 \sigma^2 V''(x) \right\}, \tag{92}$$

with boundary condition  $V(0) = 1$ .

**Theorem 4 (Verification).** *If  $w \in C_b^2$  is a solution of (92) with  $w(0) = 1$ , such that for any  $C \in \mathcal{C}$  and  $A \in \mathcal{A}$*

$$\lim_{K \rightarrow \infty} w(K) = 0, \tag{93}$$

*then,  $w \leq V$ . In addition, if*

$$C^*(x) := \left( \frac{1}{\delta\gamma} \right)^{\frac{1}{\gamma-1}} \left( \frac{w'(x)}{\int_0^\infty [w(x+y) - w(x)]p(y)dy} \right)^{\frac{1}{\gamma-1}} \text{ and } A^*(x) = -\frac{\mu w'(x)}{\sigma^2 w''(x)},$$

*are such that*

$$dX_t^* = -(\rho + C^*(X_t^*))dt + dJ_t^{C^*(X_{t-}^*)} + A^*(X_t^*)dS_t$$

*has a solution and  $C^* := C^*(X^*) \in \mathcal{C}$  and  $A^* := A^*(X^*) \in \mathcal{A}$ , then  $w = V$ .*

**Proof of Theorem 4.** We follow the supermartingale argument presented in (Rogers 2013, Theorem 1.1). Since  $w$  is bounded and continuously differentiable with bounded derivative, by Itô lemma for jump processes, we have

$$\begin{aligned} \mathbb{E} \left[ w \left( X_t^{C,A} \right) \middle| \mathcal{F}_s \right] &= w \left( X_s^{C,A} \right) + \mathbb{E} \left[ \int_s^t \left( -(\rho + C_u)w' \left( X_u^{C,A} \right) \right. \right. \\ &\quad \left. \left. + (\lambda + \delta C_u^\gamma) \int_0^\infty \left[ w \left( X_u^{C,A} + y \right) - w \left( X_u^{C,A} \right) \right] p(y) dy \right. \right. \\ &\quad \left. \left. + A_u \mu w' \left( X_u^{C,A} \right) + \frac{1}{2} A_u^2 \sigma^2 w'' \left( X_u^{C,A} \right) \right) du \middle| \mathcal{F}_s \right] \geq w \left( X_s^{C,A} \right), \end{aligned} \tag{94}$$

for any  $C \in \mathcal{C}$  and  $A \in \mathcal{A}$ . Therefore,  $w(X_t^{C,A})$  is a submartingale. Let  $\tau_K$  be the first time that the  $X_t^{C,A}$  process hits  $K > 0$ . Since  $w$  is uniformly bounded, by optional stopping theorem,

$$w(x) \leq \mathbb{E} \left[ w \left( X_{\tau_K \wedge \tau}^{C,A} \right) \right] = \mathbb{E} \left[ w \left( X_{\tau_K}^{C,A} \right) 1_{\{\tau_K < \tau\}} + 1_{\{\tau_K \geq \tau\}} \right] = w(K) \mathbb{P}(\tau_K < \tau) + \mathbb{P}(\tau < \tau_K).$$

It follows from (93) and monotone convergence theorem that the right-hand side above converges to  $\mathbb{P}(\tau < \infty)$  as  $K \rightarrow \infty$  and thus,

$$w(x) \leq \mathbb{P}(\tau < \infty).$$

By taking infimum over  $C \in \mathcal{C}$  and  $A \in \mathcal{A}$ , we obtain  $w \leq V$ . All the above inequalities change to equality for  $C_t = C^*(X_{t-}^*)$  and  $A_t = A^*(X_{t-}^*)$ . This completes the proof.  $\square$

**Corollary 1.**  $w(x) = e^{-\beta x}$  with  $\beta$  defined in (85) satisfies (93) and thus,  $w = V$ .

**Proof of Corollary 1.** We already showed, in Lemma 1, that  $w$  is a classical solution of the boundary value problem (92). Moreover, since  $C^*$  and  $A^*$  defined by (86) are admissible controls (constants). By Theorem 4 and because (93) trivially holds, we have  $V(x) = w(x) = e^{-\beta x}$ . The proof is complete.  $\square$

Next, we provide some asymptotic analysis.

**Remark 5.** As in Remark 2, let us discuss the dependence of  $C^*$ ,  $\beta$  and hence,  $V(x) = e^{-\beta x}$  on the parameters  $\rho$ ,  $\lambda$  and  $\delta$ . Since the results are similar to Remark 2, we omit the details and only summarize the results here. Note that  $\beta$  satisfies

$$\rho - (\delta\gamma)^{\frac{1}{1-\gamma}} \left(\frac{1}{\gamma} - 1\right) [g(\beta)]^{\frac{1}{1-\gamma}} - \lambda g(\beta) - \frac{1}{2} \frac{\mu^2}{\sigma^2} \frac{1}{\beta} = 0, \tag{95}$$

where  $g(\beta)$  is defined in (51).

(i) As  $\rho \rightarrow 0^+$ , we have  $\beta \sim \frac{\lambda + \frac{1}{2} \frac{\mu^2}{\sigma^2}}{\rho}$ , and  $C^* \sim (\delta\gamma)^{\frac{1}{1-\gamma}} \left(\frac{\rho}{\lambda + \frac{1}{2} \frac{\mu^2}{\sigma^2}}\right)^{\frac{1}{1-\gamma}}$ .

(ii) As  $\delta \rightarrow \infty$ , we have  $\beta \sim \frac{\gamma}{\rho^{1-\gamma}} \left(\frac{1}{\gamma} - 1\right)^{1-\gamma} \delta$ , and  $C^* \rightarrow \frac{\rho}{\frac{1}{\gamma} - 1}$ . As  $\delta \rightarrow 0$ , we have  $\beta \rightarrow \alpha$ , where  $\alpha$  is the unique positive value so that

$$\rho\alpha + \lambda \int_0^\infty [e^{-\alpha y} - 1] p(y) dy - \frac{1}{2} \frac{\mu^2}{\sigma^2} = 0. \tag{96}$$

Moreover, as  $\delta \rightarrow 0$ , we have  $C^* \sim (\delta\gamma)^{\frac{1}{1-\gamma}} \left(\frac{1}{\lambda} \left(\rho - \frac{1}{2\alpha} \frac{\mu^2}{\sigma^2}\right)\right)^{\frac{1}{1-\gamma}}$ .

(iii) As  $\lambda \rightarrow \infty$ , we have  $\beta \sim \frac{\lambda}{\rho}$ , and  $C^* \sim (\delta\gamma)^{\frac{1}{1-\gamma}} \left(\frac{\rho}{\lambda}\right)^{\frac{1}{1-\gamma}}$ .

**Remark 6.** Here, we investigate the asymptotic behavior of the value function and the optimal investment rate as  $\gamma \rightarrow 1^-$  and  $\gamma \rightarrow 0^+$ . Note that the optimal  $C^*$  and  $\beta$  satisfy

$$\rho - \left(\frac{1}{\gamma} - 1\right) C^* - \frac{\lambda}{\delta\gamma} (C^*)^{1-\gamma} - \frac{1}{2} \frac{\mu^2}{\sigma^2} \frac{1}{\beta} = 0, \tag{97}$$

and

$$C^* = \left(\frac{1}{\delta\gamma}\right)^{\frac{1}{\gamma-1}} \left(\frac{\beta}{1 - \int_0^\infty e^{-\beta y} p(y) dy}\right)^{\frac{1}{\gamma-1}}. \tag{98}$$

(i) As  $\gamma \rightarrow 0^+$ ,  $C^* \sim \eta\gamma$  for some  $\eta > 0$  and  $\beta \rightarrow \iota$  for some  $\iota > 0$ . It is easy to check that  $\eta, \iota > 0$  satisfy  $\eta = \frac{1 - \int_0^\infty e^{-\iota y} p(y) dy}{\iota}$  and  $\rho - \eta - \frac{\lambda}{\delta} \eta - \frac{1}{2} \frac{\mu^2}{\sigma^2} \frac{1}{\iota} = 0$ . Thus

$$\rho - \left(1 + \frac{\lambda}{\delta}\right) \frac{1 - \int_0^\infty e^{-\iota y} p(y) dy}{\iota} - \frac{1}{2} \frac{\mu^2}{\sigma^2} \frac{1}{\iota} = 0. \tag{99}$$

(ii) Next, let us consider  $\gamma \rightarrow 1^-$ .

If  $\delta \mathbb{E}[Y_1] > 1$ , then there exists a unique value  $\iota > 0$  such that  $\delta = \frac{\iota}{1 - \int_0^\infty e^{-iy} p(y) dy}$ . Assume further that  $\rho - \frac{\lambda}{\delta} - \frac{1}{2} \frac{\mu^2}{\sigma^2 \iota} > 0$ . Then, we have  $C^* \sim \frac{\eta}{1-\gamma}$  and  $\beta \rightarrow \iota$  as  $\gamma \rightarrow 1^-$ , where  $\eta = \rho - \frac{\lambda}{\delta} - \frac{1}{2} \frac{\mu^2}{\sigma^2 \iota}$ .

If  $\rho - \frac{\lambda}{\delta} - \frac{1}{2} \frac{\mu^2}{\sigma^2 \iota} < 0$ , the optimal  $C^* \rightarrow 0$  as  $\gamma \rightarrow 1^-$  and  $C^* \sim \left( \frac{\delta \gamma}{\lambda} \left( \rho - \frac{1}{2} \frac{\mu^2}{\sigma^2} \frac{1}{\beta} \right) \right)^{\frac{1}{1-\gamma}}$  and  $\beta \rightarrow \iota$  as  $\gamma \rightarrow 1^-$ . We can check that  $\eta, \iota$  satisfy the equations  $\eta = \frac{\lambda}{\delta \left( \rho - \frac{1}{2} \frac{\mu^2}{\sigma^2 \iota} \right)}$  and

$$\frac{\iota}{\delta} = 1 - \int_0^\infty e^{-iy} p(y) dy. \text{ As } \gamma \rightarrow 1^-, \text{ we have } C^* \sim \frac{1}{e} \left( \frac{\delta}{\lambda} \left( \rho - \frac{1}{2} \frac{\mu^2}{\sigma^2 \iota} \right) \right)^{\frac{1}{1-\gamma}}.$$

If  $\rho - \frac{\lambda}{\delta} - \frac{1}{2} \frac{\mu^2}{\sigma^2 \iota} = 0$ , then, as  $\gamma \rightarrow 1^-$ , we have that  $C^*$  converges to the unique positive solution to the equation  $\delta x + \lambda(1 + \log x) = 0$ .

#### 4.2. The $\gamma = 1$ Case

Consider the case where  $\gamma = 1$ , i.e., for  $x > 0$ . Then, we have a singular control problem on  $C \in \mathcal{C}$  (see, e.g., Fleming and Soner 1993) and the value function  $V(x)$  satisfies the Hamilton–Jacobi–Bellman equation:

$$0 = \min \left\{ -\rho V'(x) + \lambda \int_0^\infty [V(x+y) - V(x)] p(y) dy + \inf_{A \in \mathbb{R}} \left\{ A \mu V'(x) + \frac{1}{2} A^2 \sigma^2 V''(x) \right\}, \right. \\ \left. \delta \int_0^\infty [V(x+y) - V(x)] p(y) dy - V'(x) \right\}, \tag{100}$$

with boundary condition  $V(0) = 1$ . Optimizing over  $A$ , it reduces to the following equation:

$$0 = \min \left\{ -\rho V'(x) + \lambda \int_0^\infty [V(x+y) - V(x)] p(y) dy - \frac{\mu^2 (V')^2}{2\sigma^2 V''}, \right. \\ \left. \delta \int_0^\infty [V(x+y) - V(x)] p(y) dy - V'(x) \right\}, \tag{101}$$

with boundary condition  $V(0) = 1$ .

For  $w \in C_b^2$ , we define

$$\mathcal{P} := \left\{ x \in \mathbb{R}_+ : \delta \int_0^\infty [w(x+y) - w(x)] p(y) dy - w'(x) > 0 \right\}.$$

According to (Fleming and Soner 1993, Chapter 8),  $w$  is a classical solution of (101) if

(i) On  $\mathcal{P}$ ,  $w$  satisfies

$$0 = -\rho w'(x) + \lambda \int_0^\infty [w(x+y) - w(x)] p(y) dy - \frac{\mu^2 (w')^2}{2\sigma^2 w''}.$$

(ii) On  $\mathbb{R}_+$ ,  $w$  satisfies

$$0 \leq -\rho w'(x) + \lambda \int_0^\infty [w(x+y) - w(x)] p(y) dy - \frac{\mu^2 (w')^2}{2\sigma^2 w''}, \\ 0 \leq \delta \int_0^\infty [w(x+y) - w(x)] p(y) dy - w'(x). \tag{102}$$

(iii)  $w(0) = 1$ .

**Lemma 2.**  $w(x) = e^{-(\beta_1 \vee \beta_2)x}$  is a classical solution of (101) where  $\beta_1$  is the unique positive solutions of  $F(\beta) = 0$  and  $\beta_2$  is the unique positive solution of  $G(\beta) = 0$  if it exists or zero otherwise. Here,  $F$  and  $G$  are given by

$$F(\beta) := \rho\beta + \lambda \int_0^\infty [e^{-\beta y} - 1]p(y)dy - \frac{1}{2} \frac{\mu^2}{\sigma^2},$$

$$G(\beta) := \beta + \delta \int_0^\infty [e^{-\beta y} - 1]p(y)dy.$$

**Proof of Lemma 2.** If  $G'(0) = 1 - \delta\mathbb{E}[Y_1] \geq 0$ , then  $\beta_2 = 0$  and  $G(\beta_1) > 0$ . This implies that  $\mathcal{P} = \mathbb{R}_+$ . By straightforward calculations,

$$-\rho w'(x) + \lambda \int_0^\infty [w(x+y) - w(x)]p(y)dy - \frac{\mu^2(w')^2}{2\sigma^2 w''} = wF(\beta_1) = 0,$$

$$\delta \int_0^\infty [w(x+y) - w(x)]p(y)dy - w'(x) = wG(\beta_1) > 0.$$

If  $G'(0) = 1 - \delta\mathbb{E}[Y_1] < 0$  and  $\beta_1 > \beta_2$ , then  $G(\beta_1) > 0$  and we have  $\mathcal{P} = \mathbb{R}_+$ . Similar to the previous paragraph, we obtain that  $w$  is a classical solution. If  $G'(0) = 1 - \delta\mathbb{E}[Y_1] < 0$  and  $\beta_1 \leq \beta_2$ , then  $F(\beta_2) \geq 0$  and we have  $\mathcal{P} = \emptyset$ . Thus,

$$-\rho w'(x) + \lambda \int_0^\infty [w(x+y) - w(x)]p(y)dy - \frac{\mu^2(w')^2}{2\sigma^2 w''} = wF(\beta_2) \geq 0,$$

$$\delta \int_0^\infty [w(x+y) - w(x)]p(y)dy - w'(x) = wG(\beta_2) = 0.$$

The proof is complete.  $\square$

A Verification Theorem

**Theorem 5 (Verification).** Let  $w \in C_b^2$  be a decreasing classical solution of problem (101) such that condition (93) holds. Then,  $w(x) \leq V(x)$ , where  $V(x)$  is the value function of the ruin probability minimization problem with investment. In addition, if  $\mathcal{P} = \mathbb{R}_+$ , then  $w(x) = V(x)$ .

**Proof of Theorem 5.** Let  $A = \{A_s\}_{s \geq 0}$  be an admissible strategy and  $C := \{C_t\}_{t \geq 0}$  be a non-decreasing singular function, i.e.,  $C_t := \int_0^t dc_s$  where  $c_s$  is a non-negative measure. Then,

$$X_t^{C,A} = x - \rho t - C_t + J_t^C + \int_0^t A_s dS_s,$$

where  $J_t^C = \sum_{i=1}^{N_t^C} Y_i$  where  $N_t^C$  is a simple point process with compensator  $\lambda t + \delta C_t$ . Then, by Itô's formula for  $C_b^2$  functions, we have

$$\mathbb{E} \left[ w \left( X_t^{C,A} \right) \middle| \mathcal{F}_s \right] = w \left( X_s^{C,A} \right) + \mathbb{E} \left[ \int_s^t \left( -\rho w' + \lambda \int_0^\infty [w(x+y) - w(x)]p(y)dy \right. \right. \\ \left. \left. + A_u \mu w' + \frac{1}{2} A_u^2 \sigma^2 w'' \right) \left( X_u^{C,A} \right) du \right. \\ \left. + \int_s^t \left( -w' + \delta \int_0^\infty [w(x+y) - w(x)]p(y)dy \right) \left( X_u^{C,A} \right) dC_u^0 \right. \\ \left. + \sum_{s \leq u \leq t} \left( w \left( X_u^{C,A} - \Delta C_u \right) - w \left( X_u^{C,A} \right) \right) \right].$$

Here,  $C_u = C_u^0 + \Delta C_u$  where  $C_u^0$  is the continuous part of  $C$  and  $\Delta C_u$  is the pure jump part of  $C_u$ . Notice that by the definition of classical solution, (102) holds and therefore,

the first two terms inside the expectation above are non-negative. In addition, since  $w$  is non-increasing, we have  $w(X_u^{C,A} - \Delta C_u) - w(X_u^{C,A}) \geq 0$ . Thus,  $\mathbb{E}[w(X_t^{C,A})|\mathcal{F}_s] \geq w(X_s^{C,A})$  and  $w(X_t^{C,A})$  is a submartingale. Similar to the arguments in the proof of Theorem 4, (93) implies that  $w(x) \leq \mathbb{P}(\tau < \infty)$ . By taking the infimum over  $(C, A)$ , we obtain  $w \leq V$ .

Now, assume that  $\mathcal{P} = \mathbb{R}_+$  and set  $C \equiv 0$ . It follows from the definition of  $A^*$  and Itô's formula that

$$\mathbb{E}[w(X_{t \wedge \tau}^*)] = w(x) + \mathbb{E} \left[ \int_0^{t \wedge \tau} \left( -\rho w' + \lambda \int_0^\infty [w(x+y) - w(x)]p(y)dy + A^* \mu w' + \frac{1}{2}(A^*)^2 \sigma^2 w'' \right) (X_s^*) ds \right] = w(x),$$

In the above,  $X^*$  satisfies  $X_t^* = x - \rho t + J_t^\lambda + \int_0^t A^*(X_s^*)dW_s$ . If we let  $t \rightarrow \infty$ , we obtain  $w(x) = \mathbb{P}(\tau^* < \infty) \geq V(x)$  where  $\tau^*$  is the ruin time for process  $X^*$ . The proof is complete.  $\square$

**Corollary 2.** *The classical solution  $w(x) = e^{-(\beta_1 \vee \beta_2)x}$  of boundary value problem (92) satisfies the assumption of the verification and thus,  $w = V$ .*

**Proof of Corollary 2.** First, the condition (93) trivially holds. Therefore, if  $\beta_1 > \beta_2$ , then  $\mathcal{P} = \mathbb{R}_+$  and  $w = V$  is followed by Theorem 5. It remains to show the result for the case that when  $\beta_1 \leq \beta_2$ , i.e.,  $\mathcal{P} = \emptyset$ . For  $c > 0$ , let  $w_c(x) = \mathbb{P}(\tau_c < \infty)$  with  $X_t = x - (\rho + c)t + J_t^c + \int_0^t A^*dW_s$  with  $A^* = \frac{\mu}{\sigma^2 \beta_2}$ . Then, immediately, we obtain  $w_c \geq V$ . We want to show that  $w_c(x) \rightarrow w(x) = e^{-\beta_2 x}$  as  $c \rightarrow \infty$ . Notice that  $w_c$  satisfies the equation

$$0 = -(\rho + c)w'_c(x) + (\lambda + \delta c) \int_0^\infty [w_c(x+y) - w_c(x)]p(y)dy - \frac{\mu^2(w'_c)^2}{2\sigma^2 w''_c},$$

with the boundary condition  $w_c(0) = 1$ . The unique bounded solution of the above equation is given by  $w_c(x) = e^{-\beta(c)x}$  where  $\beta(c)$  satisfies

$$-(\rho + c)\beta(c) + (\lambda + \delta c) \int_0^\infty [e^{-\beta(c)y} - 1]p(y)dy - \frac{\mu^2}{2\sigma^2} = 0. \tag{103}$$

Notice that for any  $c > 0$ ,  $\beta(c)$  is uniquely determined and is continuous on  $c$ . In addition, straightforward calculations show that  $\beta(c)$  is increasing, i.e.,

$$\beta'(c) = \frac{1}{c} \frac{\rho + \lambda \int_0^\infty [1 - e^{-\beta(c)y}]p(y)dy + \frac{\mu^2}{2\sigma^2}}{\rho + c + (\lambda + \delta c) \int_0^\infty e^{-\beta(c)y}yp(y)dy} > 0.$$

Thus,  $\bar{\beta} := \lim_{c \rightarrow \infty} \beta(c)$  exists and  $\bar{\beta} > 0$  and after dividing (103) by  $c$  and taking limit when  $c \rightarrow \infty$ , we obtain

$$G(\bar{\beta}) = -\bar{\beta} + \lambda \int_0^\infty [e^{-\bar{\beta}y} - 1]p(y)dy = 0.$$

Since  $G$  has a unique positive solution, we must have  $\bar{\beta} = \beta_2$  and therefore, we obtain  $V(x) \leq \lim_{c \rightarrow \infty} w_c(x) = e^{-\beta_2 x}$ . This completes the proof.  $\square$

### 5. Numerical Studies

In this section, we carry out numerical studies to illustrate and understand better how the minimized ruin probability and the optimal investment rate depend on the parameters in the dual risk model.

### 5.1. State-Independent Ruin Probability with Optimal Investment

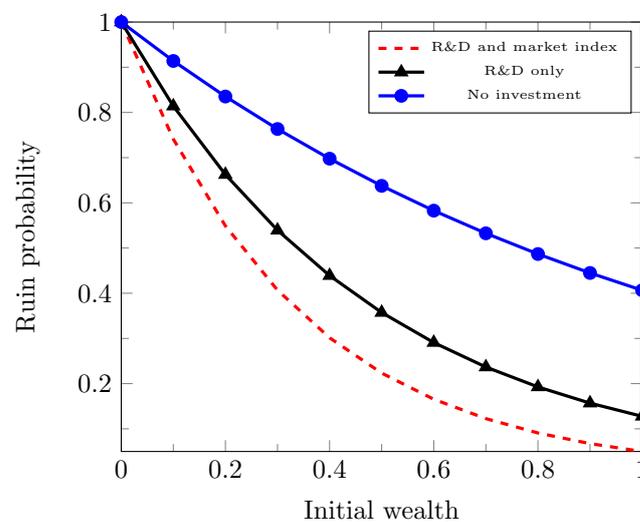
In this section, we assume that the dual risk model is state-independent, and in particular, we assume that  $\rho(\cdot) \equiv \rho$ ,  $\lambda(\cdot) \equiv \lambda$ , and  $F(\cdot, c) \equiv \lambda + \delta c^\gamma$ . We also assume that  $Y_i$  are i.i.d. exponentially distributed so that  $p(y) = \nu e^{-\nu y}$  for some  $\nu > 0$ . We also assume that  $\lambda \mathbb{E}[Y_1] = \frac{\lambda}{\nu} > \rho$  so that the ruin probability is less than 1 without any investment in research and development. Indeed, the ruin probability is given by  $e^{-\alpha x}$ , where, according to (4),  $\alpha$  satisfies the equation:

$$\rho\alpha + \lambda \int_0^\infty [e^{-\alpha y} - 1] \nu e^{-\nu y} dy = \rho\alpha - \lambda \frac{\alpha}{\nu + \alpha} = 0, \tag{104}$$

which implies that  $\alpha = \frac{\lambda}{\rho} - \nu$ .

In Figure 1, we compare the ruin probability without any investment, the minimized ruin probability with investment in research and development, and the minimized ruin probability when investment in both research and development and a market index are allowed. For simplicity, we assume that  $\gamma = \frac{1}{2}$  so that as in Example 3, the minimized ruin probability is  $V(x) = e^{-\beta x}$ , where  $\beta = \frac{\lambda + \sqrt{\lambda^2 + \rho\delta^2}}{2\rho} - \nu$ , and by investing in research and development, it reduces the ruin probability. Now, if additional investment in a risky asset, e.g., a market index is allowed, then the ruin probability can be further reduced and the minimized ruin probability becomes  $V(x) = e^{-\beta x}$ , where by letting  $p(y) = \nu e^{-\nu y}$  and  $\gamma = \frac{1}{2}$  in (85), we deduce that  $\beta > 0$  is the unique solution to the equation

$$\beta\rho - \frac{\beta\delta^2}{4} \frac{1}{(\nu + \beta)^2} - \frac{\lambda\beta}{\nu + \beta} - \frac{1}{2} \frac{\mu^2}{\sigma^2} = 0. \tag{105}$$



**Figure 1.** Illustration of the ruin probability without any investment (blue curve with circle markers), the minimized ruin probability with investment in research and development (black curve with triangle markers), and the minimized ruin probability when investment in both research and development and a market index are allowed (red dashed curve). The  $x$ -axis denotes the initial wealth of the underlying company and the  $y$ -axis denotes the (minimized) ruin probability. Here, we take  $\gamma = \frac{1}{2}$ ,  $\rho = 0.1$ ,  $\nu = 0.1$ ,  $\lambda = 0.1$ ,  $\delta = 1$ ,  $\mu = 0.1$  and  $\sigma = 0.2$ .

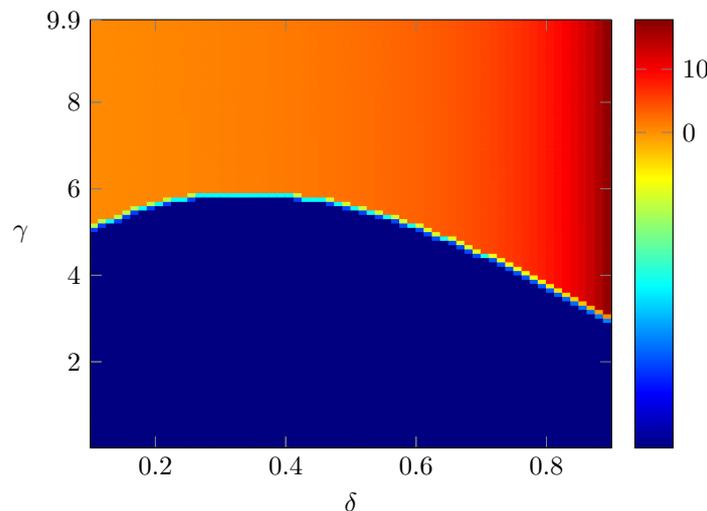
In Figure 2, we investigate the dependence of the optimal  $C^*$  on the parameters  $\gamma$  and  $\delta$  given  $\rho = 2$ ,  $\nu = 2$  and  $\lambda = 0.1$ . Let us recall that when investment in research and development is allowed, the optimal investment rate  $C^*$  is the unique positive solution to the following equation:

$$\lambda + (1 - \gamma)\delta(C^*)^\gamma = \rho\delta\gamma(C^*)^{\gamma-1}. \tag{106}$$

When additional investment in a market index is allowed, the optimal investment rate  $C^*$  for the investment in research and development remains the same. Notice that from (106), the optimal  $C^*$  is independent of the distribution of  $Y_i$ . Therefore, the definition of  $C^*$  is independent of the condition (44) under which the minimized ruin probability is less than 1. Intuitively, that is because  $C^*$  optimizes over the drift term by the random time change technique, but when the condition (44) is violated, even the optimal  $C^*$  still gives the ruin probability equal to 1. In Figure 2, we give the heat map plot of the optimal  $C^*$  as function of  $\gamma$  and  $\delta$ . Note that for  $p(y) = ve^{-\nu y}$  the condition (44) is equivalent to

$$\rho - \frac{\lambda}{\nu} - (\delta\gamma)^{\frac{1}{1-\gamma}} \left( \frac{1}{\gamma} - 1 \right) \frac{1}{\nu^{\frac{1}{1-\gamma}}} < 0. \tag{107}$$

When this condition is violated, then it corresponds to the darker region in the bottom half of the plot in Figure 2. The boundary is achieved when the left-hand side of (107) is zero. In this region, the ruin probability is always 1 regardless of the investment in research and development. When the condition (107) is satisfied, it corresponds to the upper half of the plot in Figure 2. In this region, it is easy to observe that as  $\delta$  increases,  $C^*$  increases. For the plot in Figure 2, the optimal  $C^*$  is less sensitive to the change of the parameter  $\gamma$ .

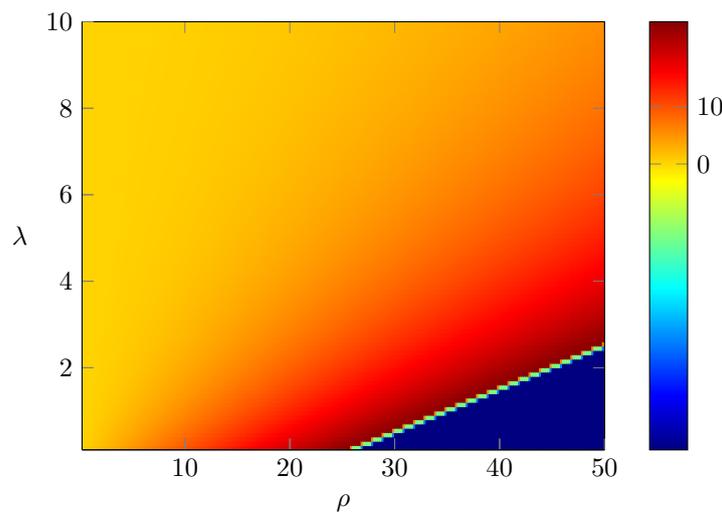


**Figure 2.** This shows  $C^*$  as a function of  $\gamma$  and  $\delta$ . In the darker region in the bottom half of the plot, this is where ruin probability is always 1 regardless of the investment. In the upper half of the plot, the minimized ruin probability is less than 1 and it shows the heat map. Here, we take  $\rho = 2, \nu = 2$  and  $\lambda = 0.1$ .

In Figure 3, we investigate the dependence of the optimal  $C^*$  on the parameters  $\rho$  and  $\lambda$  given  $\delta = 1, \nu = 0.1$  and  $\gamma = \frac{1}{2}$ . For  $\gamma = \frac{1}{2}$ , we showed in Example 3 that the optimal  $C^*$  is given by

$$C^* = \frac{\delta^2 \rho^2}{(\lambda + \sqrt{\lambda^2 + \rho \delta^2})^2}. \tag{108}$$

When  $p(y) = ve^{-\nu y}$  and  $\gamma = \frac{1}{2}$ , the condition (44) reduces to  $\rho - \frac{\lambda}{\nu} - \frac{\delta^2}{4\nu^2} < 0$ . When this condition is violated, the ruin probability is always 1 regardless of the investment and it corresponds to the dark region in the bottom right corner of the plot in Figure 3. When this condition is satisfied, the heat map plot of the optimal  $C^*$  as a function of  $\rho$  and  $\lambda$  is illustrated in Figure 3. We can see that as  $\rho$  increases, the optimal  $C^*$  increases, and as  $\lambda$  increases, the optimal  $C^*$  decreases.



**Figure 3.** This shows  $C^*$  as a function of  $\rho$  and  $\lambda$ . In the darker region in the bottom right corner of the plot, this is where ruin probability is always 1 regardless of the investment. In the rest of the plot, the minimized ruin probability is less than 1. Here, we take  $\nu = 0.1$ ,  $\gamma = 0.5$  and  $\delta = 1$ .

5.2. State-Dependent Ruin Probability with Optimal Investment

In this section, we assume that the dual risk model is state-dependent, and in particular, we assume that  $F(x, c) = \lambda(x) + \delta(x)c^\gamma$ . We also assume that  $Y_i$  are i.i.d. exponentially distributed so that  $p(y) = \nu e^{-\nu y}$  for some  $\nu > 0$ .

First, let us consider a special example in the case of  $0 < \gamma < 1$ . Let us consider the model in Example 1. For simplicity, let us assume that  $\gamma = \frac{1}{2}$ . Recall that in Example 1,  $\rho(x) = \rho_0$ ,  $\lambda(x) = \lambda_0(c_1x + c_2)$ , and  $\delta(x) = \delta_0(c_1x + c_2)$ . The optimal investment rate  $C^*(x) \equiv C_0$  is a constant and is given by:

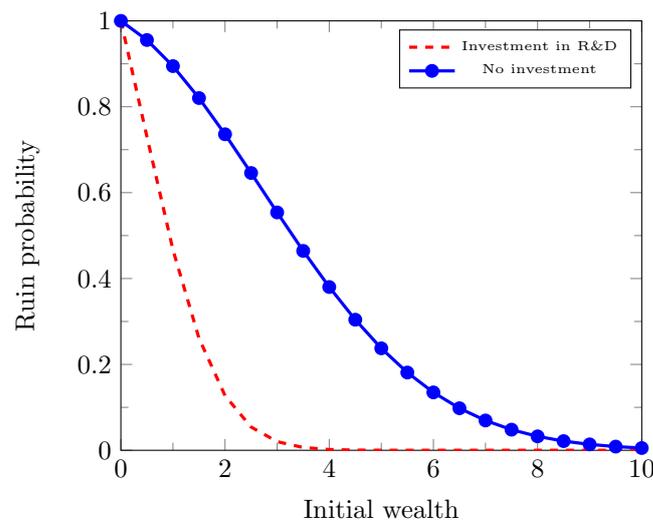
$$C_0 = \frac{\delta_0^2 \rho_0^2}{(\lambda_0 + \sqrt{\lambda_0^2 + \rho_0 \delta_0^2})^2}. \tag{109}$$

The minimized ruin probability is given by

$$\frac{2a\sqrt{d}e^{cx-dx^2} + \sqrt{\pi}e^{\frac{c^2}{4d}}(ac + 2bd)\operatorname{erfc}\left(\frac{2dx-c}{2\sqrt{d}}\right)}{2a\sqrt{d} + \sqrt{\pi}e^{\frac{c^2}{4d}}(ac + 2bd)\operatorname{erfc}\left(\frac{-c}{2\sqrt{d}}\right)}, \tag{110}$$

where  $x$  is the initial wealth,  $a := c_1$ ,  $b := c_2$ ,  $c := \nu - \frac{\lambda_0 + \delta_0 C_0^{1/2}}{\rho_0 + C_0} c_2$ , and  $d := \frac{\lambda_0 + \delta_0 C_0^{1/2}}{\rho_0 + C_0} \frac{c_1}{2}$ . By setting  $C_0 = 0$  in (110), we obtain the ruin probability without any investment in research and development.

In Figure 4, the blue curve with circle markers stands for the ruin probability without investment and the red dashed curve stands for the minimized ruin probability with investment. These two curves differ from exponential decays, which is due to the flexibility of the state-dependent model. As observed in Zhu (2015b), for the state-dependent dual risk model, the ruin probability can have subexponential, exponential and superexponential decays in terms of the initial wealth. Additionally, for the state-dependent dual risk model, the ruin probability may not be convex in the initial wealth (as we can see from the blue curve with circle markers in Figure 4).



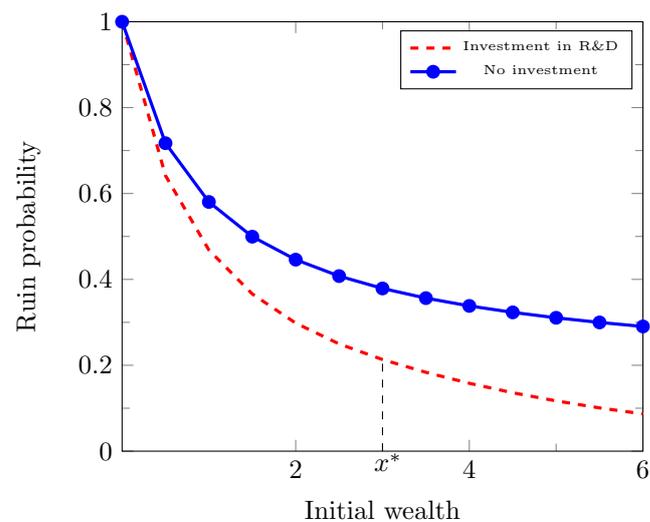
**Figure 4.** Illustration of the ruin probability without any investment (blue curve with circle markers), the minimized ruin probability with investment in research and development (red dashed curve). The  $x$ -axis denotes the initial wealth of the underlying company and the  $y$ -axis denotes the (minimized) ruin probability. Here, we take  $\gamma = 0.5, \rho_0 = 1, \nu = 0.1, \lambda_0 = 0.1, \delta = 1, c_1 = 1$  and  $c_2 = 1$ .

Next, let us consider an example for  $\gamma = 1$  for the state-dependent dual risk model. Let us recall that in Example 2,  $\rho(x) = \rho_0(c_1x + c_2), \lambda(x) = \left(\nu + \frac{\lambda_0}{1+x}\right)\rho(x)$ , and  $\delta(x) = \delta_0$ , and under the assumption that  $\nu < \delta_0 < \nu + \lambda_0$ , the optimal  $C^*$  is given by  $C^* = 0$  if  $x \leq x^*$  and  $C^* = \infty$  if  $x > x^*$ , where  $x^* := \frac{\lambda_0 - \delta_0 + \nu}{\delta_0 - \nu}$ . From Example 2, with optimal investment, the minimized ruin probability is given by  $V(x)$  in (34) if  $x > x^*$  and the minimized ruin probability is given by  $V(x)$  in (35) if  $x \leq x^*$ , where  $x$  is the initial wealth. Without any investment, as in Theorem 2, under the assumption that  $\lambda_0 > 1$ , we can compute that the ruin probability is given by

$$V(x) = \frac{\int_x^\infty \left(\nu + \frac{\lambda_0}{1+y}\right) \frac{1}{(1+y)^{\lambda_0}} dy}{\int_0^\infty \left(\nu + \frac{\lambda_0}{1+y}\right) \frac{1}{(1+y)^{\lambda_0}} dy} = \frac{\nu(1+x)^{-\lambda_0+1} + (\lambda_0 - 1)(1+x)^{-\lambda_0}}{\lambda_0 + \nu - 1}, \tag{111}$$

which is strictly between 0 and 1.

In Figure 5, we plot the curve of the ruin probability as a function of the initial wealth without investment (blue curve with circle markers) and the minimized ruin probability as a function of the initial wealth with the optimal investment in research and development (red dashed curve) as in the example of the state-dependent dual risk model we described above. In Figure 5, the critical threshold for the optimal investment strategy is  $x^* = 3$  in the plot. When the wealth process is below this threshold  $x^*$ , the optimal strategy for investment in R&D is not to invest, and when the wealth process is above this threshold  $x^*$ , the optimal strategy for investment in R&D is to invest as aggressively as possible. When  $x < x^*$ , from (35), we can see that  $V(x)$  decays polynomially in  $x$ , and when  $x > x^*$ , from (34), we can see that  $V(x)$  decays exponentially in  $x$ .



**Figure 5.** Illustration of the ruin probability without any investment (blue curve with circle markers), the minimized ruin probability with investment in research and development (red dashed curve). The  $x$ -axis denotes the initial wealth of the underlying company and the  $y$ -axis denotes the (minimized) ruin probability.  $x^*$  on the  $x$ -axis is the critical threshold above which the optimal strategy is to invest as much as possible in R&D, and below which the optimal strategy is not to invest at all in R&D. Here, we take  $\rho_0 = 1$  (irrelevant),  $\nu = 0.1$ ,  $\lambda_0 = 1.2$ ,  $\delta_0 = 0.4$  and  $c_1 = c_2 = 1$  (irrelevant) and  $\gamma = 1$ .

**Author Contributions:** Conceptualization, A.F. and L.Z.; methodology, A.F. and L.Z.; formal analysis, A.F. and L.Z.; investigation, A.F. and L.Z.; resources, A.F. and L.Z.; writing—original draft preparation, A.F. and L.Z.; writing—review and editing, A.F. and L.Z. All authors have read and agreed to the published version of the manuscript.

**Funding:** Arash Fahim gratefully acknowledges support from the National Science Foundation via the award NSF-DMS-1447067. Lingjiong Zhu is grateful to the support from the National Science Foundation via the awards NSF-DMS-1613164, NSF-DMS-2053454, NSF-DMS-2208303.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** The authors are grateful to two anonymous referees for their helpful comments and suggestions.

**Conflicts of Interest:** The authors declare no conflict of interest.

## Notes

<sup>1</sup> Available at <https://www.macro trends.net/> (accessed on 8 February 2023).

<sup>2</sup> See Supporting innovation and economic growth: The broad impact of the R&D credit in 2005. Prepared by Ernst & Young LLP for the R&D Coalition. April 2008. Available at <https://www.scribd.com/document/207312025/e-y-RatiosR-DTaxCreditStudy2008final> (accessed on 28 December 2022).

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