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Recursive Approaches for Multi-Layer Dividend Strategies in a Phase-Type Renewal Risk Model

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Abstract: In this paper we consider a risk model with two independent classes of insurance risks in the presence of a multi-layer dividend strategy. We assume that both of the claim number processes are renewal processes with phase-type inter-arrival times. By analysing the Markov chains associated with the two given phase-type distributions of the inter-arrival times, algorithmic schemes for the determination of explicit expressions for the Gerber–Shiu expected discounted penalty function, as well as the expected discounted dividend payments are derived, using two different approaches.

Keywords: two-classes of claim; renewal risk processes; multi-layer dividend strategy; phase-type distribution; Gerber–Shiu function; expected discounted dividend payments

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Citation: Papaioannou, Apostolos D., and Lewis Ramsden. 2022. Recursive Approaches for Multi-Layer Dividend Strategies in a Phase-Type Renewal Risk Model. *Risks* 11: 1. https://doi.org/10.3390/risks11010001

Academic Editor: Stéphane Loisel

Received: 1 November 2022
Revised: 25 November 2022
Accepted: 12 December 2022
Published: 20 December 2022

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1. Introduction

Dividend strategies for insurance risk models were first proposed by De Finetti (1957) to describe more realistically the surplus cash flows in insurance portfolios. Most of the dividend strategies within the literature are of the following two kinds: constant barrier strategy or threshold strategy. Strategies involving a single horizontal barrier have been studied by Lin et al. (2003) for the classical compound Poisson risk model and by Li and Garrido (2004) for the renewal generalized Erlang risk model. On the other hand, strategies involving a single dividend threshold have been studied by Lin and Pavlova (2006) for the classical compound Poisson risk model, by Albrecher et al. (2005) for the generalized Erlang renewal risk model and Badescu et al. (2007a) for the Markovian arrival risk model.

The multi-layer dividend strategy, as an extension of the threshold dividend strategy, has been investigated in several papers. For example, Zhou (2006), Lin and Sendova (2008) and Albrecher and Hartinger (2007) considered a multi-layer setting within the framework of the classical risk model, Yang and Zhang (2008) for a generalized Erlang renewal risk model and more recently, Jiang et al. (2012) for a phase-type renewal model. Moreover, Badescu et al. (2007b) consider a general framework for the multi-layer model via a Markovian arrival process for which they derive the Laplace–Stieltjes transform of the distribution of the time to ruin as well as the discounted joint density of the surplus prior and deficit at ruin. Finally, Zhou et al. (2015) fill in some gaps in the paper of Badescu et al. (2007b) by considering a special case of the Markov arrival process in the form of a Markov-dependent risk model in which the claim frequency and severity distributions are influenced by an external Markov chain.

The Gerber–Shiu (G-S) function, first introduced in Gerber and Shiu (1998), and other risk related quantities, such as the moments of the expected dividend payments have been extensively studied for the aforementioned models under the multi-layer dividend strategy based on a *layer-by-layer* recursive approach, for which certain disadvantages have been identified (risk quantities in the top layers must be calculated in order to obtain results for the lower layers). In particular, Badescu and Landriault (2008) use this approach to calculate the moments of the dividend payments within the Markovian arrival risk model.

To overcome the disadvantages of the aforementioned technique, recursive approaches with respect to the number of layers (*number of layers* recursive approach) have been developed. Albrecher and Hartinger (2007) considered the classical risk model and derived a number of layers based scheme for the determination of explicit expressions for the G-S discounted penalty function and the expected discounted dividend payments. To a certain extent, this approach allows one to improve upon computational disadvantages of the layer-by-layer techniques that have been proposed throughout the literature.

In this paper, we consider a multi-layer risk model with two classes of claims, in which the two claim number processes are independent renewal processes with phase-type inter-arrival times (for details on how this can be re-composed to a correlated (dependent) risk model, see Yuen et al. (2002) or Li and Garrido (2005)). Although this risk model lies within the class of Markovian arrival risk models, we aim to fill in some gaps for the G-S function using the layer-by-layer approach and further provide an alternative number of layers based approach for the aforementioned risk quantities.

Consider a surplus process with ν layers $0 = b_0 < b_1 < \cdots < b_\nu = \infty$, such that whenever the surplus is in layer k, i.e., in the interval $[b_{k-1}, b_k)$ for $k = 1, \ldots, \nu$, the insurer collects premiums at a constant rate p_k, with $p_1 > p_2 > \cdots > p_\nu > 0$, and pays dividends at a constant rate $0 \leq d_k \leq p_k$. Hence the surplus process increases with rate $c_k = p_k - d_k$ until a claim causes a jump to a lower layer or the surplus grows to the next layer [see e.g., Albrecher and Hartinger (2007)]. Let $\{U_\nu(t)\}_{t\geq 0}$ be the surplus process at time $t \geq 0$ under the multi-layer dividend strategy, with initial surplus $U_\nu(0) = u$. Then, for $k = 1, \ldots, \nu$, the dynamics of $U_\nu(t)$ are given by

$$dU_\nu(t) = c_k dt - dS(t), \qquad b_{k-1} \leq U_\nu(t) < b_k, \tag{1}$$

where the aggregate claim amount process, $\{S(t)\}_{t\geq 0}$, is generated by two classes of insurance risks, namely

$$S(t) = S_1(t) + S_2(t) = \sum_{i=1}^{N_1(t)} X_i + \sum_{i=1}^{N_2(t)} Y_i, \qquad t \geq 0, \tag{2}$$

where $S_1(t)$ and $S_2(t)$ are stochastically independent processes, representing the aggregate claims up to time $t \geq 0$ from the first and second class, respectively. The inclusion of the second class of claims allows us to model the different characteristics of two different risk groups within a portfolio. For example, 'Good' and 'Bad' drivers for auto-mobile insurance. The random variables (r.v.) $\{X_i\}_{i\geq 1}$ denote the positive claim severities from the first class, which are independent and identically distributed (i.i.d.) r.v. with common distribution function (d.f.) $F_1(x) = \mathbb{P}(X \leq x)$, probability density function (p.d.f.) $f_1(x)$, mean m_1 and Laplace transform (LT) $\widehat{f}_1(s) = \int_0^\infty e^{-sx} f_1(x)dx$. Similarly $\{Y_i\}_{i\geq 1}$ denote the positive claim severities from the second class, also assumed to be i.i.d. r.v., with common d.f. $F_2(x) = \mathbb{P}(Y \leq x)$, p.d.f. $f_2(x)$, mean m_2 and LT $\widehat{f}_2(s) = \int_0^\infty e^{-sx} f_2(x)dx$. The claim number process $\{N_1(t)\}_{t\geq 0}$ is considered to be a renewal process with inter-claim arrival times $\{V_i\}_{i\geq 1}$ which are assumed to be i.i.d. r.v. with common d.f. $Q_1(x) = \mathbb{P}(V \leq x)$. Similarly, the claim number process $\{N_2(t)\}_{t\geq 0}$ is considered to be a renewal process with inter-claim arrival times $\{W_i\}_{i\geq 1}$ which are assumed to be i.i.d. r.v. with common d.f. $Q_2(x) = \mathbb{P}(W \leq x)$. Finally, we assume that $\{N_1(t)\}_{t\geq 0}$, $\{N_2(t)\}_{t\geq 0}$, $\{Y_i\}_{i\geq 1}$ and $\{X_i\}_{i\geq 1}$ are all mutually independent.

In this paper, the distribution of the inter-arrival times Q_1 is considered to be a Phase-type distribution with representation $(\vec{\alpha}^\top, \mathbf{A}, \vec{a})$, where $\mathbf{A} = \left(a_{ij}\right)_{i,j=1}^n$ is a matrix of order $n \times n$ with $a_{ii} < 0$, $a_{ij} \geq 0$ for $i \neq j$, $\sum_{j=1}^n a_{ij} \leq 0$ for $i = 1, \ldots, n$, $\vec{\alpha}^\top = \left(\alpha_1, \ldots, \alpha_n\right)$ with $\alpha_i \geq 0$, $\sum_{i=1}^n \alpha_i = 1$ and $\vec{a} = \left(a_1, \ldots, a_n\right)^\top$ with $\vec{a} = -\mathbf{A}\vec{e}_n$, where $\vec{e}_n$ denotes a column vector of length $n \in \mathbb{N}^+$ with all its elements equal to one. Then, (from Theorem 1.5 of Chapter VIII in Asmussen (2000)) we have that

$$Q_1(t) = 1 - \vec{\alpha}^\top e^{\mathbf{A}t}\vec{e}_n, \qquad t \geq 0.$$

In fact, the r.v's $\{V_i\}_{i\geq 1}$ are known to be equivalent in distribution to the time to absorption in a terminating continuous-time Markov chain (CTMC), say $\{I_t^{(i)}\}_{t\geq 0}$, $i = 1, 2, \ldots$, with transient states $\{E_1, \ldots, E_n\}$, and absorbing state $\{E_0\}$ [see Asmussen (2000) for details]. Similarly, the distribution of the inter-claim arrival times Q_2 is considered to be a Phase-type distribution with representation $(\vec{\beta}^\top, \mathbf{B}, \vec{b})$, where $\mathbf{B} = (b_{ij})_{i,j=1}^m$ is a matrix of order $m \times m$, with $b_{ii} < 0$, $b_{ij} \geq 0$ for $i \neq j$, $\sum_{j=1}^n b_{ij} \leq 0$ for $i = 1, \ldots, m$, $\vec{\beta}^\top = ({}_1, \ldots, \beta_m)$ with $\beta_i \geq 0$, $\sum_{i=1}^m \beta_i = 1$ and $\vec{b} = (b_1, \ldots, b_m)^\top$ with $\vec{b} = -\mathbf{B}\vec{e}_m$. Then,

$$Q_2(t) = 1 - \vec{\beta}^\top e^{\mathbf{B}t}\vec{e}_m, \qquad t \geq 0,$$

and let $\{J_t^{(i)}\}_{t\geq 0}$, $i = 1, 2, \ldots$, be the corresponding terminating CTMC of $\{W_i\}_{i\geq 1}$, with transient states $\{G_1, \ldots, G_m\}$, and absorbing state $\{G_0\}$.

Now, let $\{(I(t), J(t))\}_{t\geq 0}$ be the underlying state process, which [see for details in Ji and Zhang (2010)] is defined by

$$\begin{aligned} I(t) &= I_t^{(1)}, \quad 0 \leq t < V_1, \qquad I(t) = I_{t-V_1}^{(2)}, \quad V_1 \leq t < V_1 + V_2, \ldots, \\ J(t) &= J_t^{(1)}, \quad 0 \leq t < W_1, \qquad J(t) = J_{t-W_1}^{(2)}, \quad W_1 \leq t < W_1 + W_2, \ldots \end{aligned}$$

Then, $\{(I(t), J(t))\}_{t\geq 0}$ is a two-dimensional CTMC with states $\{(E_1, G_1), \ldots, (E_n, G_1), (E_1, G_2), \ldots, (E_n, G_2), \ldots, (E_1, G_m), \ldots, (E_n, G_m)\}$, intensity matrix $\mathbf{K} = \mathbf{I}_m \otimes \mathbf{A} + \mathbf{B} \otimes \mathbf{I}_n + \mathbf{I}_m \otimes (\vec{a}\vec{\alpha}^\top) + (\vec{b}\vec{\beta}^\top) \otimes \mathbf{I}_n$ and initial distribution $\vec{\rho}^\top = \vec{\beta}^\top \otimes \vec{\alpha}^\top$, where $\mathbf{I}_n$ is the identity matrix of order $n \times n$ and $\otimes$ is the Kronecker product for matrices.

The main interest of this paper is to derive expressions for the G-S penalty function, which provides access to a variety of ruin related quantities for our risk model defined in Equations (1) and (2). To define the G-S function, let $\tau_\nu(u) = \inf\{t \geq 0 : U_\nu(t) < 0 | U_\nu(0) = u\}$ ($\tau_\nu(u) = \infty$ if the set is empty) be the time of ruin from initial surplus level $u \geq 0$. Then, the G-S function denoted by $\phi(u, \nu)$, is defined as

$$\phi(u, \nu) = \mathbb{E}\Big(e^{-\delta\tau_\nu(u)} w(U_\nu(\tau_\nu(u)), |U_\nu(\tau_\nu(u))|) 1_{(\tau_\nu(u) < \infty)}\Big), \quad u \geq 0, \tag{3}$$

where $\delta \geq 0$ is interpreted as the force of interest, $U_\nu(\tau_\nu(u)-)$ is the surplus immediately before ruin, $|U_\nu(\tau_\nu(u))|$ is the deficit at ruin, $\tau_\nu(u)-$ is the left limit of $\tau_\nu(u)$, $w(x, y)$ is a non-negative bivariate penalty function of x, $y \geq 0$ and $1_{(\cdot)}$ represents the indicator function. It is well known that the expected discounted penalty function (3) provides a unified approach to many important quantities related to the ruin time.

The paper is organised as follows. In Section 2, we derive results for the G-S function based on a layer-by-layer approach. In more details, we derive a system of integro-differential equations for the G-S function, conditioned on the layer from which the process begins and the class of claim causing ruin. We then discuss the so-called generalized Lundberg's equation and, based on its solution, derive a bi-directional recursive approach to provide a solution for the aforementioned system of integro-differential equations. In Section 3, we derive a number of layer approach for the G-S function and the expected discounted dividend payments. We derive results for some upper exit-type problems, which are then used to derive a classical (forward)-type recursive relationship for the G-S functions and the expected discounted dividends until ruin.

2. Layer-by-Layer Approach

Let us first introduce the notation $\phi_k(u,\nu)$ to denote the G-S function $\phi(u,\nu)$ when $b_{k-1} \le u < b_k$ for $k = 1, \ldots, \nu$ such that

$$\phi(u,\nu) = \begin{cases} \phi_1(u,\nu), & 0 \le u < b_1 \\ \phi_2(u,\nu), & b_1 \le u < b_2 \\ \quad\vdots & \\ \phi_\nu(u,\nu), & u \ge b_{\nu-1}. \end{cases} \tag{4}$$

Moreover, in a similar way to Ji and Zhang (2010), we consider the function

$$\phi_{k,\ell}(u,\nu) = \mathbb{E}\Big(e^{-\delta\tau_\nu(u)} w(U_\nu(\tau_\nu(u)-), |U_\nu(\tau_\nu(u))|) 1_{(\tau_\nu(u)<\infty, R=\ell)}\Big), \quad b_{k-1} \le u < b_k,$$

to be the G-S function at ruin from the k-th layer, if ruin is caused by a claim from class $\ell = 1,2$, where R is defined as the cause of ruin r.v., i.e., $R = \ell$ if ruin is caused by class $\ell = 1,2$. Then, the G-S functions defined in (34) can be decomposed as $\phi_k(u,\nu) = \phi_{k,1}(u,\nu) + \phi_{k,2}(u,\nu)$, for $b_{k-1} \le u < b_k$ and $k = 1, \ldots, \nu$.

For notational convenience, throughout the rest of this paper, let $\mathbb{P}_{ij} = \mathbb{P}(\cdot|(I(0), J(0)) = (E_i, G_j))$ and $\mathbb{E}_{ij}$ to be the expectation w.r.t. $\mathbb{P}_{ij}$, $1 \le i \le n$, $1 \le j \le m$. Then, for $1 \le i \le n$, $1 \le j \le m$, let for $k = 1, \ldots, \nu$, $\ell = 1,2$

$$\phi_{ij,k,\ell}(u,\nu) = \mathbb{E}_{ij}\Big(e^{-\delta\tau_\nu(u)} w(U_\nu(\tau_\nu(u)-), |U_\nu(\tau_\nu(u))|) 1_{(\tau_\nu(u)<\infty, R=\ell)}\Big), \quad b_{k-1} \le u < b_k, \tag{5}$$

to be the G-S function from the k-th layer if ruin is caused by a claim from class $\ell = 1,2$, given the initial state (E_i, G_j). Consequently, we have that

$$\phi_k(u,\nu) = \vec{\rho}^\top (\vec{\boldsymbol{\phi}}_{k,1}(u,\nu) + \vec{\boldsymbol{\phi}}_{k,2}(u,\nu)), \quad b_{k-1} \le u < b_k, \quad k = 1, \ldots, \nu,$$

where $\vec{\boldsymbol{\phi}}_{k,\ell}(u,\nu) = \big(\phi_{11,k,\ell}(u,\nu), \ldots, \phi_{n1,k,\ell}(u,\nu), \phi_{12,k,\ell}(u,\nu), \ldots, \phi_{n2,k,\ell}(u,\nu), \ldots, \phi_{1m,k,\ell}(u,\nu), \ldots, \phi_{nm,k,\ell}(u,\nu)\big)^\top$.

2.1. *Piecewise Integro-Differential Equation for $\vec{\boldsymbol{\phi}}_{k,\ell}(u,\nu)$*

In this subsection, using arguments as in Song (2008) [see also Ji and Zhang (2010)], we obtain a piecewise integro-differential equation system for the Gerber–Shiu functions $\phi_{ij,k,\ell}(u,\nu)$ and, consequently, using matrix notations obtain a piecewise integro-differential equation for $\vec{\boldsymbol{\phi}}_{k,\ell}(u,\nu)$.

Theorem 1. *For $k = 1, \ldots, \nu$ and $\ell = 1,2$, $\vec{\boldsymbol{\phi}}_{k,\ell}(u,\nu)$ satisfies the following piecewise integro-differential equation*

$$\begin{aligned} c_k \vec{\boldsymbol{\phi}}'_{k,\ell}(u,\nu) &= (\delta \mathbf{I}_{nm} - \mathbf{I}_m \otimes \mathbf{A} - \mathbf{B} \otimes \mathbf{I}_n) \vec{\boldsymbol{\phi}}_{k,\ell}(u,\nu) - \int_0^{u-b_{k-1}} \Big(\mathbf{I}_m \otimes (\vec{a}\vec{\alpha}^\top) f_1(x) \\ &\quad + (\vec{b}\vec{\beta}^\top) \otimes \mathbf{I}_n f_2(x) \Big) \vec{\boldsymbol{\phi}}_{k,\ell}(u-x,\nu) dx - \vec{\zeta}_{k,\ell}(u), \qquad b_{k-1} \le u < b_k, \end{aligned} \tag{6}$$

where

$$\begin{aligned} \vec{\zeta}_{k,\ell}(u) &= \sum_{i=1}^{k-1} \int_{u-b_i}^{u-b_{i-1}} \Big(\mathbf{I}_m \otimes (\vec{a}\vec{\alpha}^\top) f_1(x) + (\vec{b}\vec{\beta}^\top) \otimes \mathbf{I}_n f_2(x) \Big) \vec{\boldsymbol{\phi}}_{i,\ell}(u-x,\nu) dx \\ &\quad + (\vec{e}_m \otimes \vec{a}) w_1(u) 1_{(\ell=1)} + (\vec{b} \otimes \vec{e}_n) w_2(u) 1_{(\ell=2)}, \\ w_j(u) &= \int_u^\infty w(u, x-u) f_j(x) dx, \quad j = 1,2, \end{aligned} \tag{7}$$

with boundary conditions

$$\vec{\boldsymbol{\phi}}_{k,\ell}(b_k-,\nu) = \vec{\boldsymbol{\phi}}_{k+1,\ell}(b_k+,\nu) \quad \text{and} \quad c_k\vec{\boldsymbol{\phi}}'_{k,\ell}(b_k-,\nu) = c_{k+1}\vec{\boldsymbol{\phi}}'_{k+1,\ell}(b_k+,\nu). \tag{8}$$

Proof. Consider an infinitesimal time interval $(0,dt)$. Then, conditioning on whether or not the state of $\{I(t),J(t)\}_{t\geq 0}$, associated with the surplus process in the layer $[b_{k-1},b_k)$, changes accompanied by a claim or by no claim in $(0,dt)$, we have that

$$\begin{aligned}
e^{\delta dt}\phi_{ij,k,\ell}(u,\nu) \quad = \quad & \Big(1+(a_{ii}+b_{jj})dt\Big)\phi_{ij,k,\ell}(u+c_k dt,\nu) \\
& +(1+b_{jj}dt)\sum_{\nu_1=1,\nu_1\neq i}^{n} a_{i\nu_1}dt\phi_{\nu_1 j,k,\ell}(u+c_k dt,\nu) \\
& +(1+a_{ii}dt)\sum_{\nu_2=1,\nu_2\neq j}^{m} b_{j\nu_2}dt\phi_{i\nu_2,k,\ell}(u+c_k dt,\nu) \\
& +(1+b_{jj}dt)a_i dt\sum_{\nu_1=1}^{m}\alpha_{\nu_1}\Big(\int_0^{u+c_k dt-b_{k-1}}\phi_{\nu_1 j,k,\ell}(u+c_k dt-x,\nu)f_1(x)dx \\
& +\sum_{l=1}^{k-1}\int_{u+c_k dt-b_l}^{u+c_k dt-b_{l-1}}\phi_{\nu_1 j,l,\ell}(u+c_k dt-x,\nu)f_1(x)dx+w_1(u+c_k dt)1_{(\ell=1)}\Big) \\
& +(1+a_{ii}dt)b_j dt\sum_{\nu_2=1}^{m}\beta_{\nu_2}\Big(\int_0^{u+c_k dt-b_{k-1}}\phi_{i\nu_2,k,\ell}(u+c_k dt-x,\nu)f_2(x)dx \\
& +\sum_{l=1}^{k-1}\int_{u+c_k dt-b_l}^{u+c_k dt-b_{l-1}}\phi_{i\nu_2,l,\ell}(u+c_k dt-x,\nu)f_2(x)dx+w_2(u+c_k dt)1_{(\ell=2)}\Big) \\
& +o(dt).
\end{aligned}$$

Dividing both sides by dt, letting $dt\to 0$ and using the fact that $w_j(u)$ are continuous functions of u, (implying that $\lim_{dt\to 0} w_j(u+c_k dt) = w_j\big(\lim_{dt\to 0}(u+c_k dt)\big) = w_j(u)$), for $j=1,2$, yields that

$$\begin{aligned}
c_k\phi'_{ij,k,\ell}(u,\nu) \quad = \quad & \delta\phi_{ij,k,\ell}(u,\nu)-\sum_{\nu_1=1}^{n}a_{i\nu_1}\phi_{\nu_1 j,k,\ell}(u,\nu)-\sum_{\nu_2=1}^{m}b_{j\nu_2}\phi_{i\nu_2,k,\ell}(u,\nu) \\
& -a_i\sum_{\nu_1=1}^{n}\alpha_{\nu_1}\left(\int_0^{u-b_{k-1}}\phi_{\nu_1 j,k,\ell}(u-x,\nu)f_1(x)dx\right. \\
& \left.+\sum_{l=1}^{k-1}\int_{u-b_l}^{u-b_{l-1}}\phi_{\nu_1 j,l,\ell}(u-x,\nu)f_1(x)dx+w_1(u)1_{(\ell=1)}\right) \\
& -b_j\sum_{\nu_2=1}^{m}\beta_{\nu_2}\left(\int_0^{u-b_{k-1}}\phi_{i\nu_2,k,\ell}(u-x,\nu)f_2(x)dx\right. \\
& \left.+\sum_{l=1}^{k-1}\int_{u-b_l}^{u-b_{l-1}}\phi_{i\nu_2,l,\ell}(u-x,\nu)f_2(x)dx+w_2(u)1_{(\ell=2)}\right).
\end{aligned}$$

Rewriting the above equations in matrix form and rearranging the resulting matrix equation we get immediately the integro-differential Equation (6). To obtain the boundary conditions (8), we can apply similar arguments as in Lin and Sendova (2008) to show that the G-S penalty function for a multi-layer model continuous for all $u\geq 0$, even at the even at the seeming discontinuity points. Thus, $\vec{\boldsymbol{\phi}}_{k,\ell}(u,\nu)$ is a continuous function at each b_k, $k=1,\ldots,\nu$, i.e., it holds that $\vec{\boldsymbol{\phi}}_{k,\ell}(b_k-,\nu) = \vec{\boldsymbol{\phi}}_{k+1,\ell}(b_k+,\nu)$, $k=1,\ldots,\nu$, $\ell=1,2$. Furthermore, the G-S function itself is not differentiable at the b_i's but its left and right derivative exist. Then, from Equation (6) and the continuity of the G-S function, it is not difficult to see that the second set of boundary conditions (8) holds. □

Remark 1. *Note that the inclusion of the lower boundary b_{k-1} within the constraints of Equation (6) follows from the existence of the right derivative of $\vec{\phi}_{k,\ell}(u,\nu)$. We hereafter assume that a derivative is always a right derivative if a proper derivative does not exist. Moreover, for $\nu = 1$ and $c_1 = c$, we obtain the renewal risk model with two classes of claims in a barrier free environment, and thus Equation (6) reduces, for $\ell = 1$ and $\ell = 2$, to Equations (2.3) and (2.4), respectively, of Ji and Zhang (2010).*

2.2. Analysis of the Piecewise Integro-Differential Equation System for $u \geq b_{k-1}$

Similar to the methodology of Lin and Sendova (2008) [see also Yang and Zhang (2008)], the non-homogeneous piecewise integro-differential Equation (6) heavily depends on the solution of the corresponding piecewise non-homogeneous integro-differential equation for $u \geq b_{k-1}$. As such, in the following, we will consider an auxiliary vector function $\vec{\varphi}_{k,\ell}(u) = (\varphi_{11,k,\ell}(u), \ldots, \varphi_{n1,k,\ell}(u), \varphi_{12,k,\ell}(u), \ldots, \varphi_{n2,k,\ell}(u), \ldots, \varphi_{1m,k,\ell}(u), \ldots, \varphi_{nm,k,\ell}(u))^\top$ defined as the solution, for $k = 1, \ldots, \nu$ and $\ell = 1, 2$, to the non-homogenous integro-differential equation

$$\begin{aligned} c_k\vec{\varphi}'_{k,\ell}(u) &= (\delta\mathbf{I}_{nm} - \mathbf{I}_m \otimes \mathbf{A} - \mathbf{B} \otimes \mathbf{I}_n)\vec{\varphi}_{k,\ell}(u) - \int_0^{u-b_{k-1}} \Big(\mathbf{I}_m \otimes (\vec{a}\vec{\alpha}^\top) f_1(x) \\ &\quad + (\vec{b}\vec{\beta}^\top) \otimes \mathbf{I}_n f_2(x)\Big)\vec{\varphi}_{k,\ell}(u-x)dx - \vec{\zeta}_{k,\ell}(u), \qquad u \geq b_{k-1}. \end{aligned} \tag{9}$$

A change of variable $y = u - b_{k-1}$ and $g_{ij,k\ell}(y) = \varphi_{ij,k\ell}(y + b_{k-1})$, $\vec{\mathbf{g}}_{k,\ell}(y) = \vec{\varphi}_{k,\ell}(y + b_{k-1})$ and $\vec{\mathbf{z}}_{k,\ell}(y) = \vec{\zeta}_{k,\ell}(y + b_{k-1})$ for $k = 1, \ldots, \nu$, $\ell = 1, 2$, brings the non-homogeneous integro-differential Equation (9) into the form

$$\begin{aligned} c_k\vec{\mathbf{g}}'_{k,\ell}(y) &= (\delta - \mathbf{I}_m \otimes \mathbf{A} - \mathbf{B} \otimes \mathbf{I}_n)\vec{\mathbf{g}}_{k,\ell}(y) - \int_0^{y} \Big(\mathbf{I}_m \otimes (\vec{a}\vec{\alpha}^\top) f_1(x) \\ &\quad + (\vec{b}\vec{\beta}^\top) \otimes \mathbf{I}_n f_2(x)\Big)\vec{\mathbf{g}}_{k,\ell}(y-x)dx - \vec{\mathbf{z}}_{k,\ell}(y), \qquad y \geq 0. \end{aligned} \tag{10}$$

In order to find the solution for $\vec{\mathbf{g}}_{k,\ell}(y)$, we first consider the corresponding homogeneous integro-differential equation of (10), with solution $\vec{\mathbf{h}}_{k,\ell}(y)$ satisfying

$$\begin{aligned} c_k\vec{\mathbf{h}}'_{k,\ell}(y) &= (\delta\mathbf{I}_{nm} - \mathbf{I}_m \otimes \mathbf{A} - \mathbf{B} \otimes \mathbf{I}_n)\vec{\mathbf{h}}_{k,\ell}(y) - \int_0^{y} \Big(\mathbf{I}_m \otimes (\vec{a}\vec{\alpha}^\top) f_1(x) \\ &\quad + (\vec{b}\vec{\beta}^\top) \otimes \mathbf{I}_n f_2(x)\Big)\vec{\mathbf{h}}_{k,\ell}(y-x)dx, \qquad y \geq 0. \end{aligned} \tag{11}$$

Let $\hat{\vec{\mathbf{h}}}_{k,\ell}(s) = \int_0^\infty e^{-sy}\vec{\mathbf{h}}_{k,\ell}(y)dy$, for $k = 1, \ldots, \nu$, $\ell = 1, 2$ and $\mathrm{Re}(s) \geq 0$, to be the (element-wise) LT of the vector $\vec{\mathbf{h}}_{k,\ell}(y)$. Then, taking LT on both sides of Equation (11) and rearranging, we have

$$\mathbf{L}_{k,\delta}(s)\hat{\vec{\mathbf{h}}}_{k,\ell}(s) = c_k\vec{\mathbf{h}}_{k,\ell}(0), \tag{12}$$

where

$$\mathbf{L}_{k,\delta}(s) = (c_k s - \delta)\mathbf{I}_{nm} + \mathbf{I}_m \otimes \mathbf{A} + \mathbf{B} \otimes \mathbf{I}_n + \mathbf{I}_m \otimes (\vec{a}\vec{\alpha}^\top)\hat{f}_1(s) + (\vec{b}\vec{\beta}^\top) \otimes \mathbf{I}_n\hat{f}_2(s).$$

The equation $\det \mathbf{L}_{k,\delta}(s) = 0$ is called the *characteristic equation* for the risk process (1) and (2), and its roots, given by the following proposition, play an important role in determining the solution of the integro-differential Equation (9).

Proposition 1. *Given the value of δ, the following hold:*

(i) In the complex plane, for $\delta > 0$, the characteristic equation $\det \mathbf{L}_{k,\delta}(s) = 0$ has exactly nm roots with positive real parts.

(ii) In the complex plane, for $\delta = 0$, the characteristic equation $\det \mathbf{L}_{k,0}(s) = 0$ has exactly one root at zero and $nm - 1$ roots with positive real parts.

Proof. (i) The proof for the case when $\delta > 0$ is equivalent to that of Theorem 3.1 in Ji and Zhang (2010). (ii) For the case when $\delta = 0$, it is clear to see that $s = 0$ is a root of the characteristic equation. The proof that the remaining $nm - 1$ roots all have positive real parts follows similar arguments to that of Adan and Kulkarni (2003) and, as such, is omitted here. □

We turn our attention back now to the determination of the solution of the non-homogeneous integro-differential Equation (9). In the following, we will denote the nm roots of the equation $\det \mathbf{L}_{k,\delta}(s) = 0$ by $r_{i,k}(\delta) \equiv r_{i,k}$, $i = 1, \ldots, nm$ and will assume that $r_{1,k} \ldots, r_{nm,k}$ are distinct, for each $k = 1, \ldots, \nu$.

Let the matrix $\mathbf{\Gamma}_k(y) = \left(\gamma_{ij,k}(y)\right)_{i,j=1}^{nm}$, $0 \leq y < \infty$, be the $nm \times nm$ matrix whose columns are the linearly independent solutions to the homogeneous Equation (11) (determined later on in this section), with $\mathbf{\Gamma}_k(0) = \mathbf{I}_{nm}$. Then, by Theorem 2.3.1 of Burton (2005), the solution to Equation (10) is given by

$$\vec{\mathbf{g}}_{k,\ell}(y) = \mathbf{\Gamma}_k(y)\vec{\mathbf{g}}_{k,\ell}(0) - \int_0^y \mathbf{\Gamma}_k(x)\vec{\mathbf{z}}_{k,\ell}(y-x)dx, \qquad y \geq 0. \tag{13}$$

Replacing the variable $y = u - b_{k-1}$, we obtain the solution of the integro-differential Equation (9) as it is given in the following proposition.

Proposition 2. *For $k = 1, \ldots, \nu$, $\ell = 1, 2$, let $\mathbf{\Gamma}_k(y) = \left(\gamma_{ij,k}(y)\right)_{i,j=1}^{nm}$, $y \geq 0$, be a square matrix of order nm, whose columns are the linearly independent solutions to the homogeneous Equation (11) with $\mathbf{\Gamma}_k(0) = \mathbf{I}_{nm}$. Then the solution to the integro-differential Equation (9) it is given by*

$$\vec{\boldsymbol{\varphi}}_{k,\ell}(u) = \mathbf{\Gamma}_k(u - b_{k-1})\vec{\boldsymbol{\varphi}}_{k,\ell}(b_{k-1}) - \int_0^{u-b_{k-1}} \mathbf{\Gamma}_k(x)\vec{\boldsymbol{\zeta}}_{k,\ell}(u-x)dx, \qquad u \geq b_{k-1}, \tag{14}$$

where $\vec{\boldsymbol{\zeta}}_{k,\ell}(u)$ is given in Theorem 1.

In order to complete the solution for $\vec{\boldsymbol{\varphi}}_{k,\ell}(u)$, given by Equation (14) of Proposition 2, it remains to determine the initial values $\vec{\boldsymbol{\varphi}}_{k,\ell}(b_{k-1})$ for $k = 1, \ldots, \nu$, $\ell = 1, 2$, which is considered in the next subsection, as well as to calculate the form of the matrix $\mathbf{\Gamma}_k(y)$, which is done by using LT as follows:

Let $\widehat{\mathbf{\Gamma}}_k(s) = \left(\widehat{\gamma}_{ij,k}(s)\right)_{i,j=1}^{nm}$ be the square matrix of order $nm \times nm$, with elements $\widehat{\gamma}_{ij,k}(s) = \int_0^\infty e^{-sy}\gamma_{ij,k}(y)dy$ denoting the LT of $\gamma_{ij,k}(y)$. Then, since the columns of $\mathbf{\Gamma}_k(y)$ are solutions to Equation (11) and consequently their LT solutions to Equation (12), it follows that

$$\widehat{\mathbf{\Gamma}}_k(s) = c_k\left[\mathbf{L}_{k,\delta}(s)\right]^{-1} = \frac{c_k\mathbf{L}^*_{k,\delta}(s)}{\det \mathbf{L}_{k,\delta}(s)}, \tag{15}$$

using the fact that $\mathbf{\Gamma}_k(0) = \mathbf{I}_{nm}$ and $\mathbf{L}^*_{k,\delta}(s)$ denotes the adjoint matrix of $\mathbf{L}_{k,\delta}(s)$. To invert the above LT, let us consider the case where the claim amount densities (f_1 and f_2) belong to the rational family of distributions which include, the Erlang, Coxian, Phase-type and their mixtures, among others. That is, the LT $\widehat{f}_i$ for $i = 1, 2$ is of the form

$$\widehat{f}_i(s) = \frac{p_{k_{i-1}(s)}}{p_{k_i(s)}}, \qquad p_{k_{i-1}}(0) = p_{k_i}(0), \qquad i = 1, 2, \tag{16}$$

where $p_{k_{i-1}}(s)$ is a polynomial of degree k_{i-1} and $p_{k_i}(s)$ is a polynomial of degree k_i with leading coefficient 1 and with only negative roots.

Proposition 3. *Assume that the LT of the claim amount densities, $\widehat{f}_i(s)$, $i = 1, 2$, are defined according to Equation (16). Then, the elements of the matrix $\mathbf{\Gamma}_k(y)$, $k = 1, \ldots, \nu$, are given by*

$$\gamma_{ij,k}(y) = \sum_{\ell_1=1}^{nm} \bar{\alpha}_{ij,k}(\ell_1) e^{r_{\ell_1,k} y} + \sum_{\ell_2=1}^{(k_1+k_2)nm} \bar{\beta}_{ij,k}(\ell_2) e^{-R_{\ell_2,k} y}, \qquad y \geq 0,$$

with

$$\begin{aligned} \bar{\alpha}_{ij,k}(\ell_1) &= \frac{\left(\prod_{z=1}^{2} p_{k_z}(r_{\ell_1,k})\right)^{nm} \left(\mathbf{L}^*_{k,\delta}(r_{\ell_1,k})\right)_{i,j}}{c_k^{nm-1} \prod_{\nu=1,\nu\neq\ell_1}^{nm} (r_{\ell_1,k} - r_{\nu,k}) \prod_{z=1}^{(k_1+k_2)nm} (r_{\ell_1,k} + R_{z,k})}, \\ \bar{\beta}_{ij,k}(\ell_2) &= \frac{\left(\prod_{z=1}^{2} p_{k_z}(-R_{\ell_2,k})\right)^{nm} \left(\mathbf{L}^*_{k,\delta}(-R_{\ell_2,k})\right)_{i,j}}{(-1)^{(k_1+k_2)nm} c_k^{nm-1} \prod_{z=1}^{nm} (r_{z,k} + R_{\ell_2,k}) \prod_{\nu=1,\nu\neq\ell_2}^{(k_1+k_2)nm} (R_{\nu,k} - R_{\ell_2,k})}, \end{aligned}$$

where $\left(\mathbf{L}^*_{k,\delta}(s)\right)_{i,j}$ *is the* (i,j)*-th element of the matrix* $\mathbf{L}^*_{k,\delta}(s)$*,* $r_{\ell_1,k}$ *with* $Re(r_{\ell_1,k}) > 0$ *for* $\ell_1 = 1, \ldots, nm$*, and* $-R_{\ell_2,k}$ *with* $Re(R_{\ell_2,k}) > 0$ *for* $\ell_2 = 1, \ldots, (k_1+k_2)nm$ *are the roots of the characteristic equation* $\det \mathbf{L}_{k,\delta}(s) = 0$*.*

Proof. Multiplying both the numerator and the denominator of (15) with $\left(\prod_{z=1}^{2} p_{k_z}(s)\right)^{nm}$, the (i,j)-th element of the matrix $\widehat{\mathbf{\Gamma}}_k(s)$ is given by

$$\widehat{\gamma}_{ij,k}(s) = \frac{c_k \left(\prod_{z=1}^{2} p_{k_z}(s)\right)^{nm} \left(\mathbf{L}^*_{k,\delta}(s)\right)_{i,j}}{C_k(s)}, \quad k = 1, \ldots, \nu, \tag{17}$$

where $C_k(s) = \left(\prod_{z=1}^{2} p_{k_z}(s)\right)^{nm} \det \mathbf{L}_{k,\delta}(s)$ is a polynomial of degree $(k_1+k_2+1)nm$ in s with leading coefficient c_k^{nm} and thus the equation $C_k(s) = 0$ has $(k_1+k_2+1)nm$ roots. Since, from Proposition 1, the $\det \mathbf{L}_{k,\delta}(s) = 0$ has exactly nm roots $r_{i,k}$, $i = 1, \ldots, nm$, with positive real parts, then $C_k(s) = 0$ has the above $r_{i,k}$ and $(k_1+k_2)nm$ roots, say $-R_{j,k}$, $j = 1, \ldots, (k_1+k_2)nm$ with negative real parts, since the polynomial $p_{k_1}(s)$ and $p_{k_2}(s)$ have only roots with negative real parts. Thus, $C_k(s)$ may be written as

$$C_k(s) = c_k^{nm} \left(\prod_{i=1}^{nm} (s - r_{i,k})\right) \left(\prod_{j=1}^{(k_1+k_2)nm} (s + R_{j,k})\right).$$

Inserting the above equation into Equation (17) and using partial fraction techniques, we get that

$$\widehat{\gamma}_{ij,k}(s) = \sum_{l_1=1}^{nm} \frac{\bar{\alpha}_{ij,k}(l_1)}{s - r_{l_1,k}} + \sum_{l_2=1}^{(k_1+k_2)nm} \frac{\bar{\beta}_{ij,k}(l_2)}{s + R_{l_2,k}},$$

from which by inverting w.r.t. s, we obtain the required result. □

2.3. Initial Values

In this subsection we use an approach similar to Badescu (2008), to determine the unknown vector $\vec{\boldsymbol{\varphi}}_{k,\ell}(b_{k-1})$ for $k = 1, \ldots, \nu$, $\ell = 1, 2$. To do this, we require the roots of the characteristic equation $\det \mathbf{L}_{k,\delta}(s) = 0$ given by Proposition 1 and a matrix/vector operator T_r, the scalar version of which was first introduced in the actuarial literature by Dickson and Hipp (2001).

As in Lu and Li (2009) , let $\mathbf{P}(x)$ be a matrix/vector whose elements are real-valued integrable functions of x. Then, the operator $T_r\mathbf{P}(x)$ w.r.t. a complex number $r \in \mathbb{C}$ is defined by

$$T_r\mathbf{P}(x) = \int_x^{\infty} e^{-r(u-x)} \mathbf{P}(u) du, \qquad \text{Re}(r) \geq 0.$$

Pre-multiplying Equation (9) with $e^{-s(u-b_{k-1})}$ and integrating the resulting equation w.r.t. u from b_{k-1} to ∞, yields

$$\mathbf{L}_{k,\delta}(s)T_s\vec{\boldsymbol{\varphi}}_{k,\ell}(b_{k-1}) = c_k\vec{\boldsymbol{\varphi}}_{k,\ell}(b_{k-1}) - T_s\vec{\boldsymbol{\zeta}}_{k,\ell}(b_{k-1}), \tag{18}$$

for $k = 1, \ldots, \nu$ and $\ell = 1, 2$.

Note, that for each $s = r_{i,k}$ the matrix $\mathbf{L}_{k,\delta}(r_{i,k})$, $i = 1, \ldots, nm$, has a zero eigenvalue. Hence, for each $r_{i,k}$, let $\vec{q}_{i,k}^{\top}$ be the left eigenvector (of order $1 \times nm$) of the matrix $\mathbf{L}_{k,\delta}(r_{i,k})$ w.r.t. the zero eigenvalue, such that $\vec{q}_{i,k}^{\top}\mathbf{L}_{k,\delta}(r_{i,k}) = \vec{0}_{nm}^{\top}$, for every $i = 1, \ldots, nm$.

Substituting $s = r_{i,k}$ and then left-multiplying both sides of Equation (18) with $\vec{q}_{i,k}^{\top}$, we have that for $k = 1, \ldots, \nu$, $\ell = 1, 2$,

$$\vec{q}_{i,k}^{\top}\Big(c_k\vec{\boldsymbol{\varphi}}_{k,\ell}(b_{k-1}) - T_{r_{i,k}}\vec{\boldsymbol{\zeta}}_{k,\ell}(b_{k-1})\Big) = \vec{q}_{i,k}^{\top}\mathbf{L}_{k,\delta}(r_{i,k})T_{r_{i,k}}\vec{\boldsymbol{\varphi}}_{k,\ell}(b_{k-1}) = 0,$$

which gives

$$\vec{q}_{i,k}^{\top}c_k\vec{\boldsymbol{\varphi}}_{k,\ell}(b_{k-1}) = \vec{q}_{i,k}^{\top}T_{r_{i,k}}\vec{\boldsymbol{\zeta}}_{k,\ell}(b_{k-1}).$$

Equivalently, if we let $\mathbf{Q}_k = \left(\vec{q}_{1,k}, \ldots, \vec{q}_{nm,k}\right)^{\top}$ be a square matrix of order $nm \times nm$ denoting the (left) eigenvector matrix, then the above equation can be written in matrix form as

$$\mathbf{Q}_k\,\vec{\boldsymbol{\varphi}}_{k,\ell}(b_{k-1}) = \frac{1}{c_k}\mathrm{diag}\Big(\mathbf{Q}_k \cdot \Big(T_{r_{1,k}}\vec{\boldsymbol{\zeta}}_{k,\ell}(b_{k-1}), \ldots, T_{r_{nm,k}}\vec{\boldsymbol{\zeta}}_{k,\ell}(b_{k-1})\Big)\Big)\vec{e}_{nm}.$$

Finally, the assumption that $r_{i,k}$ are distinct implies that $\vec{q}_{1,k}, \ldots, \vec{q}_{nm,k}$ are linearly independent and thus $\mathbf{Q}_k$ is invertible. As such, for $k = 1, \ldots, \nu$, $\ell = 1, 2$, $\vec{\boldsymbol{\varphi}}_{k,\ell}(b_{k-1})$ is given by

$$\vec{\boldsymbol{\varphi}}_{k,\ell}(b_{k-1}) = \frac{1}{c_k}\mathbf{Q}_k^{-1}\mathrm{diag}\Big(\mathbf{Q}_k \cdot \Big(T_{r_{1,k}}\vec{\boldsymbol{\zeta}}_{k,\ell}(b_{k-1}), \ldots, T_{r_{nm,k}}\vec{\boldsymbol{\zeta}}_{k,\ell}(b_{k-1})\Big)\Big)\vec{e}_{nm}. \tag{19}$$

which fully identifies the solution given in Proposition 2.

2.4. Recursive Expressions for $\vec{\boldsymbol{\phi}}_{k,\ell}(u,\nu)$

Finally, in this subsection we use the results for the auxiliary function $\vec{\boldsymbol{\varphi}}_{k,\ell}(u)$, i.e., with 'relaxed' constraints $u \geq b_{k-1}$ given in Proposition 2, to derive a recursive expression for $\vec{\boldsymbol{\phi}}_{k,\ell}(u,\nu)$ with $b_{k-1} \leq u < b_k$.

We first note that if, in Equation (6), we extend the co-domain from $b_{k-1} \leq u < b_k$ to $u \geq b_{k-1}$, then $\vec{\boldsymbol{\phi}}_{k,\ell}(u,\nu)$ satisfies Equation (9) and thus, its solution can be derived from Proposition 2 after restricting the general co-domain $u \geq b_{k-1}$ to $b_{k-1} \leq u < b_k$, i.e., for $k = 1, \ldots, \nu$

$$\vec{\boldsymbol{\phi}}_{k,\ell}(u,\nu) = \mathbf{\Gamma}_k(u - b_{k-1})\vec{\boldsymbol{\phi}}_{k,\ell}(b_{k-1},\nu) - \int_0^{u-b_{k-1}} \mathbf{\Gamma}_k(x)\vec{\boldsymbol{\zeta}}_{k,\ell}(u-x)dx, \quad b_{k-1} \leq u < b_k, \tag{20}$$

where the case $u = b_{k-1}$ holds from the continuity conditions of Equation (8).

Remark 2. *Note from Equation (9) that for the function $\vec{\boldsymbol{\varphi}}_{k,\ell}(u)$, the dynamics of the underlying process remain constant above the level b_{k-1}. However, for the multi-layered risk model (1) and (2), the dynamics change after crossing the level b_k. As such, $\vec{\boldsymbol{\phi}}_{k,\ell}(b_{k-1},\nu)$ and $\vec{\boldsymbol{\varphi}}_{k,\ell}(b_{k-1})$ are not necessarily equal for all $k = 1, \ldots, \nu - 1$. For the case $k = \nu$, the dynamics of both underlying processes are equivalent above the level $b_{\nu-1}$ and thus for $\ell = 1, 2$, we have*

$$\vec{\boldsymbol{\phi}}_{\nu,\ell}(u,\nu) = \vec{\boldsymbol{\varphi}}_{\nu,\ell}(u), \qquad u \geq b_{\nu-1}. \tag{21}$$

On the other hand, for $b_{k-1} \leq u < b_k$, the term $\int_0^{u-b_{k-1}} \mathbf{\Gamma}_k(x)\vec{\zeta}_{k,\ell}(u-x)dx$ in Equations (14) and (20) are identical for all $k = 1, \ldots, \nu$ and thus, taking the difference of these equations, yields

$$\begin{aligned}
\vec{\boldsymbol{\phi}}_{k,\ell}(u,\nu) &= \mathbf{\Gamma}_k(u-b_{k-1})\vec{\boldsymbol{\phi}}_{k,\ell}(b_{k-1},\nu) + \vec{\boldsymbol{\varphi}}_{k,\ell}(u) - \mathbf{\Gamma}_k(u-b_{k-1})\vec{\boldsymbol{\varphi}}_{k,\ell}(b_{k-1}) \\
&= \vec{\boldsymbol{\varphi}}_{k,\ell}(u) + \mathbf{\Gamma}_k(u-b_{k-1})\Big(\vec{\boldsymbol{\phi}}_{k,\ell}(b_{k-1},\nu) - \vec{\boldsymbol{\varphi}}_{k,\ell}(b_{k-1})\Big) \\
&= \vec{\boldsymbol{\varphi}}_{k,\ell}(u) + \mathbf{\Gamma}_k(u-b_{k-1})\vec{\eta}_{k,\ell}, \quad b_{k-1} \leq u < b_k,
\end{aligned} \tag{22}$$

where $\vec{\eta}_{k,\ell} = \vec{\boldsymbol{\phi}}_{k,\ell}(b_{k-1},\nu) - \vec{\boldsymbol{\varphi}}_{k,\ell}(b_{k-1})$ is an unknown vector which can be determined as follows:

Using the continuity conditions in Equations (8) and (22), it follows that for $k = 1, \ldots, \nu-1$, $\ell = 1, 2$,

$$\vec{\boldsymbol{\varphi}}_{k,\ell}(b_k) + \mathbf{\Gamma}_k(b_k - b_{k-1})\vec{\eta}_{k,\ell} = \vec{\boldsymbol{\varphi}}_{k+1,\ell}(b_k) + \mathbf{\Gamma}_{k+1}(0)\vec{\eta}_{k+1,\ell}$$

and thus, since $\mathbf{\Gamma}_{k+1}(0) = \mathbf{I}_{nm}$, $\vec{\eta}_{k+1,\ell}$ satisfies a recursive equation of the form

$$\vec{\eta}_{k+1,\ell} = \vec{\boldsymbol{\varphi}}_{k,\ell}(b_k) - \vec{\boldsymbol{\varphi}}_{k+1,\ell}(b_k) + \mathbf{\Gamma}_k(b_k - b_{k-1})\vec{\eta}_{k,\ell},$$

where $\vec{\boldsymbol{\varphi}}_{k,\ell}(b_k)$ can be determined from Equation (14) and $\vec{\boldsymbol{\varphi}}_{k+1,\ell}(b_k)$ is given by Equation (19). Moreover, from Equations (22) and (21) we obtain that $\vec{\eta}_{\nu,\ell} = \vec{0}_{nm}$.

Finally, to summarize our results for $\vec{\boldsymbol{\phi}}_{k,\ell}(u,\nu)$ we have the following theorem.

Theorem 2. *For $k = 1, \ldots, \nu$, $\ell = 1, 2$, the vector of the expected discounted penalty functions, $\vec{\boldsymbol{\phi}}_{k,\ell}(u,\nu)$, is given by*

$$\vec{\boldsymbol{\phi}}_{k,\ell}(u,\nu) = \vec{\boldsymbol{\varphi}}_{k,\ell}(u) + \mathbf{\Gamma}_k(u-b_{k-1})\vec{\eta}_{k,\ell}, \quad b_{k-1} \leq u < b_k, \tag{23}$$

where $\vec{\boldsymbol{\varphi}}_{k,\ell}(u)$ satisfies

$$\vec{\boldsymbol{\varphi}}_{k,\ell}(u) = \mathbf{\Gamma}_k(u-b_{k-1})\vec{\boldsymbol{\varphi}}_{k,\ell}(b_{k-1}) - \int_0^{u-b_{k-1}} \mathbf{\Gamma}_k(x)\vec{\zeta}_{k,\ell}(u-x)dx, \qquad u \geq b_{k-1}$$

and $\vec{\eta}_{k,\ell}$ are obtained recursively by

$$\begin{cases} \vec{\eta}_{k+1,\ell} = \vec{\boldsymbol{\varphi}}_{k,\ell}(b_k) - \vec{\boldsymbol{\varphi}}_{k+1,\ell}(b_k) + \mathbf{\Gamma}_k(b_k - b_{k-1})\vec{\eta}_{k,\ell}, & k = 1, \ldots, \nu, \\ \vec{\eta}_{\nu,\ell} = \vec{0}_{nm}. \end{cases}$$

Remark 3. *We point out here that as well providing a recursive expression for the G-S function of a multi-layer dividend strategy, Theorem 2 can also be used to approximate the G-S function for a risk model with a general level dependent premium rate function $c(\cdot)$, such that*

$$dU_t = c(U_t) - dS_t, \tag{24}$$

where $c(\cdot)$ is a non-decreasing and locally Lipschitz continuous function (this guarantees that the above SDE has a weak solution). In this case, it is known that one can use piecewise constant functions for the premium rate (as in Equation (1)) to (strongly) approximate the above, general level model [see Czarna et al. (2019) for details].

Theorem 2 provides a recursive approach to calculating $\vec{\boldsymbol{\phi}}_{k,\ell}(u,\nu)$ starting from the first layer, i.e., $k = 1$. In this case, $\vec{\boldsymbol{\varphi}}_{1,\ell}(u)$ corresponds to the expected discounted penalty function in the absence of a dividend barrier strategy and can be calculated from Theorem 4 of Ji and Zhang (2010) or equivalently from Proposition 2 (above). However, to calculate $\vec{\eta}_{1,\ell}$, it is necessary to apply the recursive approach for $\vec{\eta}_{k,\ell}$ from the final layer and solve backwards or equivalently, by solving for $\vec{\eta}_{1,\ell}$ in terms of $\vec{\eta}_{\nu,\ell}$. The result is a 'bi-directional' type recursion ('up and down' as phrased by Albrecher and Hartinger (2007)), which clearly produces a computational disadvantage and makes the method rather tedious and computationally heavy, even for the first layer, for large ν. Hence, in the next section, by adopting the alternative methodology of Albrecher and Hartinger (2007), we will pursue an alternative approach based on level crossings, in which explicit solutions for the G-S function and, additionally, the expected discounted dividends until ruin, for a model with ν layers can be obtained solely in term of quantities from a model with $\nu-1$ layers. The advantage of this method is that the recursive method derived is only in one direction, which reduces the complexity present in the method above.

3. Number of Layers Approach

The method presented in this section is based on a recursive approach in terms of the number of layers within the model rather than the initial starting layer given in the previous section. In this case, a model with ν layers can be fully characterised by determination of quantities in a model with $\nu-1$ layers alone. Within this approach, the event of up-crossing or 'upper-exit' from a given layer is of vital importance and as such, we will first present some results relating to these quantities.

3.1. Time Value of "Upper Exit"

Let us introduce stopping times for our risk model with ν layers, such that for $a \le b$ and $u \in [a,b]$

$$\begin{aligned} \tau_\nu^+(u,a,b) &= \inf\{t \ge 0 : U_\nu(t) \ge b | U_\nu(0) = u\} \\ \tau_\nu^-(u,a,b) &= \inf\{t \ge 0 : U_\nu(t) < a | U_\nu(0) = u\}. \end{aligned} \tag{25}$$

We remark that the stopping time $\tau_\nu^+(u,a,b)$ can be interpreted as the first time of exiting through the upper level b (which can only occur via the continuous premium income), whilst $\tau_\nu^-(u,a,b)$ represents the first time of dropping below the lower level a (which can only occur by downward jumps due to the occurrence of a claim). In particular, the time of ruin in a risk model with ν layers can now be denoted by $\tau_\nu(u) \equiv \tau_\nu^-(u,0,\infty)$.

For $1 \le i,k \le n$, $1 \le j,\ell \le m$, let

$$B_{ij,k\ell,\nu}(u,b) = \mathbb{E}_{ij}\left(e^{-\delta\tau_\nu^+(u,0,b)} 1_{[I(\tau_\nu^+(u,0,b))=k, J(\tau_\nu^+(u,0,b))=\ell, \tau_\nu^+(u,0,b)<\tau_\nu^-(u,0,b)]}\right), \tag{26}$$

denote the LT of the stopping time $\tau_\nu^+(u,0,b)$, given that the surplus process reaches the upper level b (for the first time) in states (E_k, G_ℓ) from initial states (E_i, G_j) and initial surplus u, provided that ruin has not occurred. In the remainder of this paper, we will omit the latter condition from the notation for conciseness but remind the reader this is implicitly included within the quantities. Moreover, let $\mathbf{B}_\nu(u,b)$ denote the $nm \times nm$ square matrix with elements $B_{ij,k\ell,\nu}(u,b)$ defined above.

Remark 4. *In a similar manner to that pointed out by Albrecher and Hartinger (2007), we note here that for a model with ν layers, by shifting the top barrier, $b_{\nu-1}$, to infinity then $\mathbf{B}_\nu(u,b)$ is equivalent to $\mathbf{B}_{\nu-1}(u,b)$.*

The main result of this subsection (Proposition 4) shows that the matrices $\mathbf{B}_\nu(u,b)$ for a risk model with ν layers can be expressed solely in terms of matrices for risk model(s) with $\nu-1$ *layers* and a corresponding model with only a single *layer* (see below).

In the following, when we consider a model with $k < \nu$ layers, we mean a risk model which has k layers equivalent to those of the first k layers of our ν layer model, i.e., the barriers b_i for $0 \le i \le k-1$ are equivalent, as are the dynamics of the process within these layers, and $b_k = \infty$. In addition, we introduce a corresponding one-layer risk model with surplus process denoted by $U_{(1,\nu)}(t)$, having premium income p_ν and dividend rate d_ν, i.e., net premium $c_\nu = p_\nu - d_\nu$. That is, a one-layer risk model whose dynamics behave in the same manner as in the final (upper) layer of the model with ν layers. We also introduce the subscript $\{\cdot\}_{(1,\nu)}$, which refers to the corresponding one-layer model equivalent of previously defined quantities, e.g., $\tau_{(1,\nu)}(u)$ denotes the time of ruin for the surplus process $U_{(1,\nu)}(t)$.

Remark 5. *In the case of a single layer, the risk model defined in Equations* (1) *and* (2) *reduces to a special case of a Markov Additive Process, which have been extensively studied in the literature and a number of results regarding exit times have been derived. See, for example, Ivanovs and Palmowski (2012) who provide expressions for a number of exit problems in terms of so-called scale matrices.*

Lemma 1. *For $\nu \in \mathbb{N}^+$ and $\delta \geq 0$ it holds that:*

(i)

$$\begin{aligned} \mathbf{B}_\nu(u,b) &= \mathbf{I}_{nm}, \quad \text{if} \quad u \geq b, \\ \mathbf{B}_\nu(u,b) &= \mathbf{0}_{nm}, \quad \text{if} \quad u < 0, \end{aligned}$$

where $\mathbf{0}_{nm}$ is a square matrix of order $nm \times nm$ with all its elements being equal to zero.

(ii) *For $0 \leq u < b_{\nu-1}$*

$$\mathbf{B}_\nu(u,b) = \begin{cases} \mathbf{B}_{\nu-1}(u,b), & \text{if} \quad b \leq b_{\nu-1}, \\ \mathbf{B}_{\nu-1}(u,b_{\nu-1})\mathbf{B}_\nu(b_{\nu-1},b), & \text{if} \quad b > b_{\nu-1} \end{cases}. \tag{27}$$

(iii) *For $b_{\nu-1} \leq u \leq b$*

$$\begin{aligned} \mathbf{B}_\nu(u,b) &= \mathbf{B}_{(1,\nu)}(u-b_{\nu-1}, b-b_{\nu-1}) + \mathbf{G}_\nu(u-b_{\nu-1}) \\ &\quad -\mathbf{B}_{(1,\nu)}(u-b_{\nu-1}, b-b_{\nu-1})\mathbf{G}_\nu(b-b_{\nu-1}), \end{aligned}$$

where $\mathbf{G}_\nu(u)$ is an $nm \times nm$ square matrix with elements, for $1 \leq i,k \leq n$, $1 \leq j,\ell \leq m$

$$\begin{aligned} G_{ij,k\ell,\nu}(u) &= \mathbb{E}_{ij}\Big(\sum_{k_1=1}^{n}\sum_{k_2=1}^{m} e^{-\delta\tau_{(1,\nu)}(u)} B_{k_1k_2,k\ell,\nu}\big(b_{\nu-1} - \big|U_{(1,\nu)}(\tau_{(1,\nu)}(u))\big|, b\big) \\ &\quad \times 1_{[I(\tau_{(1,\nu)}(u))=k_1, J(\tau_{(1,\nu)}(u))=k_2]}\Big) \end{aligned}$$

and $\mathbf{G}_1(u) \equiv \mathbf{0}_{nm}$, for all $u \geq 0$.

Proof. *(i)* Follows directly from the definitions of $\tau_\nu^+(u,0,b)$ and $\mathbf{B}_\nu(u,b)$, given in Equations (25) and (26) respectively.

(ii) For $0 \leq u < b_{\nu-1}$, first note that the surplus process with ν layers coincides with the surplus process with $(\nu-1)$ layers before the first exit of the interval $[0, b_{\nu-1})$. Thus, it follows that $\tau_\nu^+(u,0,b) = \tau_{\nu-1}^+(u,0,b)$ and hence $B_{ij,k\ell,\nu}(u,b) = B_{ij,k\ell,\nu-1}(u,b)$ for $0 < b \leq b_{\nu-1}$, from which the first part of (ii) follows. For $b \geq b_{\nu-1}$, in order for the surplus process to up-cross the level b, it must first up-cross $b_{\nu-1}$ at which point the process renews with initial surplus $b_{\nu-1}$, given the states of $I(t)$ and $J(t)$ at this time. Thus, we have

$$\begin{aligned} B_{ij,k\ell,\nu}(u,b) &= \mathbb{E}_{ij}\Big(e^{-\delta[\tau_{\nu-1}^+(u,0,b_{\nu-1})+\tau_k^+(b_{\nu-1},0,b)]} 1_{[I(\tau_\nu^+(u,0,b))=k, J(\tau_\nu^+(u,0,b))=\ell]}\Big) \\ &= \sum_{k_1=1}^{n}\sum_{k_2=1}^{m} \mathbb{E}_{ij}\Big(e^{-\delta\tau_{\nu-1}^+(u,0,b_{\nu-1})} 1_{[I(\tau_{\nu-1}^+(u,0,b_{\nu-1}))=k_1, J(\tau_{\nu-1}^+(u,0,b_{\nu-1}))=k_2]}\Big) \\ &\quad \times \mathbb{E}_{k_1k_2}\Big(e^{-\delta\tau_\nu^+(b_{\nu-1},0,b)} 1_{[I(\tau_\nu^+(b_{\nu-1},0,b))=k, J(\tau_\nu^+(b_{\nu-1},0,b))=\ell]}\Big) \\ &= \sum_{k_1=1}^{n}\sum_{k_2=1}^{m} B_{ij,k_1k_2,\nu-1}(u,b_{\nu-1}) B_{k_1k_2,k\ell,\nu}(b_{\nu-1},b), \end{aligned}$$

or equivalently, in matrix form

$$\mathbf{B}_\nu(u,b) = \mathbf{B}_{\nu-1}(u,b_{\nu-1})\mathbf{B}_\nu(b_{\nu-1},b), \qquad b \geq b_{\nu-1}.$$

(iii) For $b_{\nu-1} \leq u \leq b$, there are two possible events that can occur in order to reach the level b: (1) the surplus process up-crosses the level b before dropping below the level $b_{\nu-1}$, and (2) the process drops below $b_{\nu-1}$ before crossing the level b and does so without causing ruin at which point the process renews, given the states of $I(t)$ and $J(t)$, and then crosses b from this new level. Thus, we have

$$\begin{aligned} B_{ij,k\ell,\nu}(u,b) &= \mathbb{E}_{ij}\Big(e^{-\delta\tau_\nu^+(u,b_{k-1},b)}1_{[\tau_\nu^+(u,b_{\nu-1},b)<\tau_\nu^-(u,b_{\nu-1},b),I(\tau_\nu^+(u,b_{\nu-1},b))=k,J(\tau_\nu^+(u,b_{\nu-1},b))=\ell]}\Big) \\ &\quad +\mathbb{E}_{ij}\Big(e^{-\delta\tau^*(u,b_{\nu-1},b)}1_{[\tau_\nu^-(u,b_{\nu-1},b)<\tau_\nu^+(u,b_{\nu-1},b),I(\tau^*(u,b_{\nu-1},b))=k,J(\tau^*(u,b_{\nu-1},b))=\ell]}\Big) \\ &= M_{1,\nu}(u) + M_{2,\nu}(u), \end{aligned} \tag{28}$$

where $\tau^*(u,b_{\nu-1},b) := \tau_\nu^-(u,b_{\nu-1},b) + \tau_\nu^+(U_b(\tau_\nu^-(u,b_{\nu-1},b)),0,b)$.

Now, in a similar way to Albrecher and Hartinger (2007), by using a shifting argument we can re-write Equation (28) in terms of the one-layer risk model. That is, if we shift the surplus process down by $b_{\nu-1}$, such that the lower level $b_{\nu-1} = 0$, the upper level becomes $b - b_{\nu-1}$ and the initial surplus $u - b_{\nu-1}$, it follows that

$$\begin{aligned} M_{1,\nu}(u) &= \mathbb{E}_{ij}\Big(e^{-\delta\tau_{(1,\nu)}^+(u-b_{\nu-1},0,b-b_{\nu-1})}1_{[I(\tau_{(1,\nu)}^+(u-b_{\nu-1},0,b-b_{\nu-1}))=k,J(\tau_{(1,\nu)}^+(u-b_{\nu-1},0,b-b_{\nu-1}))=\ell]}\Big) \\ &= B_{ij,k\ell,(1,\nu)}(u-b_{\nu-1},b-b_{\nu-1}). \end{aligned} \tag{29}$$

Moreover, for $M_{2,\nu}(u)$, we note that after shifting the process down by $b_{\nu-1}$ event (2) is equivalent to the process $U_{(1,\nu)}(t)$, with initial surplus $u - b_{\nu-1}$, experiencing ruin before hitting the level $b - b_{\nu-1}$ and then continuing with a new initial surplus which is related to the deficit at ruin, i.e., $\left|U_{(1,\nu)}(\tau_{(1,\nu)}(u-b_{\nu-1}))\right|$. Note that the magnitude of the deficit cannot exceed $b_{\nu-1}$ as this would cause ultimate ruin in the corresponding model with ν layers. At this point, if we shift the process back up by $b_{\nu-1}$, we remain with a situation described in part (ii). Hence, $M_{2,\nu}(u)$ can be alternatively expressed by

$$\begin{aligned} M_{2,\nu}(u) &= \mathbb{E}_{ij}\Bigg\{\sum_{k_1=1}^{n}\sum_{k_2=1}^{m} e^{-\delta\tau_{(1,\nu)}(u-b_{\nu-1})}B_{k_1k_2,k\ell,\nu}\big(b_{\nu-1} - \left|U_{(1,\nu)}(\tau_{(1,\nu)}(u-b_{\nu-1}))\right|\big) \\ &\quad \times 1_{[I(\tau_{(1,\nu)}(u-b_{\nu-1}))=k_1,J(\tau_{(1,\nu)}(u-b_{\nu-1}))=k_2]}\Bigg\} \\ &\quad -\sum_{k_3=1}^{n}\sum_{k_4=1}^{m} B_{ij,k_3k_4,(1,\nu)}(u-b_{\nu-1},b-b_{\nu-1}) \\ &\quad \times\mathbb{E}_{k_3k_4}\Bigg\{e^{-\delta\tau_{(1,\nu)}(b-b_{\nu-1})}B_{k_1k_2,k\ell,\nu}\big(b_{\nu-1} - \left|U_{(1,\nu)}(\tau_{(1,\nu)}(u-b_{\nu-1}))\right|,b\big) \\ &\quad \times 1_{[I(\tau_{(1,k)}(b-b_{k-1}))=k_1,J(\tau_{(1,k)}(b-b_{k-1}))=k_2]}\Bigg\} \\ &= G_{ij,k\ell,\nu}(u-b_{\nu-1}) - \sum_{k_3=1}^{n}\sum_{k_4=1}^{m} B_{ij,k_3k_4,(1,\nu)}(u-b_{\nu-1},b-b_{\nu-1})G_{k_3k_4,k\ell,\nu}(b-b_{\nu-1}), \end{aligned} \tag{30}$$

where the last term corrects for those trajectories of $U_{(1,\nu)}(t)$ reaches $b - b_{\nu-1}$ before ruin. Then from Equations (28)–(31) we find that

$$\begin{aligned} B_{ij,k\ell,\nu}(u) &= M_{1,\nu}(u) + M_{2,\nu}(u) \\ &= B_{ij,k\ell,(1,\nu)}(u - b_{\nu-1}, b - b_{\nu-1}) + G_{ij,k\ell,\nu}(u - b_{\nu-1}) \\ &\quad - \sum_{k_3=1}^{n}\sum_{k_4=1}^{m} B_{ij,k_3k_4,(1,\nu)}(u - b_{\nu-1}, b - b_{\nu-1}) G_{k_3k_4,k\ell,\nu}(b - b_{\nu-1}), \end{aligned}$$

or, equivalently, in matrix form

$$\mathbf{B}_\nu(u,b) = \mathbf{B}_{(1,\nu)}(u - b_{\nu-1}, b - b_{\nu-1}) + \mathbf{G}_\nu(u - b_{\nu-1}) - \mathbf{B}_{(1,\nu)}(u - b_{\nu-1}, b - b_{\nu-1})\mathbf{G}_\nu(b - b_{\nu-1}).$$

Finally, note that in the single-layer model $b_{\nu-1} = b_0 = 0$ and thus, $\tau_{(1,\nu)}(u) \equiv \tau^-_{u,0,b}(u)$. Hence, it is impossible for the surplus to drop below the $b_{\nu-1}$ barrier without causing ultimate ruin and consequently, $\mathbf{G}_1(u) \equiv \mathbf{0}_{nm}$, for all $u \geq 0$. This completes the proof. □

Lemma 1 provides a method for determining the upper exit quantities for a model with ν layers, depending on the value of the initial surplus $u \geq 0$. However, by applying the results of this Lemma, we can derive recursive expressions for which the upper exit for a model with ν layers can be completely determined via quantities from a model with $\nu - 1$ layers only. The following proposition corresponds to the matrix generalisation of Proposition 3.1 in Albrecher and Hartinger (2007).

Proposition 4. *For $\nu \in \mathbb{N}^+$ and $\delta \geq 0$ it holds that:*

(i) *For $0 \leq u < b_{\nu-1}$*

$$\mathbf{B}_\nu(u,b) = \begin{cases} \mathbf{B}_{\nu-1}(u,b), & \text{if } \ b \leq b_{\nu-1}, \\ \mathbf{B}_{\nu-1}(u,b_{\nu-1})[\mathbf{I}_{nm} - \mathbf{K}_\nu(b_{\nu-1})]^{-1}\mathbf{B}_{(1,\nu)}(0, b - b_{\nu-1}), & \text{if } \ b > b_{\nu-1}, \end{cases}$$

where

$$\mathbf{K}_\nu(u) = \mathbf{H}_\nu(u - b_{\nu-1}) - \mathbf{B}_{(1,\nu)}(u - b_{\nu-1}, b - b_{\nu-1})\mathbf{H}_\nu(b - b_{\nu-1}),$$

and $\mathbf{H}_\nu(u)$ is a square matrix of order $nm \times nm$ with elements

$$\begin{aligned} H_{ij,k\ell,\nu}(u) &= \mathbb{E}_{ij}\Big(\sum_{k_1=1}^{n}\sum_{k_2=1}^{m} e^{-\delta\tau_{(1,\nu)}(u)} B_{k_1k_2,k\ell,\nu-1}\big(b_{\nu-1} - \big|U_{(1,\nu)}(\tau_{(1,\nu)}(u))\big|, b\big) \\ &\quad \times 1_{[I(\tau_{(1,\nu)}(u))=k_1, J(\tau_{(1,\nu)}(u))=k_2]}\Big). \end{aligned} \tag{31}$$

(ii) *For $b_{\nu-1} \leq u \leq b$*

$$\mathbf{B}_\nu(u,b) = \mathbf{B}_{(1,\nu)}(u - b_{\nu-1}, b - b_{\nu-1}) + \mathbf{H}_\nu(u)[\mathbf{I}_{nm} - \mathbf{K}_\nu(b_{\nu-1})]^{-1}\mathbf{B}_{(1,\nu)}(0, b - b_{\nu-1}).$$

Proof. To begin, first note that if we insert result (ii) of Lemma 1 into the definition of the elements $G_{ij,k\ell,\nu}(u)$, then we have

$$\begin{aligned} G_{ij,k\ell,\nu}(u) &= \mathbb{E}_{ij}\Big(\sum_{k_1=1}^{n}\sum_{k_2=1}^{m} e^{-\delta\tau_{(1,\nu)}(u)} \sum_{k_3=1}^{n}\sum_{k_4=1}^{m} B_{k_1k_2,k_3k_4,\nu-1}\big(b_{\nu-1} - \big|U_{(1,\nu)}(\tau_{(1,\nu)}(u))\big|, b\big) \\ &\quad \times B_{k_3k_4,k\ell,\nu}(b_{\nu-1}, b) 1_{[I(\tau_{(1,\nu)}(u))=k_1, J(\tau_{(1,\nu)}(u))=k_2]}\Big) \\ &= \sum_{k_3=1}^{n}\sum_{k_4=1}^{m} \mathbb{E}_{ij}\Big(\sum_{k_1=1}^{n}\sum_{k_2=1}^{m} e^{-\delta\tau_{(1,\nu)}(u)} B_{k_1k_2,k_3k_4,\nu-1}\big(b_{\nu-1} - \big|U_{(1,\nu)}(\tau_{(1,\nu)}(u))\big|, b\big) \\ &\quad \times 1_{[I(\tau_{(1,\nu)}(u))=k_1, J(\tau_{(1,\nu)}(u))=k_2]}\Big) B_{k_3k_4,k\ell,\nu}(b_{\nu-1}, b), \end{aligned}$$

or, alternatively, in matrix form

$$\mathbf{G}_\nu(u) = \mathbf{H}_\nu(u)\mathbf{B}_\nu(b_{\nu-1}, b),$$

where $\mathbf{H}_\nu(u)$ is a square matrix of order $nm \times nm$ with elements defined in Equation (31). As such, (iii) of Lemma 1, can be written in the alternative form

$$\begin{aligned} \mathbf{B}_\nu(u,b) &= \mathbf{B}_{(1,\nu)}(u-b_{\nu-1}, b-b_{\nu-1}) \\ &\quad + \Big[\mathbf{H}_\nu(u-b_{\nu-1}) - \mathbf{B}_{(1,\nu)}(u-b_{\nu-1}, b-b_{\nu-1})\mathbf{H}_\nu(b-b_{\nu-1})\Big]\mathbf{B}_\nu(b_{\nu-1}, b). \\ &\quad = \mathbf{B}_{(1,\nu)}(u-b_{\nu-1}, b-b_{\nu-1}) + \mathbf{K}_\nu(u)\mathbf{B}_\nu(b_{\nu-1}, b), \end{aligned} \tag{32}$$

where $\mathbf{K}_\nu(u) = \mathbf{H}_\nu(u-b_{\nu-1}) - \mathbf{B}_{(1,\nu)}(u-b_{\nu-1}, b-b_{\nu-1})\mathbf{H}_\nu(b-b_{\nu-1})$. Now, letting $u = b_{\nu-1}$, in the above equation, gives

$$\mathbf{B}_\nu(b_{\nu-1}, b) = \mathbf{B}_{(1,\nu)}(0, b-b_{\nu-1}) + \mathbf{K}_\nu(b_{\nu-1})\mathbf{B}_\nu(b_{\nu-1}, b), \tag{33}$$

which, after solving for $\mathbf{B}_\nu(b_{\nu-1}, b)$, yields that

$$\mathbf{B}_\nu(b_{\nu-1}, b) = [\mathbf{I}_{nm} - \mathbf{K}_\nu(b_{\nu-1})]^{-1}\mathbf{B}_{(1,\nu)}(0, b-b_{\nu-1}),$$

where the existence of the inverse matrix follows due to diagonal dominance. Finally, substituting this result into Equations (27) and (32), prove results (i) and (ii), respectively. □

Remark 6. *We point out that the matrix $\mathbf{H}_\nu(u)$, and consequently $\mathbf{K}_\nu(u)$, are defined in terms of $\tau_{(1,\nu)}(u)$ and elements of $\mathbf{B}_{\nu-1}(\cdot, b)$ only. As such, these matrices also solely depend on quantities of risk model(s) with $\nu - 1$ layers or fewer.*

3.2. The Expected Discounted Penalty Function

In this subsection, we show how to use the 'upper-exit' quantities of the previous subsection to derive a layer-based approach for the expected discounted penalty function $\phi(u,\nu)$, for a model with ν layers.

Let $\phi_{ij,\ell}(u,\nu)$ denote the general collection (over k) of the individual G-S functions $\phi_{ij,k,\ell}(u,\nu)$ defined in Equation (5). That is,

$$\phi_{ij,\ell}(u,\nu) = \begin{cases} \phi_{ij,1,\ell}(u,\nu), & 0 \le u < b_1 \\ \phi_{ij,2,\ell}(u,\nu), & b_1 \le u < b_2 \\ \quad\vdots & \\ \phi_{ij,\nu,\ell}(u,\nu), & u \ge b_{\nu-1}. \end{cases} \tag{34}$$

Then, for $0 \le u < b_{\nu-1}$, by conditioning on the events $\{\tau_\nu^+(u,0,b_{\nu-1}) < \tau_\nu^-(u,0,b_{\nu-1})\}$ or $\{\tau_\nu^+(u,0,b_{\nu-1}) > \tau_\nu^-(u,0,b_{\nu-1})\}$ respectively, we obtain

$$\begin{aligned}\phi_{ij,\ell}(u,\nu) \;&=\; \mathbb{E}_{ij}\Bigg(\sum_{k_1=1}^{n}\sum_{k_2=1}^{m} e^{-\delta\tau_\nu^+(u,0,b_{\nu-1})}\phi_{k_1k_2,\ell}(b_{\nu-1},\nu) \\ &\quad \times 1_{\left[\tau_\nu^+(u,0,b_{\nu-1})<\tau_\nu^-(u,0,b_{\nu-1}),I(\tau_\nu^+(u,0,b_{\nu-1}))=k_1,J(\tau_\nu^+(u,0,b_{\nu-1}))=k_2\right]}\Bigg) \\ &\quad +\mathbb{E}_{ij}\Bigg(e^{-\delta\tau_\nu^-(u,0,b_{\nu-1})}w\big(U_\nu(\tau_\nu^-(u,0,b_{\nu-1}))-,\big|U_\nu(\tau_\nu^-(u,0,b_{\nu-1}))\big|\big) \\ &\quad \times 1_{\left[\tau_\nu^+(u,0,b_{\nu-1})>\tau_\nu^-(u,0,b_{\nu-1}),R=\ell\right]}\Bigg) \\ &\quad = \sum_{k_1=1}^{n}\sum_{k_2=1}^{m} B_{ij,k_1k_2,\nu}(u,b_{\nu-1})\phi_{k_1k_2,\ell}(b_{\nu-1},\nu)+C_{ij,\ell}(u,\nu),\end{aligned} \tag{35}$$

where

$$\begin{aligned}C_{ij,\ell}(u,\nu) \;&=\; \mathbb{E}_{ij}\Bigg(e^{-\delta\tau_\nu^-(u,0,b_{\nu-1})}w\big(U_\nu(\tau_\nu^-(u,0,b_{\nu-1}))-,\big|U_\nu(\tau_\nu^-(u,0,b_{\nu-1}))\big|\big) \\ &\quad \times 1_{\left[\tau_\nu^+(u,0,b_{\nu-1})>\tau_\nu^-(u,0,b_{\nu-1}),R=\ell\right]}\Bigg) \\ &\quad = \phi_{ij,\ell}(u,\nu-1)-\sum_{k_1=1}^{n}\sum_{k_2=1}^{m} B_{ij,k_1k_2,\nu}(u,b_{\nu-1})\phi_{k_1k_2,\ell}(b_{\nu-1},\nu-1).\end{aligned} \tag{36}$$

Combining Equations (35) and (36) and using result (ii) of Lemma 1, we have

$$\begin{aligned}\phi_{ij,\ell}(u,\nu) \;&=\; \phi_{ij,\ell}(u,\nu-1)+\sum_{k_1=1}^{n}\sum_{k_2=1}^{m} B_{ij,k_1k_2,\nu-1}(u,b_{\nu-1})\big[\phi_{k_1k_2,\ell}(b_{\nu-1},\nu) \\ &\quad -\phi_{k_1k_2,\ell}(b_{\nu-1},\nu-1)\big],\end{aligned} \tag{37}$$

or equivalently, in matrix/vector notation

$$\vec{\boldsymbol{\phi}}_\ell(u,\nu)=\vec{\boldsymbol{\phi}}_\ell(u,\nu-1)+\mathbf{B}_{\nu-1}(u,b_{\nu-1})\big[\vec{\boldsymbol{\phi}}_\ell(b_{\nu-1},\nu)-\vec{\boldsymbol{\phi}}_\ell(b_{\nu-1},\nu-1)\big], \tag{38}$$

where $\vec{\boldsymbol{\phi}}_\ell(u,\nu)=(\phi_{11,\ell}(u,\nu),\ldots,\phi_{nm,\ell}(u,\nu))^\top$.

On the other hand, for $u\geq b_{\nu-1}$, we can condition on the size of first drop below the level $b_{\nu-1}$ which, by applying a similar shifting argument as in the proof of (iii) in Lemma 1 (shifting down by $b_{\nu-1}$), for $1\leq i\leq n, 1\leq j\leq m$ and $\ell=1,2$, gives

$$\begin{aligned}\phi_{ij,\ell}(u,\nu) \;&=\; \mathbb{E}_{ij}\Big(\sum_{k_1=1}^{n}\sum_{k_2=1}^{m} e^{-\delta\tau_{(1,\nu)}(u-b_{\nu-1})}\phi_{k_1k_2,\ell}(b_{\nu-1}-|U_{(1,\nu)}(\tau_{(1,\nu)}(u-b_{\nu-1}))|,\nu) \\ &\quad \times 1_{[I(\tau_{(1,\nu)}(u-b_{\nu-1}))=k_1,J(\tau_{(1,\nu)}(u-b_{\nu-1}))=k_2,|U_{(1,\nu)}(\tau_{(1,\nu)}(u-b_{\nu-1}))|\leq b_{\nu-1}]}\Big) \\ &\quad +\mathbb{E}_{ij}\Big(\sum_{k_1=1}^{n}\sum_{k_2=1}^{m} e^{-\delta\tau_{(1,\nu)}(u-b_{\nu-1})}w\big(b_{\nu-1}+U_{(1,\nu)}(\tau_{(1,\nu)}(u-b_{\nu-1})-), \\ &\quad b_{\nu-1}-|U_{(1,\nu)}(\tau_{(1,\nu)}(u-b_{\nu-1}))|\big)1_{[I(\tau_{(1,\nu)}(u-b_{\nu-1}))=k_1,J(\tau_{(1,\nu)}(u-b_{\nu-1}))=k_2]} \\ &\quad \times 1_{[U_{(1,\nu)}(\tau_{(1,\nu)}(u-b_{\nu-1}))>b_{\nu-1},R=\ell]}\Big),\end{aligned}$$

which, after using Equation (37) in the first term, becomes

$$\begin{aligned}\phi_{ij,\ell}(u,v) &= A_{ij,\ell}(u-b_{v-1},v)\\ &+\mathbb{E}_{ij}\Big(\sum_{k_1=1}^{n}\sum_{k_2=1}^{m} e^{-\delta\tau_{(1,v)}(u-b_{v-1})}1_{[I(\tau_{(1,v)}(u-b_{v-1}))=k_1,J(\tau_{(1,v)}(u-b_{v-1}))=k_2]}\\ &\times\sum_{k_3=1}^{n}\sum_{k_4=1}^{m} B_{k_1k_2,k_3k_4,v-1}(b_{v-1}-|U_{(1,v)}(\tau_{(1,v)}(u-b_{v-1}))|,b_{v-1})\\ &\times\big(\phi_{k_3k_4,\ell}(b_{v-1},v)-\phi_{k_3k_4,\ell}(b_{v-1},v-1)\big)\Big),\end{aligned}$$

where

$$\begin{aligned}A_{ij,\ell}(u-b_{v-1},v) &= \mathbb{E}_{ij}\Big(\sum_{k_1=1}^{n}\sum_{k_2=1}^{m} e^{-\delta\tau_{(1,v)}(u-b_{v-1})}\\ &\times\phi_{k_1k_2,\ell}(b_{v-1}-|U_{(1,v)}(\tau_{(1,v)}(u-b_{v-1}))|,v-1)\\ &\times 1_{[I(\tau_{(1,v)}(u-b_{v-1}))=k_1,J(\tau_{(1,v)}(u-b_{v-1}))=k_2,|U_{(1,v)}(\tau_{(1,v)}(u-b_{v-1}))|\le b_{v-1}]}\Big)\\ &+\mathbb{E}_{ij}\Big(\sum_{k_1=1}^{n}\sum_{k_2=1}^{m} e^{-\delta\tau_{(1,v)}(u-b_{v-1})}\\ &\times w\big(b_{v-1}+U_{(1,v)}(\tau_{(1,v)}(u-b_{v-1})-),b_{v-1}-|U_{(1,v)}(\tau_{(1,v)}(u-b_{v-1}))|\big)\\ &\times 1_{[I(\tau_{(1,v)}(u-b_{v-1}))=k_1,J(\tau_{(1,v)}(u-b_{v-1}))=k_2]}\\ &\times 1_{[U_{(1,v)}(\tau_{(1,v)}(u-b_{v-1}))>b_{v-1},R=\ell]}\Big).\end{aligned}$$

Equivalently, in matrix/vector form, for $u\ge b_{v-1}$, we have

$$\vec{\boldsymbol{\phi}}_\ell(u,v)=\vec{A}_\ell(u-b_{v-1},v)+\mathbf{H}_v(u-b_{v-1})\big(\vec{\boldsymbol{\phi}}_\ell(b_{v-1},v)-\vec{\boldsymbol{\phi}}_\ell(b_{v-1},v-1)\big). \quad (39)$$

Now, due to the continuity of $\vec{\boldsymbol{\phi}}_\ell(u,v)$ at $u=b_{v-1}$, from Equations (38) and (39), we obtain

$$\begin{aligned}&\vec{\boldsymbol{\phi}}_\ell(b_{v-1},v-1)+\big(\vec{\boldsymbol{\phi}}_\ell(b_{v-1},v)-\vec{\boldsymbol{\phi}}_\ell(b_{v-1},v-1)\big)\\ &\qquad=\vec{A}_\ell(0,v)+\mathbf{H}_v(0)\big(\vec{\boldsymbol{\phi}}_\ell(b_{v-1},v)-\vec{\boldsymbol{\phi}}_\ell(b_{v-1},v-1)\big),\end{aligned}$$

since, from Lemma 1 $\mathbf{B}_{v-1}(b_{v-1},b_{v-1})=\mathbf{I}_{nm}$, and thus

$$\vec{\boldsymbol{\phi}}_\ell(b_{v-1},v)-\vec{\boldsymbol{\phi}}_\ell(b_{v-1},v-1)=[\mathbf{I}_{nm}-\mathbf{H}_v(0)]^{-1}\Big(\vec{A}_\ell(0,v)-\vec{\boldsymbol{\phi}}_\ell(b_{v-1},v-1)\Big),$$

where the inverse matrix exists by diagonal dominance. The above results are summarised in the following proposition.

Proposition 5. *For a risk model with v layers and $\ell=1,2$, the G-S function $\vec{\boldsymbol{\phi}}_\ell(u,v)$ is given by*

$$\vec{\boldsymbol{\phi}}_\ell(u,v)=\begin{cases}\vec{\boldsymbol{\phi}}_\ell(u,v-1)+\mathbf{B}_{v-1}(u,b_{v-1})[\mathbf{I}_{nm}-\mathbf{H}_v(0)]^{-1}\Big(\vec{A}_\ell(0,v)-\vec{\boldsymbol{\phi}}_\ell(b_{v-1},v-1)\Big), & \text{for } 0\le u<b_{v-1},\\ \vec{A}_\ell(u-b_{v-1},v)+\mathbf{H}_v(u-b_{v-1})[\mathbf{I}_{nm}-\mathbf{H}_v(0)]^{-1}\Big(\vec{A}_\ell(0,v)-\vec{\boldsymbol{\phi}}_\ell(b_{v-1},v-1)\Big), & \text{for } u\ge b_{v-1}.\end{cases}$$

Remark 7. *In a similar way to Remark 6, we point out that the vector $\vec{A}_\ell(u,v)$ is actually defined solely in terms of quantities for a model with $v-1$ layers and a single layer. As such,* *Proposition 5 provides a recursive formula for $\vec{\boldsymbol{\phi}}_\ell(u,\nu)$ in terms of quantities from models with $\nu-1$ (or fewer) layers.*

Remark 8. *To see the difference in the two methods ('layer-by-layer' and 'number of layers') via a numerical example, we direct the reader to Albrecher and Hartinger (2007) who demonstrate the results for a compound Poisson risk model with exponential claims in a $\nu=2$ barrier model.*

3.3. The Expected Discounted Dividends

For the multi-layer risk model defined in Equations (1) and (2), recall that whilst the surplus is in layer k, i.e., in the interval $[b_{k-1},b_k)$, the insurer pays out dividends at a constant rate $d_k\geq 0$. Hence, in this subsection we will use the results of Lemma 1 and apply similar techniques as above, to obtain a recursive expression for the expected discounted dividend payments until ruin.

Let $\{D(t)\}_{t\geq 0}$ denote the accumulated dividends paid up to time $t\geq 0$ having dynamics

$$dD(t)=d_k\,dt, \qquad \text{whilst} \qquad b_{k-1}\leq U_\nu(t)<b_k,$$

and define

$$D_{u,\nu}=\int_0^{\tau_\nu(u)}e^{-\delta t}dD(t),$$

to be the present value of the total discounted dividends until ruin for a risk model with ν layers and $D_{u,\nu}=0$ for $u<0$. Then, the expected discounted dividends until ruin is defined by $V(u,\nu)=\mathbb{E}(D_{u,\nu})$, which can be decomposed, for $1\leq i\leq n, 1\leq j\leq m$ and $\ell=1,2$, into quantities of the form

$$V_{ij,\ell}(u,\nu)=\mathbb{E}_{ij}\big(D_{u,\nu}1_{(R=\ell)}\big)$$

such that

$$V(u,\nu)=\vec{\rho}^\top\left(\vec{V}_1(u,\nu)+\vec{V}_2(u,\nu)\right),$$

where $\vec{V}_\ell(u,\nu)=\big(V_{11,\ell}(u,\nu),\ldots,V_{nm,\ell}(u,\nu)\big)^\top$ for $\ell=1,2$.

Then, in a similar way as in Section 3.2, by conditioning on the events $\{\tau_\nu^+(u,0,b_{\nu-1})<\tau_\nu^-(u,0,b_{\nu-1})\}$ or $\{\tau_\nu^+(u,0,b_{\nu-1})>\tau_\nu^-(u,0,b_{\nu-1})\}$ respectively, for $0\leq u<b_{\nu-1}, 1\leq i\leq n, 1\leq j\leq m$ and $\ell=1,2$, we obtain

$$\begin{aligned}
V_{ij,\ell}(u,\nu) &= \mathbb{E}_{ij}\left(\int_0^{\tau_\nu(u)}e^{-\delta t}dD(t)1_{(R=\ell)}\right)\\
&= \mathbb{E}_{ij}\left(\sum_{k_1=1}^{n}\sum_{k_2=1}^{m}\left[\int_0^{\tau_\nu^+(u,0,b_{\nu-1})}e^{-\delta t}dD(t)+\int_{\tau_\nu^+(u,0,b_{\nu-1})}^{\tau_\nu(u)}e^{-\delta t}dD(t)\right]\right.\\
&\quad \left.\times 1_{(\tau_\nu^+(u,0,b_{\nu-1})<\tau_\nu^-(u,0,b_{\nu-1}),I(\tau_\nu^+(u,0,b_{\nu-1}))=k_1,J(\tau_\nu^+(u,0,b_{\nu-1}))=k_2,R=\ell)}\right)\\
&\quad +\mathbb{E}_{ij}\left(\int_0^{\tau_\nu^-(u,0,b_{\nu-1})}e^{-\delta t}dD(t)1_{(\tau_\nu^-(u,0,b_{\nu-1})<\tau_\nu^+(u,0,b_{\nu-1}),R=\ell)}\right)\\
&= \mathbb{E}_{ij}\left(\int_0^{\tau_\nu^+(u,0,b_{\nu-1})}e^{-\delta t}dD(t)1_{(\tau_\nu^+(u,0,b_{\nu-1})<\tau_\nu^-(u,0,b_{\nu-1}),R=\ell)}\right)\\
&\quad +\sum_{k_1=1}^{n}\sum_{k_2=1}^{m}B_{ij,k_1k_2,\nu-1}(u,b_{\nu-1})V_{k_1k_2,\ell}(b_{\nu-1},\nu)\\
&\quad +\mathbb{E}_{ij}\left(\int_0^{\tau_\nu^-(u,0,b_{\nu-1})}e^{-\delta t}dD(t)1_{(\tau_\nu^-(u,0,b_{\nu-1})<\tau_\nu^+(u,0,b_{\nu-1}),R=\ell)}\right),
\end{aligned} \tag{40}$$

where we have used result (ii) of Lemma 1 in the final equality. Moreover, since this equation holds for an arbitrary number of layers, it also follows for $0 \le u < b_{\nu-1}$ that

$$\begin{aligned} V_{ij,\ell}(u,\nu-1) &= \mathbb{E}_{ij}\left(\int_0^{\tau^+_{\nu-1}(u,0,b_{\nu-1})} e^{-\delta t} dD(t) 1_{(\tau^+_{\nu-1}(u,0,b_{\nu-1})<\tau^-_{\nu-1}(u,0,b_{\nu-1}),R=\ell)}\right) \\ &+ \sum_{k_1=1}^{n}\sum_{k_2=1}^{m} B_{ij,k_1k_2,\nu-1}(u,b_{\nu-1})V_{k_1k_2,\ell}(b_{\nu-1},\nu-1) \\ &+\mathbb{E}_{ij}\Big(\int_0^{\tau^-_{\nu-1}(u,0,b_{\nu-1})} e^{-\delta t} dD(t) 1_{(\tau^-_{\nu-1}(u,0,b_{\nu-1})<\tau^+_{\nu-1}(u,0,b_{\nu-1}),R=\ell)}\Big). \end{aligned} \tag{41}$$

Now, by noting that $\tau^+_\nu(u,0,b_{\nu-1}) \equiv \tau^+_{\nu-1}(u,0,b_{\nu-1})$ and $\tau^-_\nu(u,0,b_{\nu-1}) \equiv \tau^-_{\nu-1}(u,0,b_{\nu-1})$, combining Equations (40) and (41), we get that

$$\begin{aligned} V_{ij,\ell}(u,\nu) &= V_{ij,\ell}(u,\nu-1) \\ &+ \sum_{k_1=1}^{n}\sum_{k_2=1}^{m} B_{ij,k_1k_2,\nu-1}(u,b_{\nu-1})\big(V_{k_1k_2,\ell}(b_{\nu-1},\nu) - V_{k_1k_2,\ell}(b_{\nu-1},\nu-1)\big), \end{aligned} \tag{42}$$

or in matrix/vector form

$$\vec{V}_\ell(u,\nu) = \vec{V}_\ell(u,\nu-1) + \mathbf{B}_{\nu-1}(u,b_{\nu-1})\Big(\vec{V}_\ell(b_{\nu-1},\nu) - \vec{V}_\ell(b_{\nu-1},\nu-1)\Big). \tag{43}$$

For $u \ge b_{\nu-1}$, we consider a similar argument to that of the previous section, that is by apply a shifting argument, conditioning on whether or not the process drops below the level $b_{\nu-1}$ and if so, the corresponding size of the drop, which gives

$$\begin{aligned} V_{ij,\ell}(u,\nu) &= V_{ij,\ell,(1,\nu)}(u-b_{\nu-1}) + \mathbb{E}_{ij}\Bigg(\sum_{k_1=1}^{n}\sum_{k_2=1}^{m} e^{-\delta\tau_{(1,\nu)}(u-b_{\nu-1})} \\ &\times V_{k_1k_2,\ell}(b_{\nu-1} - U_{(1,\nu)}(\tau_{(1,\nu)}(u-b_{\nu-1})),\nu) \\ &\times 1_{[I(\tau_{(1,\nu)}(u-b_{\nu-1}))=k_1, J(\tau_{(1,\nu)}(u-b_{\nu-1}))=k_2]}\Bigg), \end{aligned}$$

where $V_{ij,\ell,(1,\nu)}(u)$ denotes the expected discounted dividends until ruin for a risk model with only a single layer, paying out dividends continuously at rate $d_\nu \ge 0$. Then, after substituting the result of Equation (42) into the second term and writing in matrix/vector form, for $u \ge b_{\nu-1}$, we obtain

$$\begin{aligned} \vec{V}_\ell(u,\nu) &= \vec{V}_{\ell,(1,\nu)}(u-b_{\nu-1}) + \vec{D}_\ell(u-b_{\nu-1},\nu) \\ &+\mathbf{H}_\nu(u-b_{\nu-1})\Big(\vec{V}_\ell(b_{\nu-1},\nu) - \vec{V}_\ell(b_{\nu-1},\nu-1)\Big), \end{aligned} \tag{44}$$

where $\vec{V}_{\ell,(1,\nu)}(u) = \Big(V_{11,\ell,(1,\nu)}(u),\ldots,V_{nm,\ell,(1,\nu)}(u)\Big)^\top$ and $\vec{D}_\ell(u,\nu) = (D_{11,\ell}(u,\nu),\ldots,$ $D_{nm,\ell}(u,\nu))^\top$ with ij-th element

$$\begin{aligned} D_{ij,\ell}(u,\nu) &= \mathbb{E}_{ij}\Bigg(\sum_{k_1=1}^{n}\sum_{k_2=1}^{m} e^{-\delta\tau_{(1,\nu)}(u)} \\ &\times V_{k_1k_2,\ell}(b_{\nu-1} - U_{(1,\nu)}(\tau_{(1,\nu)}(u)),\nu-1) 1_{[I(\tau_{(1,\nu)}(u))=k_1, J(\tau_{(1,\nu)}(u))=k_2]}\Bigg). \end{aligned}$$

Finally, due to the continuity at the barrier, we can equate Equations (43) and (44) with $u = b_{\nu-1}$ and re-arrange the resulting equation to find

$$\vec{V}_\ell(b_{\nu-1},\nu) - \vec{V}_\ell(b_{\nu-1},\nu-1) = [\mathbf{I}_{nm} - \mathbf{H}_\nu(0)]^{-1}\Big(\vec{D}_\ell(0,\nu) + \vec{V}_{\ell,(1,\nu)}(0) - \vec{V}_\ell(b_{\nu-1,\nu-1})\Big).$$

To summarize our results, we give the following proposition.

Proposition 6. *For a risk model with ν layers and $\ell = 1, 2$, the expected discounted dividends until ruin, $\vec{V}_\ell(u, \nu)$, is given by*

$$\vec{V}_\ell(u,\nu) = \begin{cases} \vec{V}_\ell(u,\nu-1) + \mathbf{B}_{\nu-1}(u, b_{\nu-1})[\mathbf{I}_{nm} - \mathbf{H}_\nu(0)]^{-1} \\ \qquad \times \Big(\vec{D}_\ell(0,\nu) + \vec{V}_{\ell,(1,\nu)}(0) - \vec{V}_\ell(b_{\nu-1,\nu-1})\Big), \quad \text{for} \quad 0 \le u < b_{\nu-1}, \\ \vec{V}_\ell(u,\nu-1) + \vec{D}_\ell(u - b_{\nu-1}, \nu) + \mathbf{H}_\nu(u - b_{\nu-1})[\mathbf{I}_{nm} - \mathbf{H}_\nu(0)]^{-1} \\ \qquad \times \Big(\vec{D}_\ell(0,\nu) + \vec{V}_{\ell,(1,\nu)}(0) - \vec{V}_\ell(b_{\nu-1,\nu-1})\Big), \quad \text{for} \quad u \ge b_{\nu-1}. \end{cases}$$

Remark 9. *The vector $\vec{D}_\ell(u, \nu)$ is similar to that of $\vec{A}_\ell(u, \nu)$ in Proposition 5 in the sense that it is defined in terms of quantities from models with $\nu - 1$ layers or fewer. As such, Proposition 6 provides a recursive expression for calculating the expected discounted dividends for a risk model with ν layers, in terms of corresponding quantities for models with fewer layers, i.e., a classic one-way 'forward' type recursion.*

Author Contributions: Conceptualization, A.D.P. and L.R.; methodology, A.D.P. and L.R.; formal analysis, A.D.P. and L.R.; investigation, A.D.P. and L.R.; writing—original draft preparation, A.D.P. and L.R.; writing—review and editing, A.D.P. and L.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the anonymous referees for their insightful comments which helped improve the final version of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

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