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General Bivariate Appell Polynomials via Matrix Calculus and Related Interpolation Hints

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Abstract: An approach to general bivariate Appell polynomials based on matrix calculus is proposed. Known and new basic results are given, such as recurrence relations, determinant forms, differential equations and other properties. Some applications to linear functional and linear interpolation are sketched. New and known examples of bivariate Appell polynomial sequences are given.

Keywords: Polynomial sequences; Appell polynomials; bivariate Appell sequence

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1. Introduction

Appell polynomials have many applications in various disciplines: probability theory [1–5], number theory [6], linear recurrence [7], general linear interpolation [8–12], operators approximation theory [13–17]. In [18], P. Appell introduced a class of polynomials by the following equivalent conditions: $\{A_n\}_{n\in\mathbb{N}}$ is an Appell sequence (A_n being a polynomial of degree n) if either

$$\begin{cases} \dfrac{d\,A_n(x)}{dx} = nA_{n-1}(x), & n \geq 1, \\ A_n(0) = \alpha_n, \quad \alpha_0 \neq 0, \quad \alpha_n \in \mathbb{R}, & n \geq 0, \\ A_0(x) = 1, \end{cases}$$

or

$$A(t)e^{xt} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!},$$

where $A(t) = \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!}, \quad \alpha_0 \neq 0, \quad \alpha_k \in \mathbb{R},\ k \geq 0.$

Subsequentely, many other equivalent characterizations have been formulated. For example, in [19] [p. 87], there are seven equivalences.

Properties of Appell sequences are naturally handled within the framework of modern classic umbral calculus (see [19,20] and references therein).

Special polynomials in two variables are useful from the point of view of applications, particularly in probability [21], in physics, expansion of functions [22], etc. These polynomials allow the derivation of a number of useful identities in a fairly straightforward way and help in introducing new families of polynomials. For example, in [23] the authors introduced general classes of two variables Appell polynomials by using properties of an iterated isomorphism related to the Laguerre-type exponentials. In [24], the two-variable general polynomial (2VgP) family $p_n(x,y)$ has been considered, whose members are defined by the generating function

$$e^{xt}\phi(y,t) = \sum_{n=0}^{\infty} p_n(x,y)\frac{t^n}{n!},$$

where $\phi(y,t) = \sum_{k=0}^{\infty} \phi_k(y) \frac{t^k}{k!}$.

Later, the authors considered the two-variable general Appell polynomials (2VgAP) denoted by ${}_pA_n(x,y)$ based on the sequence $\{p_n\}^b_{n\in\mathbb{N}}$, that is

$$A(t)e^{xt}\phi(y,t) = \sum_{n=0}^{\infty} {}_pA_n(x,y)\frac{t^n}{n!},$$

where $A(t) = \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!}, \quad \alpha_0 \neq 0, \quad \alpha_k \in \mathbb{R},\ k \geq 0.$

These polynomials are framed within the context of monomiality principle [24–27].

Generalizations of Appell polynomials can be also found in [22,28–31] (see also the references therein).

In this paper, we will reconsider the 2VgAP, but with a systematic and alternative theory, that is matrix calculus-based. To the best of authors knowledge, a systematic approach to general bivariate Appell sequences does not appear in the literature. New properties are given and a general linear interpolation problem is hinted. Some applications of the previous theory are given and new families of bivariate polynomials are presented. Moreover a biorthogonal system of linear functionals and polynomials is constructed.

In particular, the paper is organized as follows: in Section 2 we give the definition and the first characterizations of general bivariate Appell polynomial sequences; in Sections 3–5 we derive, respectively, matrix form, recurrence relations and determinant forms for the elements of a general bivariate Appell polynomial sequence. These sequences satisfy some interesting differential equations (Section 6) and properties (Section 7). In Section 8 we consider the relations with linear functional of linear interpolation. Section 9 introduces new and known examples of polynomial sequences. Finally, Section 10 contains some concluding remarks.

We point out that the first recurrence formula and the determinant forms, as well as the relationship with linear functionals and linear interpolation, to the best of authors' knowledge, do not appear in the literature.

We will adopt the following notation for the derivatives of a polynomial f

$$f^{(i,j)} = \frac{\partial^{i+j} f}{\partial x^i \partial y^j}, \qquad f^{(0,0)} = f(x,y), \qquad f^{(i,j)}(\alpha,\beta) = f^{(i,j)}(x,y)\Big|_{(x,y)\equiv(\alpha,\beta)}.$$

A set of polynomials is denoted, for example, by $\{p_0,\ldots,p_n \mid n \in \mathbb{N}\}$, where the subscripts $0,\ldots,n$ represent the (total) degree of each polynomial. Moreover, for polynomial sequences, we will use the notation $\{a_n\}_{n\in\mathbb{N}}$ for univariate sequence and $\{r_n\}^b_{n\in\mathbb{N}}$ in the bivariate case. Uppercase letters will be used for particular and well-known sequences.

2. Definition and First Characterizations

Let $A(t)$ be the power series

$$A(t) = \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!}, \quad \alpha_0 \neq 0,\ \alpha_k \in \mathbb{R},\ k \geq 0, \tag{1}$$

(usually $\alpha_0 = 1$) and let $\phi(y,t)$ be the two-variable real function defined as

$$\phi(y,t) = \sum_{k=0}^{\infty} \varphi_k(y)\frac{t^k}{k!}, \tag{2}$$

where $\varphi_k(y)$ are real polynomials in the variable y, with $\varphi_0(y) = 1$.

It is known ([19], p. 78) that the power series $A(t)$ generates the univariate Appell polynomial sequence $\{A_n\}_{n\in\mathbb{N}}$ such that

$$A_0(x) = 1, \qquad A_n(x) = \sum_{k=0}^{n} \binom{n}{k} \alpha_{n-k} x^k, \quad n \geq 1. \tag{3}$$

Now we consider the bivariate polynomals r_n with real variables. We denote by $\mathcal{A}(\phi, A)$, or simply $\mathcal{A}$ where there is no possibility of misunderstanding, the set of bivariate polynomial sequences $\{r_n\}^b_{n\in\mathbb{N}}$ such that

$$\begin{cases} r_0(x,y) = 1 & \text{(4a)} \\ r_n^{(1,0)}(x,y) = n\, r_{n-1}(x,y), \quad n \geq 1 & \text{(4b)} \\ r_n(0,y) = \displaystyle\sum_{k=0}^{n} \binom{n}{k} \alpha_{n-k} \varphi_k(y). & \text{(4c)} \end{cases}$$

In the following, unless otherwise specified, the previous hypotheses and notations will always be used.

Remark 1. *We observe that in [21,32] a polynomial sequence $\{P_i\}^b_{n\in\mathbb{N}}$ is said to satisfy the Appell condition if*

$$\frac{\partial}{\partial t} P_i(t,x) = P_{i-1}(t,x), \qquad P_0(t,x) = 1.$$

This sequence in [32] is used to obtain an expansion of bivariate, real functions with integral remainder (generalization of Sard formula [33]. Nothing is said about the theory of this kind of sequences.

Proposition 1. *A bivariate polynomial sequence $\{r_n\}^b_{n\in\mathbb{N}}$ is an element of $\mathcal{A}$ if and only if*

$$r_n(x,y) = \sum_{k=0}^{n} \binom{n}{k} A_{n-k}(x) \varphi_k(y), \qquad n \geq 1. \tag{5}$$

Proof. If $\{r_n\}^b_{n\in\mathbb{N}} \in \mathcal{A}$, relations (4a) hold. Then, by induction and partial integration with respect to the variable x ([19] p. 93), we get relation (5), according to (3). Vice versa, from (5), we easily get (4a). □

Proposition 2. *A bivariate polynomial sequence $\{r_n\}^b_{n\in\mathbb{N}}$ is an element of $\mathcal{A}$ if and only if*

$$A(t) e^{xt} \phi(y,t) = \sum_{n=0}^{\infty} r_n(x,y) \frac{t^n}{n!}. \tag{6}$$

Proof. If $\{r_n\}^b_{n\in\mathbb{N}} \in \mathcal{A}$, from Proposition 1 the identity (5) holds. Then

$$\sum_{n=0}^{\infty} r_n(x,y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} A_{n-k}(x) \varphi_k(y) \right) \frac{t^n}{n!}.$$

From the Cauchy product of series, according to (1) and (2), we get (6). Vice-versa, from (6) we obtain (5). Therefore relations (4a) hold. □

We call the function $F(x,y;t) = A(t)e^{xt}\phi(y,t)$ *exponential generating function of the bivariate polynomial sequence* $\{r_n\}^b_{n\in\mathbb{N}}$.

Remark 2. *From Propositions 1 and 2 we note explicitly that relations (4a) are equivalent to the identity (6).*

Example 1. *Let $\phi(y,t) = 1$, that is $\varphi_0(y) = 1$, $\varphi_k(y) \equiv 0$, $k > 0$. Then $\{r_n\}^b_{n\in\mathbb{N}}$, constructed as in Proposition 1, or, equivalently, Proposition 2, is a polynomial sequence in one variable, with elements*

$$r_n(x,y) \equiv r_n(x) = A_n(x).$$

Therefore $\{r_n\}^b_{n\in\mathbb{N}}$ is a univariate Appell polynomial sequence [18,19].

Example 1 suggests us the following definition.

Definition 1. *A bivariate polynomial sequence $\{r_n\}^b_{n\in\mathbb{N}} \in \mathcal{A}$, that is a polynomial sequence satisfying relations (4a) or relation (6), is called* general bivariate Appell polynomial sequence*.*

Remark 3. *(Elementary general bivariate Appell polynomial sequences) Assuming $A(t) = 1$, that is $\alpha_0 = 1$, $\alpha_i = 0$, $i \geq 1$, relations (4a) become*

$$\begin{cases} r_0(x,y) = 1 & (7a) \\ r_n^{(1,0)}(x,y) = n\, r_{n-1}(x,y), \qquad n > 1 & (7b) \\ r_n(0,y) = \varphi_n(y). & (7c) \end{cases}$$

Moreover, the univariate Appell sequence is $A_n(x) = x^n$, $n \geq 0$. Hence, from (5),

$$r_n(x,y) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \varphi_k(y), \qquad n \geq 1. \tag{8}$$

Relation (6) becomes

$$e^{xt}\phi(y,t) = \sum_{n=0}^{\infty} r_n(x,y) \frac{t^n}{n!}. \tag{9}$$

In this case, we call the polynomial sequence $\{r_n\}^b_{n\in\mathbb{N}}$ elementary bivariate Appell sequence*. We will denote it by $\{p_n\}^b_{n\in\mathbb{N}}$, that is*

$$p_n(x,y) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \varphi_k(y), \qquad \forall n \in \mathbb{N}. \tag{10}$$

The set of elementary bivariate Appell sequences will be denoted by $\mathcal{A}(\phi,1)$, or $\mathcal{A}^e$. Of course, $\mathcal{A}^e \subset \mathcal{A}$. We observe that the set $\mathcal{A}^e$ coincides with the set of 2VgP considered in [24].

We note that $\{p_0, \ldots, p_n | n \in \mathbb{N}\}$ is a set of $n+1$ linearly independent polynomials in $\mathcal{A}^e$.

Proposition 3. *Let $\{r_n\}^b_{n\in\mathbb{N}} \in \mathcal{A}(\phi, A)$ and $\{p_n\}^b_{n\in\mathbb{N}} \in \mathcal{A}(\phi,1)$. Then, the following identities hold*

$$\sum_{k=0}^{n} \binom{n}{k} A_{n-k}(x)\varphi_k(y) = r_n(x,y) = \sum_{k=0}^{n} \binom{n}{k} \alpha_{n-k} p_k(x,y). \tag{11}$$

Proof. From (9), $e^{xt}\phi(y,t) = \sum_{n=0}^{\infty} p_n(x,y)\frac{t^n}{n!}$. Hence the result follows from (1), (6) and the Cauchy product of series. □

It is known that ([19] p. 11) the power series $A(t)$ is invertible and it results

$$\frac{1}{A(t)} \equiv A^{-1}(t) = \sum_{k=0}^{\infty} \beta_k \frac{t^k}{k!},$$

with $\beta_k, k \geq 0$, defined by

$$\sum_{k=0}^{n} \binom{n}{k} \alpha_{n-k}\beta_k = \delta_{n0}. \tag{12}$$

The identity (9) (with $r_n = p_n$) yelds

$$A^{-1}(t)e^{xt}\phi(y,t) = \sum_{n=0}^{\infty} \hat{r}_n(x,y)\frac{t^n}{n!},$$

with

$$\hat{r}_n(x,y) = \sum_{k=0}^{n} \binom{n}{k} \beta_{n-k} p_k(x,y). \tag{13}$$

The polynomial sequence $\{\hat{r}_n\}^b_{n\in\mathbb{N}}$ is called *conjugate bivariate Appell polynomial sequence* of $\{r_n\}^b_{n\in\mathbb{N}}$.

Observe that the bivariate polynomial sequence $\{\hat{r}_n\}^b_{n\in\mathbb{N}}$ is an element of the set $\mathcal{A}$.

3. Matrix Form

We denote by $A = (a_{i,j})_{i,j\in\mathbb{N}}$ the infinite lower triangular matrix [19,34] with

$$a_{i,j} = \binom{i}{j}\alpha_{i-j}, \qquad i,j = 0,\dots,\ j \le i, \qquad \alpha_0 \neq 0,\ \alpha_k \in \mathbb{R},\ k \ge 0,$$

and let $B = (b_{i,j})_{i,j\in\mathbb{N}}$ be the inverse matrix. It is known ([19] p. 11) that

$$b_{i,j} = \binom{i}{j}\beta_{i-j}, \qquad i,j = 0,\dots,\ j \le i,$$

where β_k are defined as in (12).

Observe that the matrices A and B can be factorized ([19] p. 11) as

$$A = D_1 T^{\alpha} D_1^{-1}, \qquad B = D_1 T^{\beta} D_1^{-1},$$

where $D_1 = diag(i!)_{i\ge 0}$ is a factorial diagonal matrix and T^{α}, T^{β} are lower triangualar Toeplitz matrices with entries, respectively, $t^{\alpha}_{i,j} = \dfrac{\alpha_{i-j}}{(i-j)!}$ and $t^{\beta}_{i,j} = \dfrac{\beta_{i-j}}{(i-j)!}$, $i \ge j$.

We denote by A_n and B_n the principal submatrices of order n of A and B, respectively.

Let P and R be the infinite vectors

$$P = [p_0(x,y),\dots,p_n(x,y),\cdots]^T \qquad \text{and} \qquad R = [r_0(x,y),\dots,r_n(x,y),\cdots]^T.$$

Moreover, for every $n \in \mathbb{N}$, let

$$P_n = [p_0(x,y),\dots,p_n(x,y)]^T \qquad \text{and} \qquad R_n = [r_0(x,y),\dots,r_n(x,y)]^T. \tag{14}$$

Proposition 4. *The following matrix identities hold:*

$$R = A\,P, \qquad \text{and } \forall n \in \mathbb{N} \quad R_n = A_n\,P_n; \tag{15a}$$

$$P = B\,R, \qquad \text{and } \forall n \in \mathbb{N} \quad P_n = B_n\,R_n. \tag{15b}$$

Proof. Identities (15a) follow directly from (11). The relations (15b) follow from (15a). □

The identities (15a) are called *matrix forms* of the bivariate general Appell sequence and we call A the related associated matrix.

Now, we consider the vectors

$$\hat{R} = [\hat{r}_0(x,y),\dots,\hat{r}_n(x,y),\cdots]^T, \qquad \text{and, } \forall n \in \mathbb{N}, \quad \hat{R}_n = [\hat{r}_0(x,y),\dots,\hat{r}_n(x,y)]^T.$$

From (13) we get

$$\hat{R}_n = B_n \, P_n, \tag{16a}$$

$$P_n = A_n \, \hat{R}_n. \tag{16b}$$

By combining (16a) and the second in (15a) we obtain

$$\hat{R}_n = B_n^2 \, R_n \qquad \text{and} \qquad R_n = \left(B_n^2\right)^{-1} \hat{R}_n = A_n^2 \, \hat{R}_n.$$

If $B_n^2 = \left(b_{i,j}^2\right)_{i,j\in\mathbb{N}}$ and $A_n^2 = \left(a_{i,j}^2\right)_{i,j\in\mathbb{N}}$, we get the inverse formulas

$$r_n(x,y) = \sum_{j=0}^{n} a_{n,j}^2 \hat{r}_j(x,y), \qquad \hat{r}_n(x,y) = \sum_{j=0}^{n} b_{n,j}^2 r_j(x,y).$$

Remark 4. *For the elementary Appell sequence $\{p_n\}_{n\in\mathbb{N}}^b$ with p_n given in (10), we observe that the associated matrix is*

$$A^* = \left(a_{i,j}^*\right)_{i,j\in\mathbb{N}} \qquad \textit{with} \quad a_{i,j}^* = \binom{i}{j},$$

that is the known Pascal matrix [12]. Hence the inverse matrix is

$$B^* = \left(b_{i,j}^*\right)_{i,j\in\mathbb{N}} \qquad \textit{with} \quad b_{i,j}^* = \binom{i}{j}(-1)^{i-j}.$$

Then we can obtain the conjugate sequence, $\{\hat{p}_n\}_{n\in\mathbb{N}}^b$. Therefore, from (16a) and (16b), we get

$$\hat{p}_n(x,y) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} p_k(x,y), \tag{17a}$$

$$p_n(x,y) = \sum_{k=0}^{n} \binom{n}{k} \hat{p}_k(x,y). \tag{17b}$$

If we introduce the vectors

$$\hat{P} = [\hat{p}_0(x,y), \ldots, \hat{p}_i(x,y), \cdots]^T, \qquad \textit{and} \qquad \forall n \in \mathbb{N}, \;\; \hat{P}_n = [\hat{p}_0(x,y), \ldots, \hat{p}_n(x,y)]^T,$$

we get the matrix identities

$$\begin{aligned} P &= A^* \, \hat{P}, \qquad \textit{and} \qquad \forall n \in \mathbb{N}, \;\; P_n = A_n^* \, \hat{P}_n, \\ \hat{P} &= B^* \, P, \qquad \textit{and} \qquad \forall n \in \mathbb{N}, \;\; \hat{P}_n = B_n^* \, P_n. \end{aligned} \tag{18}$$

Combining this with (15a) we get

$$R_n = (A_n \, A_n^*) \hat{P}_n = C_n \, \hat{P}_n, \quad \textit{with} \quad C_n = A_n \, A_n^*, \tag{19}$$

From (19) we have

$$\forall n \in \mathbb{N}, \qquad r_n(x,y) = \sum_{j=0}^{n} c_{n,j} \hat{p}_j(x,y) \tag{20}$$

with $c_{n,j} = \displaystyle\sum_{k=j}^{n} \binom{n}{k} \binom{k}{j} \alpha_{n-k}$.

Since the matrix C_n is invertible, we get from (10)

$$\hat{P}_n = C_n^{-1} R_n \tag{21}$$

that is,

$$\forall n \in \mathbb{N}, \qquad \hat{p}_n(x,y) = \sum_{j=0}^{n} \hat{c}_{n,j} r_j(x,y), \tag{22}$$

with

$$\hat{c}_{n,j} = \sum_{k=j}^{n} \binom{n}{k}\binom{k}{j}(-1)^{n-k}\beta_{k-j}, \tag{23a}$$

$$\hat{c}_{n,j} = \binom{n}{j}\hat{c}_{n-j,0} = \binom{n}{j}\hat{c}_{n-j}, \qquad \text{with} \quad \hat{c}_{n-j} \equiv \hat{c}_{n-j,0}. \tag{23b}$$

Formulas (20) and (22) are the inverse each other.

In order to determine the generating function of the sequence $\{\hat{p}_n\}_{n\in\mathbb{N}}^{b}$ we observe that

$$\frac{1}{A(t)} = \sum_{k=0}^{\infty} \beta_k \frac{t^k}{k!}, \qquad \text{and hence} \qquad \beta_k = (-1)^k.$$

Consequently, the generating function of $\{\hat{p}_n\}_{n\in\mathbb{N}}^{b}$ is

$$G(x,y;t) = e^{-t}e^{xt}\phi(y,t). \tag{24}$$

that is $\{\hat{p}_n\}_{n\in\mathbb{N}}^{b}$ is an element of $\mathcal{A}(\phi, A)$.

Proposition 5. *For the conjugate sequence $\{\hat{p}_n\}_{n\in\mathbb{N}}^{b}$ the following identity holds*

$$\forall n \in \mathbb{N}, \qquad \hat{p}_n(x,y) = \sum_{k=0}^{n} \binom{n}{k}(x-1)^k \phi_{n-k}(y). \tag{25}$$

Proof. From (24) and (17a) we get

$$e^{-t}e^{xt}\phi(y,t) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(-1)^k p_{n-k}(x,y)\right)\frac{t^n}{n!}. \tag{26}$$

By applying the Cauchy product of series to the left-hand term in (26), and substituting (17a) in the right-hand term, we obtain (25). □

Corollary 1.

$$\forall n \in \mathbb{N}, \qquad \sum_{k=0}^{n}\binom{n}{k}(-1)^k p_{n-k}(x,y) = \sum_{k=0}^{n}\binom{n}{k}(x-1)^k \phi_{n-k}(y). \tag{27}$$

4. Recurrence Relations

In [35] has been noted that recurrence relations are a very interesting tool for the study of the polynomial sequences.

Theorem 1 (Recurrence relations)**.** *Under the previous hypothesis and notations for the elements of $\{r_n\}_{n\in\mathbb{N}}^{b} \in \mathcal{A}(\phi, A)$ the following recurrence relations hold:*

$$r_0(x,y) = 1, \qquad r_n(x,y) = p_n(x,y) - \sum_{j=0}^{n-1}\binom{n}{j}\beta_{n-j} r_j(x,y), \quad n \geq 1; \tag{28}$$

$$r_0(x,y) = 1, \qquad r_n(x,y) = \hat{p}_n(x,y) - \sum_{j=0}^{n-1} \binom{n}{j} \hat{c}_{n-j} r_j(x,y), \quad n \geq 1, \tag{29}$$

with β_k defined as in (12) and $\hat{c}_k$ given as in (23b).

Proof. The proof follows easily by identities (15a) and (21). □

We call relations (28) and (29), first and second recurrence relations, respectively.

The third recurrence relations can be obtained from the generating function.

Theorem 2 (Third recurrence relation)**.** *For the elements of $\{r_n\}_{n\in\mathbb{N}}^b \in \mathcal{A}(\phi, A)$ the following identity holds: $\forall n \geq 0$*

$$r_{n+1}(x,y) = [x + b_0 + c_0(y)]r_n(x,y) + \sum_{k=0}^{n-1} \binom{n}{k} [b_{n-k} + c_{n-k}(y)]r_k(x,y), \tag{30}$$

where b_k and c_k are such that

$$\frac{A'(t)}{A(t)} = \sum_{k=0}^{\infty} b_k \frac{t^k}{k!}, \qquad \frac{\phi^{(0,1)}(y,t)}{\phi(y,t)} = \sum_{k=0}^{\infty} c_k(y) \frac{t^k}{k!}. \tag{31}$$

Proof. Partial differentiation with respect to the variable t in (6) gives

$$\left[x + \frac{A'(t)}{A(t)} + \frac{\phi^{(0,1)}(y,t)}{\phi(y,t)}\right] A(t)e^{xt}\phi(y,t) = \sum_{n=1}^{\infty} n\, r_n(x,y) \frac{t^{n-1}}{n!} = \sum_{n=0}^{\infty} r_{n+1}(x,y) \frac{t^n}{n!} \tag{32}$$

Hence we get

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} [b_{n-k} + c_{n-k}(y)]r_k(x,y) + x r_n(x,y) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} r_{n+1}(x,y) \frac{t^n}{n!},$$

and from this, relation (30) follows. □

The same techniques used previously can be used to derive recurrence relations for the conjugate sequence. Particularly, the third recurrence relation is similar to (30) by exchanging b_k with d_k, $k = 0, \ldots, n$, being d_k such that

$$\frac{\left(A^{-1}(t)\right)'}{A^{-1}(t)} = \sum_{k=0}^{\infty} d_k \frac{t^k}{k!}. \tag{33}$$

Remark 5. *Observe that if $\sum_{k=0}^{n-2} \binom{n}{k} [b_{n-k} + c_{n-k}(y)]r_k(x,y) = 0$, the recurrence relation (30) becomes a three-terms relation.*

5. Determinant Forms

The previous recurrence relations provide determinant forms [36,37], which can be useful for both numerical calculations and new combinatorial identities.

Theorem 3 (Determinant forms)**.** *For the elements of $\{r_n\}_{n\in\mathbb{N}}^b \in \mathcal{A}(\phi, A)$ the following identities hold:*

$$r_0(x,y)=1, \quad r_n(x,y)=(-1)^n \begin{vmatrix} p_0(x,y) & p_1(x,y) & p_2(x,y) & \cdots & p_n(x,y) \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n}{1}\beta_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \quad n>0. \tag{34}$$

$$r_0(x,y)=1, \quad r_n(x,y)=(-1)^n \begin{vmatrix} \hat{p}_0(x,y) & \hat{p}_1(x,y) & \hat{p}_2(x,y) & \cdots & \hat{p}_n(x,y) \\ \hat{c}_0 & \hat{c}_1 & \hat{c}_2 & \cdots & \hat{c}_n \\ 0 & \hat{c}_0 & \binom{2}{1}\hat{c}_1 & \cdots & \binom{n}{1}\hat{c}_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \hat{c}_0 & \binom{n}{n-1}\hat{c}_1 \end{vmatrix}, \quad n>0. \tag{35}$$

Proof. For $n>1$ relation (28) can be regarded as an infinite lower triangular system in the unknowns $r_0(x,y),\dots,r_n(x,y),\dots$. By solving the first $n+1$ equations by Cramer's rule, after elementary determinant operations we get (34). Relation (35) follows from (29) by the same technique. □

We note that the determinant forms are Hessenberg determinants. It is known ([19] p. 28) that Gauss elimination for the calculation of an Hessenberg determinant is stable.

Theorem 4 (Third determinant form)**.** *For the elements of $\{r_n\}^b_{n\in\mathbb{N}}\in\mathcal{A}(\phi,A)$ the following determinant form holds:*

$$r_0(x,y)=1,$$
$$r_{n+1}(x,y)=\begin{vmatrix} x+b_0+c_0(y) & -1 & 0 & \cdots & 0 \\ b_1+c_1(y) & x+b_0+c_0(y) & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -1 \\ b_n+c_n(y) & \binom{n}{1}[b_{n-1}+c_{n-1}(y)] & \cdots & \binom{n}{n-1}[b_1+c_1(y)] & x+b_0+c_0(y) \end{vmatrix}, \quad n\geq 0. \tag{36}$$

Proof. The result follows from (30) with the same technique used in the previous Theorem. □

We point out that the first and second recurrence relations and the determinant forms (34)–(36) do not appear in the literature. They will be fundamental in the relationship with linear interpolation.

Remark 6. *For the elements of $\{\hat{r}_n\}^b_{n\in\mathbb{N}}\in\mathcal{A}(\phi,A)$ an expression similar to (36) is obtained by exchanging b_k with d_k, $k=0,\dots,n$, d_k being defined as in (33).*

Remark 7. *For the elements of $\{p_n\}^b_{n\in\mathbb{N}}\in\mathcal{A}(\phi,1)$, from (17a), we get the recurrence relation*

$$p_n(x,y)=\hat{p}_n(x,y)-\sum_{k=0}^{n-1}\binom{n}{k}(-1)^{n-k}p_k(x,y). \tag{37}$$

By the same technique used in the proof of Theorem 3 we obtain the following determinant form

$$p_0(x,y)=1,\quad p_n(x,y)=(-1)^n\begin{vmatrix} \hat{p}_0(x,y) & \hat{p}_1(x,y) & \hat{p}_2(x,y) & \hat{p}_3(x,y) & \cdots & \hat{p}_n(x,y) \\ 1 & -1 & 1 & -1 & \cdots & (-1)^n \\ 0 & 1 & -2 & 3 & \cdots & \binom{n}{1}(-1)^{n-1} \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & -\binom{n}{n-1} \end{vmatrix},\quad n>0. \tag{38}$$

From (30) *we obtain*

$$p_{n+1}(x,y)=x\,p_n(x,y)+\sum_{k=0}^{n}\binom{n}{k}c_{n-k}(y)p_k(x,y),$$

where c_k *are defined as in* (31)*. The related determinant form is*

$$p_0(x,y)=1,$$
$$p_{n+1}(x,y)=\begin{vmatrix} x+c_0(y) & -1 & 0 & \cdots & 0 \\ c_1(y) & x+c_0(y) & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -1 \\ c_n(y) & \binom{n}{1}c_{n-1}(y) & \cdots & \binom{n}{n-1}c_1(y) & x+c_0(y) \end{vmatrix},\quad n\geq 0.$$

6. Differential Operators and Equations

The elements of a general bivariate Appell sequence satisfy some interesting differential equations.

Proposition 6. *For the elements of* $\{r_n\}_{n\in\mathbb{N}}^b\in\mathcal{A}(\phi,A)$ *the following identity holds*

$$\forall n,k\in\mathbb{N},\ k<n,\qquad r_{n-k}(x,y)=\frac{1}{n(n-1)\ldots(n-k+1)}r_n^{(k,0)}(x,y). \tag{39}$$

Proof. The proof follows easily after k partial differentiation of (7b) with respect to x. □

Theorem 5 (Differential equations)**.** *The elements of* $\{r_n\}_{n\in\mathbb{N}}^b\in\mathcal{A}(\phi,A)$ *satisfy the following differential equations*

$$\frac{\beta_n}{n!}\frac{\partial^n}{\partial x^n}f(x,y)+\frac{\beta_{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial x^{n-1}}f(x,y)+\ldots+f(x,y)=\sum_{i=0}^{n}\binom{n}{i}x^i\varphi_{n-i}(y);$$

$$\hat{c}_n\frac{\partial^n}{\partial x^n}f(x,y)+\frac{n\,\hat{c}_{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial x^{n-1}}f(x,y)+\frac{n(n-1)\hat{c}_{n-2}}{2(n-2)!}\frac{\partial^{n-2}}{\partial x^{n-2}}f(x,y)+\ldots+f(x,y)=\sum_{i=0}^{n}\binom{n}{i}(x-1)^i\varphi_{n-i}(y).$$

Proof. The results follow by replacing relation (39) in the first recurrence relation (28) and in the second recurrence relation (29), respectively. □

Theorem 6. *The elements of* $\{p_n\}_{n\in\mathbb{N}}^b\in\mathcal{A}(\phi,1)$ *satisfy the following differential equation*

$$\frac{(-1)^n}{n!}\frac{\partial^n}{\partial x^n}f(x,y)+\frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial x^{n-1}}f(x,y)+\ldots+f(x,y)=\sum_{i=0}^{n}\binom{n}{i}(x-1)^i\varphi_{n-i}(y).$$

Proof. The result follows by replacing relation (39) in (27). □

We observe that the results in Theorems 5 and 6 are new in the literature.

In order to make the paper as autonomous as possible, we remind that a polynomial sequence $\{q_n\}_{n\in\mathbb{N}}$ is said to be quasi-monomial if two operators $\tilde{M}$ and $\tilde{P}$, called multiplicative and derivative operators respectively, can be defined in such a way that

$$\tilde{P}\{q_n(x)\} = nq_{n-1}(x), \tag{40a}$$

$$\tilde{M}\{q_n(x)\} = q_{n+1}(x). \tag{40b}$$

If these operators have a differential realization, some important consequences follow:

- differential equation: $\tilde{M}\tilde{P}\{q_n(x)\} = nq_n(x)$;
- it $q_0(x) = 1$, then $q_n(x) = \tilde{M}^n\{1\}$, and this yields the series definition for $q_n(x)$;
- the exponential generating function of $q_n(x)$ is $e^{t\tilde{M}\{1\}} = \sum_{n=0}^{\infty} q_n(x)\frac{t^n}{n!}$.

For the general bivariate Appell sequence $\{r_n\}^b_{n\in\mathbb{N}}$ we also have multiplicative and derivative operators.

Theorem 7 (Multiplicative and derivative operators [24])**.** *For $\{r_n\}^b_{n\in\mathbb{N}} \in \mathcal{A}(\phi, A)$ multiplicative and derivative operators are respectively*

$$\tilde{M}_r = x + \frac{A'(D_x)}{A(D_x)} + \frac{\phi'(y, D_x)}{\phi(y, D_x)}, \tag{41a}$$

$$\tilde{P}_r = D_x. \tag{41b}$$

where $\phi'(y,t) = \phi^{(0,1)}(y,t)$ and $D_x = \dfrac{\partial}{\partial x}$.

Thus the set $\{r_n\}^b_{n\in\mathbb{N}}$ is quasi-monomial under the action of the operators $\tilde{M}_r$ and $\tilde{P}_r$.

Proof. Relations (41a) and (41b) follow from (32) and (4b), respectively [24,38]. □

Theorem 8 (Differential identity)**.** *The elements of a general bivariate Appell sequence $\{r_n\}^b_{n\in\mathbb{N}}$ satisfy the following differential identity*

$$\sum_{k=0}^{n} \frac{b_k + c_k(y)}{k!} r_n^{(k,0)}(x,y) + x r_n(x,y) \equiv \tilde{M}_r\{r_n(x,y)\} = r_{n+1}(x,y).$$

Proof. From (41a) we get the first identity. The second equality follows by (40b), according to Theorem 7. □

Remark 8. *The operators (41a) and (41b) satisfy the commutation relation [24] $\tilde{P}_r\tilde{M}_r - \tilde{M}_r\tilde{P}_r = I$, and this shows the structure of a Weyl group.*

Remark 9. *From Theorem 7 and Remark 8 we get $\tilde{M}_r\tilde{P}_r\{r_n(x,y)\} = n\, r_n(x,y)$ that can be interpreted as a differential equation.*

7. General Properties

The general bivariate Appell polynomial sequences satisfy some properties.

Proposition 7 (Binomial identity)**.** *Let $\{r_n\}^b_{n\in\mathbb{N}} \in \mathcal{A}(\phi, A)$. The following identity holds*

$$\forall n \in \mathbb{N}, \qquad r_n(x_1 + x_2, y) = \sum_{k=0}^{n} \binom{n}{k} r_k(x_1, y) x_2^{n-k}. \tag{42}$$

Proof. From the generating function

$$A(t)e^{(x_1+x_2)t}\phi(y,t) = A(t)e^{x_1 t}\phi(y,t)e^{x_2 t} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k} r_k(x_1,y)x_2^{n-k}\right]\frac{t^n}{n!}.$$

Thus the result follows. □

Corollary 2. *For $n \in \mathbb{N}$ we get*

$$\sum_{k=0}^{n}\binom{n}{k} r_k(x,y)(-x)^{n-k} = \sum_{k=0}^{n}\binom{n}{k}\alpha_{n-k}\varphi_k(y).$$

Proof. The proof follows from Proposition 7 for $x_2 = -x_1$ and $x_1 = x$ and from (4c). □

Corollary 3 (Forward difference)**.** *For $n \in \mathbb{N}$ we get*

$$\Delta_x r_n(x,y) \equiv r_n(x+1,y) - r_n(x,y) = \sum_{k=0}^{n-1}\binom{n}{k} r_k(x,y).$$

Remark 10. *Proposition 7 suggests us to consider general Appell polynomial sequences with three variables. In fact, setting in (42) $x_1 = x,\ x_2 = z$ and*

$$v_n(x,y,z) = \sum_{k=0}^{n}\binom{n}{k} r_k(x,y)z^{n-k},$$

the sequence $\{v_n\}_n$ can be consider a general Appell polynomial sequence in three variables. Analogously, we can consider Appell polynomial sequences in d variables with $d \geq 3$.

Proposition 8 (Integration with respect to the variable x)**.** *For $n \in \mathbb{N}$ we get*

$$\int_0^x r_n(t,y)dt = \frac{1}{n+1}[r_{n+1}(x,y) - r_{n+1}(0,y)] \tag{43}$$

$$\int_0^1 p_n(x,y)dx = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}\varphi_k(y). \tag{44}$$

Proof. Relation (43) follows from (4b). The (44) is obtained from (7c), (7b) and Proposition 7 for $x_1 = 0$, $x_2 = 1$. □

Proposition 9 (Partial matrix differentiation with respect to the variable x)**.** *Let R_n be the vector defined in (14). Then*

$$R_n^{(1,0)} = D\ R_n,$$

where D is the matrix with entries

$$d_{i,j} = \begin{cases} i & i = j+1 \\ 0 & \text{otherwise} \end{cases} \qquad i,j = 0,\dots,n.$$

Proof. The proof follows from (4b). □

In order to give an algebraic structure to the set $\mathcal{A}$, we consider two elements $\{r_n\}_{n\in\mathbb{N}}^b$ and $\{s_n\}_{n\in\mathbb{N}}^b$. From (11) we get, $\forall n \in \mathbb{N}$,

$$r_n(x,y) = \sum_{k=0}^{n}\binom{n}{k}\alpha_{n-k}p_k(x,y), \qquad s_n(x,y) = \sum_{k=0}^{n}\binom{n}{k}\overline{\alpha}_{n-k}p_k(x,y).$$

That is, $A_n = \left(a_{i,j}\right)_{i,j\le n}$ with $a_{i,j} = \binom{i}{j}\alpha_{i-j}$ is the associated matrix to $\{r_n\}^b_{n\in\mathbb{N}}$, and $\overline{A}_n = \left(\overline{a}_{i,j}\right)_{i,j\le n}$ with $\overline{a}_{i,j} = \binom{i}{j}\overline{\alpha}_{i-j}$ is the associated matrix to $\{s_n\}^b_{n\in\mathbb{N}}$.

Then we define

$$(r_n \circ s_n)(x,y) = r_n(s_n(x,y)) := \sum_{k=0}^{n} \binom{n}{k}\alpha_{n-k}s_k(x,y)$$

and we set

$$z_n^{r,s}(x,y) = (r_n \circ s_n)(x,y). \tag{45}$$

Proposition 10 (Umbral composition)**.** *The polynomial sequence $\{z_n^{r,s}\}^b_{n\in\mathbb{N}}$, with $z_n^{r,s}$ defined as in (45), is a general bivariate Appell sequence and we call it umbral composition of $\{r_n\}^b_{n\in\mathbb{N}} \in \mathcal{A}(\phi, A)$ and $\{s_n\}^b_{n\in\mathbb{N}} \in \mathcal{A}(\phi, A)$.*

Proof. It's easy to verify that the matrix associated to the sequence $\{z_n^{r,s}\}^b_{n\in\mathbb{N}}$ is $V = A\,\overline{A}$. In fact

$$z_n^{r,s}(x,y) = \sum_{k=0}^{n}\binom{n}{k}\alpha_{n-k}\sum_{i=0}^{k}\binom{k}{i}\overline{\alpha}_{k-i}p_i(x,y) = \sum_{k=0}^{n}\binom{n}{k}\, v_{n,k}\, p_k(x,y)$$

with $v_{n,k} = \sum_{i=0}^{n-k}\binom{n-k}{i}\alpha_{n-i-k}\overline{\alpha}_i$.

Moreover V is an Appell-type matrix [19]. In fact

$$V = D_1 T^{\alpha} D_1^{-1} D_1 T^{\overline{\alpha}} D_1^{-1} = D_1 T^{\alpha} T^{\overline{\alpha}} D_1^{-1}.$$

□

The set $\mathcal{A}(\phi, A)$ with the umbral composition operation is an algebraic structure $(\mathcal{A}(\phi, A), \circ)$.

Let $(\mathcal{L}, \cdot)$ be the group of infinite, lower triangular matrix with the usual product operation.

Proposition 11 (Algebraic structure)**.** *The algebraic structure $(\mathcal{A}(\phi, A), \circ)$ is a group isomorphic to $(\mathcal{L}, \cdot)$.*

Proof. We have observed that $\mathcal{A}(\phi, A)$ is an algebraic structure. Then we have

(i) the elementary Appell sequence $\{p_n\}^b_{n\in\mathbb{N}}$ is the identity in $(\mathcal{A}(\phi, A), \circ)$.

(ii) for every $\{r_n\}^b_{n\in\mathbb{N}} \in \mathcal{A}(\phi, A)$ the conjugate sequence $\{\hat{r}_n\}^b_{n\in\mathbb{N}}$ is its inverse.

□

Remark 11. *Given $\lambda, \mu \in \mathbb{R}$, with $(\lambda, \mu) \neq (0, 0)$, if $\{r_n\}^b_{n\in\mathbb{N}}$ and $\{s_n\}^b_{n\in\mathbb{N}}$ are two elements of $\mathcal{A}(\phi, A)$, the sequence $\{\lambda r_n + \mu s_n\}^b_{n\in\mathbb{N}}$ is also an element of $\mathcal{A}(\phi, A)$. Hence the algebraic structure $(\mathcal{A}(\phi, A), \circ, +, \cdot)$ is an algebra on $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$.*

8. Relations with Linear Functional and Linear Interpolation

Let $\{p_n\}^b_{n\in\mathbb{N}} \in \mathcal{A}(\phi, 1)$. We consider the set of polynomials

$$\mathcal{S}_n = \operatorname{span}\{p_0, \ldots, p_n \mid n \in \mathbb{N}\}.$$

where p_i, $i = 0, \ldots, n$, are defined as in (10). Let L be a linear functional on $\mathcal{S}_n^*$. If we set

$$L(p_k) = \beta_k, \qquad k = 0, \ldots, n, \quad \beta_0 = 1, \ \beta_k \in \mathbb{R}, \ k \ge 1, \ \forall p_k \in \mathcal{S},$$

we can consider the general bivariate Appell polynomial sequence in $\mathcal{A}(\phi, A)$ as in (34) and we call it the *polynomial sequence related to the functional* L. We denote it by $\left\{r_n^{L,p}\right\}_{n\in\mathbb{N}}^b$.

Now we define the $n+1$ linear functionals L_i, $i = 0, \dots, n$, in $\mathcal{S}_n^*$ as

$$L_0(p_k) = L(p_k) = \beta_k, \qquad L_i(p_k) = L\left(p_k^{(i,0)}\right) = i!\binom{k}{i}\beta_{k-i}, \quad i = 1, \dots, k,\ k = 0, \dots, n,$$

where in the second relation we have applied (7b).

Theorem 9. *For the elements of the bivariate general Appell sequence* $\left\{r_n^{L,p}\right\}_{n\in\mathbb{N}}^b$ *the following identity holds*

$$L_i\left(r_n^{L,p}\right) = n!\delta_{ni}, \qquad i = 0, \dots, n,$$

where δ_{ni} *is the known Kronecker symbol.*

Proof. The proof follows from the first determinant form (Theorem 3). □

Corollary 4. *The bivariate general Appell polynomial sequence* $\left\{r_n^{L,p}\right\}_{n\in\mathbb{N}}^b$ *is the solution of the following general linear interpolation problem on* $\mathcal{S}_n$

$$L_i(z_n) = n!\delta_{ni}, \qquad i = 0, \dots, n, \quad z_n \in \mathcal{S}_n.$$

Proof. The proof follows from Theorem 9 and the known theorems on general linear interpolation problem [39] since L_i, $i = 0, \dots, n$, are linearly independent functionals. □

Theorem 10 (Representation theorem). *For every* $z_n \in \mathcal{S}_n$ *the following relation holds*

$$z_n(x,y) = \sum_{k=0}^{n} L\left(z_n^{(k,0)}\right)\frac{r_k^{L,p}(x,y)}{k!}.$$

Proof. The proof follows from Theorem 9 and the previous definitions. □

9. Some Bivariate Appell Sequences

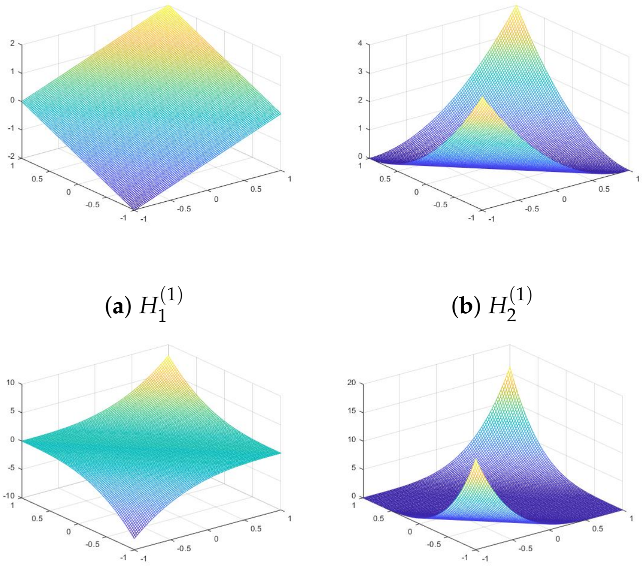
In order to illustrate the previous results, we construct some two variables Appell sequences. As we have shown, to do this, for each sequence we need two power series $A(t)$ and $\phi(y,t)$, where y is considered as a parameter.

Example 2. *Let* $\phi(y,t) = e^{yt}$. *There are several choices for* $A(t)$.

(1) $A(t) = 1$.
In this case, the elementary bivariate Appell sequence is the classical bivariate monomials. These polynomials are known in the literature also as Hermite polynomials in two variables and denoted by $H_n^{(1)}(x,y)$ *[40,41]:*

$$H_n^{(1)}(x,y) = (x+y)^n.$$

Figure 1 provides the graphs of the first four polynomials.



(**a**) $H_1^{(1)}$ (**b**) $H_2^{(1)}$

(**c**) $H_3^{(1)}$ (**d**) $H_4^{(1)}$

Figure 1. Plot of $H_i^{(1)}$, $i = 1, \ldots, 4$, in $[-1,1] \times [-1,1]$.

The matrix form is obtained by using the known Pascal matrix [34].
From (25) we get the conjugate sequence

$$\hat{H}_n^{(1)}(x,y) = \sum_{k=0}^{n} \binom{n}{k} (x-1)^{n-k} y^k = [(x-1)+y]^n,$$

hence, from (17a) and (17b), the inverse relations are

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} [(x-1)+y]^k \tag{46a}$$

$$[(x-1)+y]^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (x+y)^k. \tag{46b}$$

Note that from (46a) and (46b) we obtain the basic relations for binomial coefficients ([42] p. 3).
From (46b) we get the second recurrence relation

$$(x+y)^n = (x+y-1)^n - \sum_{j=0}^{n-1} \binom{n}{j} (-1)^{n-j} (x+y)^j, \qquad n \geq 1.$$

and the related determinant form for $n > 0$

$$(x+y)^n = (-1)^n \begin{vmatrix} 1 & x+y-1 & (x+y-1)^2 & \cdots & (x+y-1)^n \\ 1 & -1 & 1 & \cdots & (-1)^n \\ 0 & 1 & -2 & \cdots & (-1)^{n-1} n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -n \end{vmatrix}.$$

From this we can derive many identities. For example, for $n > 0$,

$$1 = (-1)^n \begin{vmatrix} -1 & 1 & -1 & \cdots & (-1)^n \\ 1 & -2 & \cdots & \cdots & (-1)^{n-1} n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -n \end{vmatrix},$$

and

$$x^n = (-1)^n \begin{vmatrix} 1 & x-1 & (x-1)^2 & \cdots & (x-1)^n \\ 1 & -1 & 1 & \cdots & (-1)^n \\ 0 & 1 & -2 & \cdots & (-1)^{n-1} n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -n \end{vmatrix}.$$

(2) $A(t) = \dfrac{t}{e^t - 1}$.

It is known ([19] p. 107) that this power series generates the univariate Bernoulli polynomials. Hence, directly from (11) we obtain a general bivariate Appell sequence which we call natural bivariate Bernoulli polynomials and denote it by $\{\mathcal{B}_n\}^b_{n\in\mathbb{N}}$, where

$$\mathcal{B}_n(x,y) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(x) y^k = \sum_{k=0}^{n} \binom{n}{k} (x+y)^k B_{n-k}. \tag{47}$$

$B_j(x)$ and B_j are, respectively, the Bernoulli polynomial of degree j and the j-th Bernoulli number ([19] p. 109).

We note that

$$\mathcal{B}_n(x,0) = B_n(x), \qquad \mathcal{B}_n(0,0) = B_n, \qquad n \geq 1.$$

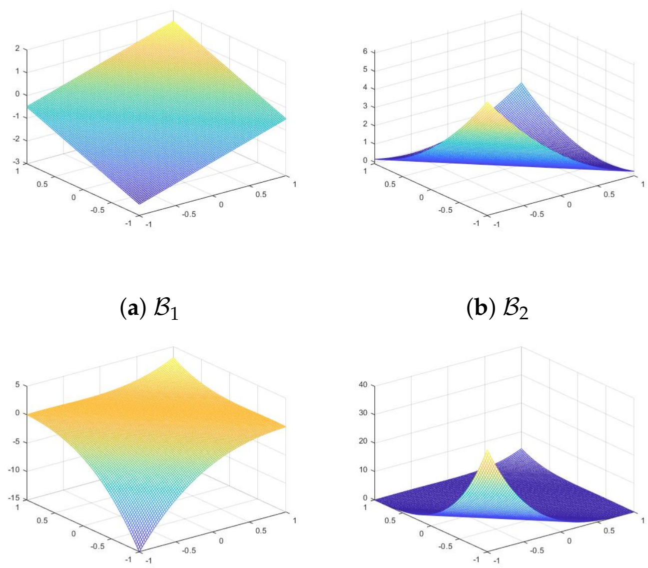
From the second equality in (47) and the known properties of Bernoulli polynomials ([19] p. 109) we have

$$\mathcal{B}_n(x,y) = B_n(x+y), \qquad n \geq 1.$$

The first natural bivariate Bernoulli polynomials are

$$\mathcal{B}_0(x,y) = 1, \qquad \mathcal{B}_1(x,y) = x + y - \frac{1}{2}, \qquad \mathcal{B}_2(x,y) = (x+y)^2 - (x+y) + \frac{1}{6},$$
$$\mathcal{B}_3(x,y) = (x+y)^3 - \frac{3}{2}(x+y)^2 + \frac{1}{2}(x+y),$$
$$\mathcal{B}_4(x,y) = (x+y)^4 - 2(x+y)^3 + (x+y)^2 - \frac{1}{30}.$$

Figure 2 shows the graphs of the first four polynomials $\mathcal{B}_i$, $i = 1, \ldots, 4$.



(**a**) $\mathcal{B}_1$ (**b**) $\mathcal{B}_2$

(**c**) $\mathcal{B}_3$ (**d**) $\mathcal{B}_4$

Figure 2. Plot of $\mathcal{B}_i$, $i = 1, \ldots, 4$, in $[-1, 1] \times [-1, 1]$.

From (11), (12) *and* (47) *we get* $\alpha_k = B_k$ *and* $\beta_k = \dfrac{1}{k+1}$, $k = 0, 1, \ldots$
Therefore the first recurrence relation is

$$\mathcal{B}_0(x,y) = 1, \qquad \mathcal{B}_n(x,y) = (x+y)^n - \sum_{j=0}^{n-1} \binom{n}{j} \frac{\mathcal{B}_j(x,y)}{n-j+1}, \qquad n \geq 1.$$

The related determinant form for $n > 0$ *is*

$$\mathcal{B}_n(x,y) = (-1)^n \begin{vmatrix} 1 & x+y & (x+y)^2 & (x+y)^3 & \cdots & (x+y)^n \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \frac{3}{2} & & \binom{n}{2}\frac{1}{n-1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & \frac{n}{2} \end{vmatrix}. \tag{48}$$

For the coefficients of $\dfrac{A'(t)}{A(t)} = \displaystyle\sum_{k=0}^{\infty} b_k \frac{t^k}{k!}$ *we find* $b_0 = B_1$, $b_k = -\dfrac{B_{k+1}}{k+1}$, $k \geq 1$. *Moreover,* $c_0(y) = y$, $c_k(y) = 0$, $k \geq 1$. *Hence the third recurrence relation is*

$$\mathcal{B}_{n+1}(x,y) = \left(x + y - \frac{1}{2}\right)\mathcal{B}_n(x,y) - \sum_{k=1}^{n-1} \binom{n}{k} \frac{B_{k+1}}{k+1} \mathcal{B}_{n-k}(x,y).$$

The related determinant form for $n > 0$ is

$$\mathcal{B}_{n+1}(x,y) = \begin{vmatrix} x-\frac{1}{2}+y & -1 & 0 & \cdots & \cdots & 0 \\ -\frac{1}{2} & x-\frac{1}{2}+y & -1 & \cdots & \cdots & 0 \\ -\frac{B_3}{3} & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & -1 \\ -\frac{B_{n+1}}{n+1} & -\binom{n}{1}\frac{B_n}{n} & \cdots & \cdots & -\binom{n}{n-1}\frac{1}{2} & x-\frac{1}{2}+y \end{vmatrix}.$$

(3) $A(t) = \dfrac{2}{e^t+1}$.

This power series generates the univariate Euler polynomials ([19] p. 123). Hence, directly from (11) we obtain a general bivariate Appell sequence which we call natural bivariate Euler polynomials and denote it by $\{\mathcal{E}_n\}^b_{n\in\mathbb{N}}$, *where*

$$\mathcal{E}_n(x,y) = \sum_{k=0}^{n} \binom{n}{k} E_{n-k}(x)y^k = \sum_{k=0}^{n} \binom{n}{k} (x+y)^k E_{n-k}(0). \tag{49}$$

$E_j(x)$ *is the Euler polynomial of degree j ([19] p. 124).*
We note that

$$\mathcal{E}_n(x,0) = E_n(x), \qquad n \geq 1,$$

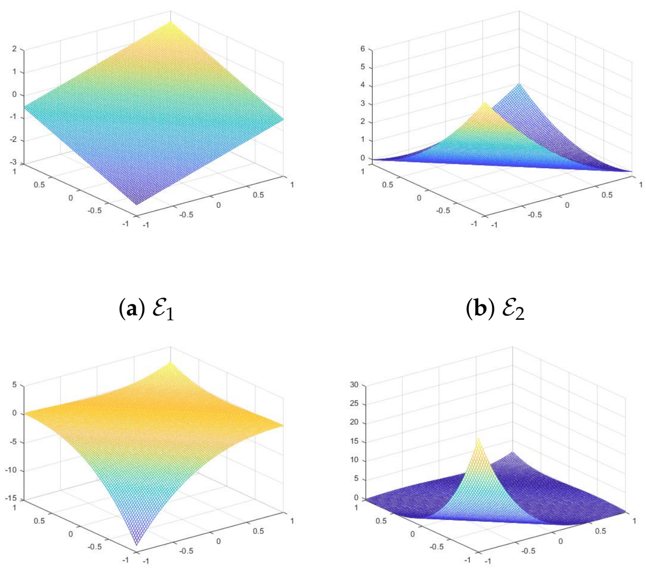
and

$$\mathcal{E}_n(x,y) = E_n(x+y), \qquad n \geq 1.$$

The first natural bivariate Euler polynomials are

$$\mathcal{E}_0(x,y) = 1, \qquad \mathcal{E}_1(x,y) = x+y-\frac{1}{2}, \qquad \mathcal{E}_2(x,y) = (x+y)^2 - (x+y),$$
$$\mathcal{E}_3(x,y) = (x+y)^3 - \frac{3}{2}(x+y)^2 + \frac{1}{4}, \quad \mathcal{E}_4(x,y) = (x+y)^4 - 2(x+y)^3 + x + y.$$

Figure 3 shows the graphs of the first four polynomials $\mathcal{E}_i$, $i = 1,\ldots,4$.



(a) $\mathcal{E}_1$ **(b)** $\mathcal{E}_2$

(c) $\mathcal{E}_3$ **(d)** $\mathcal{E}_4$

Figure 3. Plot of $\mathcal{E}_i$, $i = 1,\ldots,4$, in $[-1,1]\times[-1,1]$.

From (11), (12) *and* (49) *we get* $\alpha_k = E_k(0)$, *hence* ([19] *p. 124)* $\beta_0 = 1$ *and* $\beta_k = \frac{1}{2}$, $k \geq 1$. *Therefore the first recurrence relation is*

$$\mathcal{E}_0(x,y) = 1, \qquad \mathcal{E}_n(x,y) = (x+y)^n - \frac{1}{2}\sum_{j=0}^{n-1}\binom{n}{j}\mathcal{E}_j(x,y), \qquad n \geq 1.$$

The related determinant form for $n > 0$ *is*

$$\mathcal{E}_n(x,y) = (-1)^n \begin{vmatrix} 1 & x+y & (x+y)^2 & \cdots & (x+y)^n \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ 0 & 1 & \frac{1}{2}\binom{2}{1} & \cdots & \frac{1}{2}\binom{n}{1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \frac{1}{2}\binom{n}{n-1} \end{vmatrix}.$$

For the coefficients of the power series $\frac{A'(t)}{A(t)} = \sum_{k=0}^{\infty} b_k \frac{t^k}{k!}$ *we find* $b_0 = -\frac{1}{2}$, $b_k = -\frac{E_k(0)}{2}$, $k \geq 1$. *Hence the third recurrence relation becomes*

$$\mathcal{E}_{n+1}(x,y) = \left(x + y - \frac{1}{2}\right)\mathcal{E}_n(x,y) + \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}E_{n-k}(0)\mathcal{E}_k(x,y).$$

The related determinant form for $n > 0$ *is*

$$\mathcal{E}_{n+1}(x,y) = \begin{vmatrix} x - \frac{1}{2} + y & -1 & 0 & \cdots & 0 \\ -\frac{E_1(0)}{2} & x - \frac{1}{2} + y & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -1 \\ -\frac{E_n(0)}{2} & -\binom{n}{1}\frac{E_{n-1}(0)}{2} & \cdots & -\binom{n}{n-1}\frac{E_1(0)}{2} & x - \frac{1}{2} + y \end{vmatrix}.$$

For other choices of $A(t)$ *we proceed in a similar way.*

Example 3. *Let* $\phi(y,t) = e^{yt^2}$. *We can consider the power series* $A(t)$ *as in the previous example.*

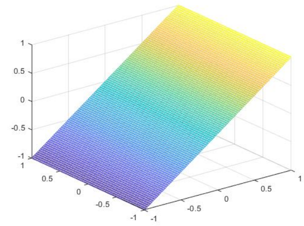
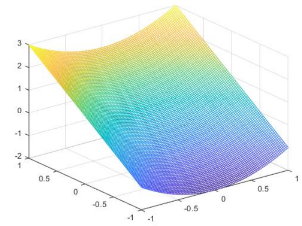
(1) $A(t) = 1$.
In this case we obtain the Hermite-Kampé de Fériet polynomials. They are denoted by $H_n^{(2)}(x,y)$, $n \geq 0$ [23,28,40]. *From* (9) *we get*

$$H_n^{(2)}(x,y) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n-2k}y^k}{k!(n-2k)!}.$$

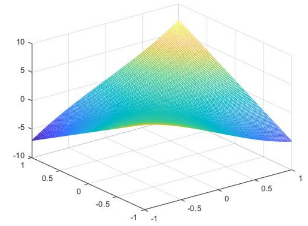
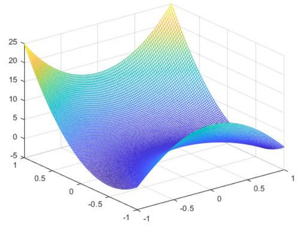
The first polynomials are:

$$H_0^{(2)}(x,y) = 1, \qquad H_1^{(2)}(x,y) = x, \qquad H_2^{(2)}(x,y) = x^2 + 2y,$$
$$H_3^{(2)}(x,y) = x^3 + 6xy, \qquad H_4^{(2)}(x,y) = x^4 + 12x^2y + 12y^2.$$

Their graphs are displayed in Figure 4.

(**a**) $H_1^{(2)}$ (**b**) $H_2^{(2)}$

(**c**) $H_3^{(2)}$ (**d**) $H_4^{(2)}$

Figure 4. Plot of $H_i^{(2)}$, $i = 1, \ldots, 4$, in $[-1,1] \times [-1,1]$.

Particular cases are

(a) $H_n^{(2)}\left(x, -\dfrac{1}{2}\right) = H_n^e(x)$*, known as probabilistic Hermite univariate polynomials [19] (p. 134);*

(b) $H_n^{(2)}(2x, -1) = H_n(x)$*, known as physicist Hermite or simply Hermite polynomials [19] (p. 134);*

(c) $H_n^{(2)}(x, 0) = x^n$*;*

(d) $H_n^{(2)}(0, y) = s_n(y) = \begin{cases} \dfrac{n!}{(\frac{n}{2})!} y^{\frac{n}{2}} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$

From (13) we obtain the conjugate sequence

$$\hat{H}_n^{(2)}(x, y) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(x-1)^{n-2k} y^k}{k!(n-2k)!},$$

and the second recurrence relation:

$$H_n^{(2)}(x, y) = \hat{H}_n^{(2)}(x, y) - \sum_{j=0}^{n-1} \binom{n}{j} (-1)^{n-j} H_j^{(2)}(x, y), \quad n \geq 1.$$

The related determinant form for $n > 0$ is

$$H_n^{(2)}(x, y) = (-1)^n \begin{vmatrix} \hat{H}_0^{(2)}(x, y) & \hat{H}_1^{(2)}(x, y) & \hat{H}_2^{(2)}(x, y) & \cdots & \hat{H}_n^{(2)}(x, y) \\ 1 & -\binom{1}{0} & \binom{2}{0}(-1)^2 & \cdots & \binom{n}{0}(-1)^n \\ 0 & 1 & -\binom{2}{1} & \cdots & \binom{n}{1}(-1)^{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -\binom{n}{n-1} \end{vmatrix}. \tag{50}$$

From (50) for $x = 1$ and $n > 0$ we have

$$n!\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\frac{y^k}{k!(n-2k)!} = (-1)^n \begin{vmatrix} 1 & s_1(y) & s_2(y) & \cdots & s_n(y) \\ 1 & -\binom{1}{0} & \binom{2}{0}(-1)^2 & \cdots & \binom{n}{0}(-1)^n \\ 0 & 1 & -\binom{2}{1} & \cdots & \binom{n}{1}(-1)^{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -\binom{n}{n-1} \end{vmatrix}. \tag{51}$$

Observe that $\dfrac{\phi^{(0,1)}(y,t)}{\phi(y,t)} = 2yt$. *Therefore the third recurrence relation becomes*

$$H_{n+1}^{(2)}(x,y) = x\,H_n^{(2)}(x,y) + 2nyH_{n-1}^{(2)}(x,y).$$

The related determinant form for $n > 0$ is

$$H_n^{(2)}(x,y) = \begin{vmatrix} x & -1 & 0 & \cdots & 0 \\ 2y & x & -1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & -1 \\ & \cdots & & 2y\binom{n}{n-1} & x \end{vmatrix}.$$

To the best of authors knowledge the first recurrence relation, the first determinant form and the last determinant form are new.
For $x = 1$ and $n > 0$ we get the identity

$$n!\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\frac{y^k}{k!(n-2k)!} = (-1)^n \begin{vmatrix} 1 & -1 & 0 & \cdots & 0 \\ 2y & 1 & -1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & -1 \\ & \cdots & & 2y\binom{n}{n-1} & 1 \end{vmatrix}.$$

From the comparison with (51) the following identity is obtained:

$$\begin{vmatrix} 1 & s_1(y) & s_2(y) & \cdots & s_n(y) \\ 1 & -\binom{1}{0} & \binom{2}{0}(-1)^2 & \cdots & \binom{n}{0}(-1)^n \\ 0 & 1 & -\binom{2}{1} & \cdots & \binom{n}{1}(-1)^{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -\binom{n}{n-1} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & \cdots & 0 \\ 2y & 1 & -1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & -1 \\ & \cdots & & 2y\binom{n}{n-1} & 1 \end{vmatrix}.$$

The Hermite-Kampé de Fériet polynomials $H_n^{(2)}(x,y)$ satisfy the following differential equations

1. $\dfrac{(-1)^n}{n!}\dfrac{\partial^n}{\partial x^n}f(x,y) + \cdots + f(x,y) = \displaystyle\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\frac{n!\,(x-1)^{n-2k}y^k}{k!(n-2k)!}$;
2. $\dfrac{\partial}{\partial y}H_n^{(2)}(x,y) = \dfrac{\partial^2}{\partial x^2}H_n^{(2)}(x,y)$ *(heat equation);*
3. $\left(2y\dfrac{\partial^2}{\partial x^2} + x\dfrac{\partial}{\partial x} - n\right)H_n^{(2)}(x,y) = 0.$

(2) $A(t) = \dfrac{t}{e^t - 1}$.

In this case we get the bivariate Appell sequence whose elements can be called Bernoulli–Hermite–Kampé de Fériet polynomials and denoted by $\mathcal{K}_n^B$.
From (6) *and* (11) *we obtain*

$$\mathcal{K}_n^B(x,y) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(x)\varphi_k(y) = \sum_{k=0}^{n} \binom{n}{k} H_k^{(2)}(x,y) B_{n-k},$$

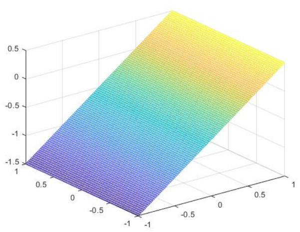
with

$$\varphi_k(y) = \frac{w_k k!}{\lfloor \frac{k}{2} \rfloor!} y^{\lfloor \frac{k}{2} \rfloor}, \quad \textit{being} \;\; w_k = \begin{cases} 1 & \textit{even } k \\ 0 & \textit{odd } k. \end{cases} \tag{52}$$

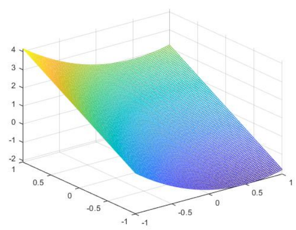
The first bivariate Bernoulli–Hermite–Kampé de Fériet polynomials are

$$\begin{aligned}
&\mathcal{K}_0^B(x,y) = 1, \qquad \mathcal{K}_1^B(x,y) = x - \frac{1}{2}, \qquad \mathcal{K}_2^B(x,y) = x^2 - x + 2y + \frac{1}{6}, \\
&\mathcal{K}_3^B(x,y) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x - 3y + 6xy, \\
&\mathcal{K}_4^B(x,y) = x^4 - 2x^3 + x^2 + 2y - 12xy + 12x^2y + 12y^2 - \frac{1}{30}.
\end{aligned}$$

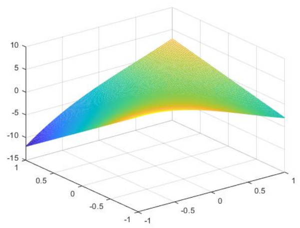
Their graphs are in Figure 5.



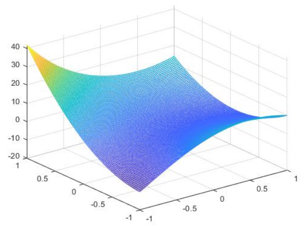
(**a**) $\mathcal{K}_1^B$



(**b**) $\mathcal{K}_2^B$



(**c**) $\mathcal{K}_3^B$



(**d**) $\mathcal{K}_4^B$

Figure 5. Plot of $\mathcal{K}_i^B$, $i = 1, \ldots, 4$, in $[-1,1] \times [-1,1]$.

In this case we observe that $\mathcal{K}_n^B(x,0) = B_n(x)$.
The first recurrence relation is

$$\mathcal{K}_0^B(x,y) = 1, \qquad \mathcal{K}_n^B(x,y) = H_n(x,y) - \sum_{j=0}^{n-1} \binom{n}{j} \frac{\mathcal{K}_j^B(x,y)}{n-j+1}, \qquad n \geq 1.$$

The related determinant form is obtained from (48) *by replacing* $(x+y)^k$ *by* $H_k^{(2)}(x,y)$, $k = 0, \ldots, n$.

As we observed, for $\phi(y,t) = e^{yt^2}$, $c_0(y) = 0$, $c_1(y) = 2y$, $c_k(y) = 0$, $k \geq 2$. *Moreover, as in the Example 2, case 2),* $b_0 = B_1$, $b_k = -\dfrac{B_{k+1}}{k+1}$, $k \geq 1$. *Hence the third recurrence relation is*

$$\mathcal{K}^B_{n+1}(x,y) = \left(x - \frac{1}{2}\right)\mathcal{K}^B_n(x,y) + n\left(2y - \frac{1}{12}\right)\mathcal{K}^B_{n-1}(x,y) - \sum_{k=1}^{n-2}\binom{n}{k}\frac{B_{n-k+1}}{n-k+1}\mathcal{K}^B_k(x,y).$$

The related determinant form for $n > 0$ *is*

$$\mathcal{K}^B_{n+1}(x,y) = \begin{vmatrix} x-\frac{1}{2} & -1 & 0 & \cdots & \cdots & 0 \\ 2y-\frac{1}{2} & x-\frac{1}{2} & -1 & \cdots & \cdots & 0 \\ -\frac{B_3}{3} & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & -1 \\ -\frac{B_{n+1}}{n+1} & -\binom{n}{1}\frac{B_n}{n} & \cdots & \cdots & \binom{n}{n-1}\left(2y-\frac{1}{2}\right) & x-\frac{1}{2} \end{vmatrix}.$$

(3) $A(t) = \dfrac{2}{e^t+1}$.

In this case we get the bivariate Appell sequence whose elements can be called Euler–Hermite–Kampé de Fériet polynomials and denoted by $\mathcal{K}^E_n$.

$$\mathcal{K}^E_n(x,y) = \sum_{k=0}^{n}\binom{n}{k}E_{n-k}(x)\varphi_k(y) = \sum_{k=0}^{n}\binom{n}{k}H^{(2)}_k(x,y)E_{n-k}(0).$$

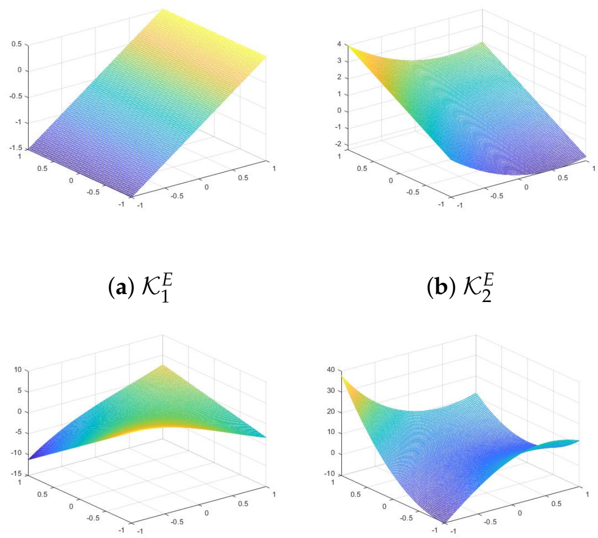
with $\varphi_k(y)$ *as in* (52).

The first polynomials of the sequence $\{\mathcal{K}^E_n\}^b_{n\in\mathbb{N}}$ *are*

$$\mathcal{K}^E_0(x,y) = 1, \qquad \mathcal{K}^E_1(x,y) = x - \frac{1}{2}, \qquad \mathcal{K}^E_2(x,y) = x^2 - x + 2y,$$

$$\mathcal{K}^E_3(x,y) = x^3 - \frac{3}{2}x^2 - 3y + 6xy + \frac{1}{4}, \qquad \mathcal{K}^E_4(x,y) = x^4 - 2x^3 + 12x^2y - 12xy + 12y^2 + x.$$

Their graphs are in Figure 6.



(**a**) $\mathcal{K}^E_1$ (**b**) $\mathcal{K}^E_2$

(**c**) $\mathcal{K}^E_3$ (**d**) $\mathcal{K}^E_4$

Figure 6. Plot of $\mathcal{K}^E_i$, $i = 1, \ldots, 4$, in $[-1,1] \times [-1,1]$.

Since $\alpha_k = E_k(0)$, $k = 0, \ldots, n$, *from* (12) *we get* $\beta_0 = 1$, $\beta_k = \dfrac{1}{2}$, $k = 1, \ldots, n$. *Therefore, the first recurrence relation is*

$$\mathcal{K}_0^E(x,y) = 1, \qquad \mathcal{K}_n^E(x,y) = H_n^{(2)}(x,y) - \frac{1}{2}\sum_{j=1}^{n-1}\binom{n}{j}\mathcal{K}_j^E(x,y), \qquad n \geq 1.$$

Since in this case $b_0 = -\dfrac{1}{2}$, $b_k = \dfrac{E_k(0)}{2}$, $k \geq 1$, *the third recurrence relation is*

$$\mathcal{K}_{n+1}^E(x,y) = \left(x - \frac{1}{2}\right)\mathcal{K}_n^E(x,y) + n\left(2y - \frac{1}{4}\right)\mathcal{K}_{n-1}^E(x,y) + \frac{1}{2}\sum_{k=0}^{n-2}\binom{n}{k}E_{n-k}(0)\mathcal{K}_k^E(x,y).$$

The related determinant form for $n > 0$ *is*

$$\mathcal{K}_{n+1}^E(x,y) = \begin{vmatrix} x+y-\frac{1}{2} & -1 & 0 & \cdots & 0 \\ \frac{E_1(0)}{2} & x+y-\frac{1}{2} & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -1 \\ \frac{E_n(0)}{2} & \binom{n}{1}\frac{E_{n-1}(0)}{2} & \cdots & \binom{n}{n-1}\frac{E_1(0)}{2} & x+y-\frac{1}{2} \end{vmatrix}.$$

Example 4. *Let* $\phi(y,t) = \dfrac{1}{1-yt}$.

(1) $A(t) = 1$.

Being $\phi(y,t) = \sum_{k=0}^{\infty} k! y^k \frac{t^k}{k!}$, *from* (10) *we get the elementary bivariate Appell sequence*

$$p_n(x,y) = \sum_{k=0}^{n}\frac{n!}{k!}x^k y^{n-k},$$

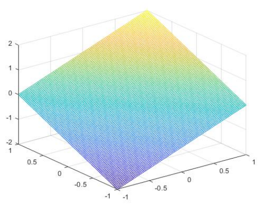
and from (25) *the conjugate sequence*

$$\hat{p}_n(x,y) = \sum_{k=0}^{n}\frac{n!}{k!}(x-1)^k y^{n-k}.$$

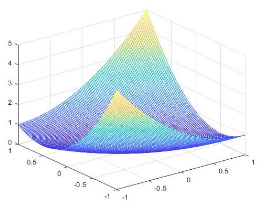
The first polynomials of the sequence $\{p_n\}_{n\in\mathbb{N}}^b$ *are*

$$p_0(x,y) = 1, \qquad p_1(x,y) = x + y, \qquad p_2(x,y) = x^2 + 2xy + 2y^2,$$
$$p_3(x,y) = x^3 + 3x^2y + 6xy^2 + 6y^3, \qquad p_4(x,y) = x^4 + 4x^3y + 12x^2y^2 + 24xy^3 + 24y^4.$$

Their graphs are in Figure 7.

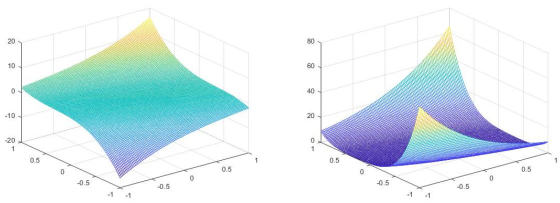


(a) p_1



(b) p_2

Figure 7. *Cont.*



(b) p_3 **(c)** p_4

Figure 7. Plot of polynomials p_i, $i = 1, \ldots, 4$, in $[-1,1] \times [-1,1]$.

For $p_n(x, y)$ relations (37) and (38) hold. Furthermore, since $\dfrac{\phi^{(0,1)}(y,t)}{\phi(y,t)} = \dfrac{y}{1-yt}$, *then* $c_k(y) = k!\, y^{k+1}$, $k \geq 0$. *Hence, from Remark (7), for $n > 0$*

$$p_{n+1}(x,y) = x\, p_n(x,y) + n! \sum_{k=0}^{n} \frac{y^{k+1}}{(n-k)!} p_{n-k}(x,y),$$

and

$$p_{n+1}(x,y) = \begin{vmatrix} x+y & -1 & 0 & \cdots & 0 \\ y^2 & x+y & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -1 \\ n!y^{n+1} & \binom{n}{1}(n-1)!y^n & \cdots & \binom{n}{n-1}y^2 & x+y \end{vmatrix}.$$

(2) $A(t) = \dfrac{t}{e^t - 1}$.
In this case we obtain

$$r_n^B(x,y) = \sum_{k=0}^{n} \frac{n!}{k!} B_k(x) y^{n-k}.$$

We note that

$$r_n^B(x,0) = B_n(x), \qquad r_n^B(0,0) = B_n, \qquad n \geq 1,$$

Moreover, $\alpha_k = B_k$ *and from (12)* $\beta_k = \dfrac{1}{k+1}$, $k = 0, 1, \ldots$
Hence the first recurrence relation is

$$r_0^B(x,y) = 1, \qquad r_n^B(x,y) = p_n(x,y) - \sum_{j=0}^{n-1} \binom{n}{j} \frac{r_j^B(x,y)}{n-j+1}, \qquad n \geq 1,$$

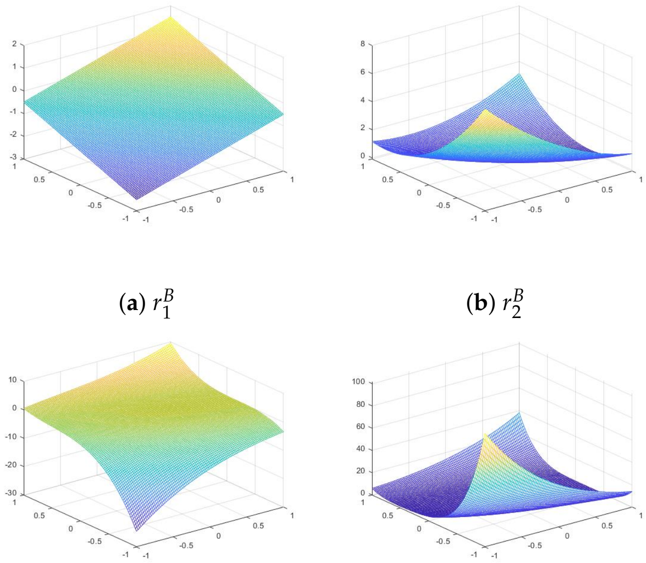
and the conjugate sequence is

$$\hat{r}_n^B(x,y) = \sum_{k=0}^{n} \frac{n!}{k!(n-k+1)!} p_k(x,y).$$

The first polynomials of the sequence $\{r_n^B\}_{n\in\mathbb{N}}^b$ *are*

$$r_0^B(x,y) = 1, \qquad r_1^B(x,y) = -\frac{1}{2} + x + y, \qquad r_2^B(x,y) = \frac{1}{6} - x + x^2 - y + 2xy + 2y^2,$$
$$r_3^B(x,y) = \frac{x}{2} - \frac{3}{2}x^2 + x^3 + \frac{y}{2} - 3xy + 3x^2y - 3y^2 + 6xy^2 + 6y^3,$$
$$r_4^B(x,y) = -\frac{1}{30} + x^2 - 2x^3 + x^4 + 2xy - 6x^2y + 4x^3y + 2y^2 - 12xy^2$$
$$+12x^2y^2 - 12y^3 + 24xy^3 + 24y^4.$$

Their graphs are in Figure 8.



(**a**) r_1^B (**b**) r_2^B (**c**) r_3^B (**d**) r_4^B

Figure 8. Plot of r_i^B, $i = 1, \ldots, 4$, in $[-1,1] \times [-1,1]$.

As in the case (2) of the previous Examples, $b_0 = B_1$, $b_k = -\dfrac{B_{k+1}}{k+1}$, $k \geq 1$. *Hence the third recurrence relation is*

$$r_{n+1}^B(x,y) = \left(x + y - \frac{1}{2}\right) r_n^B(x,y) + n! \sum_{k=0}^{n-1} \left(y^{n-k+1} - \frac{B_{n-k+1}}{(n-k+1)!} \right) \frac{r_k^B(x,y)}{k!}.$$

The related determinant form for $n > 0$ *is*

$$r_{n+1}^B(x,y) = \begin{vmatrix} x+y-\frac{1}{2} & -1 & 0 & \cdots & 0 \\ b_1 + y^2 & x+y-\frac{1}{2} & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 \\ b_n + n! y^{n+1} & \binom{n}{1}(b_{n-1} + (n-1)! y^n) & \cdots & \binom{n}{n-1}(b_1 + y^2) & x+y-\frac{1}{2} \end{vmatrix}.$$

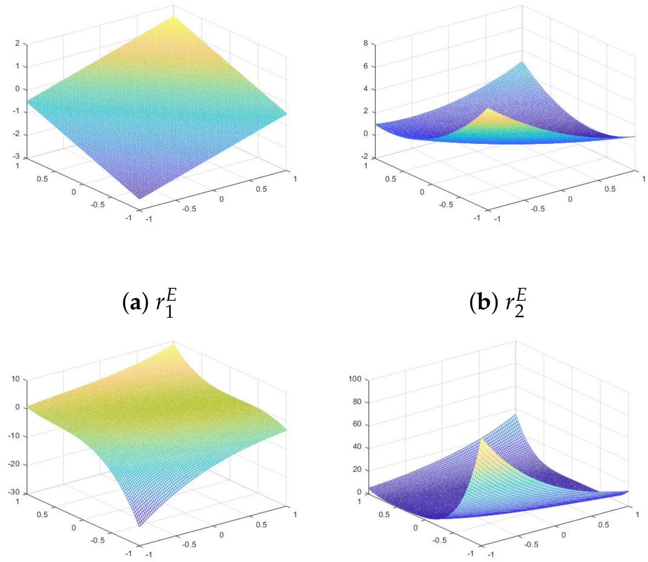
(3) $A(t) = \dfrac{2}{e^t + 1}$. *In this case we obtain*

$$r_n^E(x,y) = \sum_{k=0}^{n} \frac{n!}{k!} E_k(x) y^{n-k}.$$

The first polynomials of the sequence $\{r_n^E\}_{n \in \mathbb{N}}^b$ *are*

$$r_0^E(x,y) = 1, \qquad r_1^E(x,y) = -\frac{1}{2} + x + y, \qquad r_2^E(x,y) = -x + x^2 - y + 2xy + 2y^2,$$
$$r_3^E(x,y) = \frac{1}{4} - \frac{3}{2}x^2 + x^3 - 3xy + 3x^2y - 3y^2 + 6xy^2 + 6y^3,$$
$$r_4^E(x,y) = x - 2x^3 + x^4 + y - 6x^2y + 4x^3y - 12xy^2 + 12x^2y^2 - 12y^3 + 24xy^3 + 24y^4.$$

Their graphs are in Figure 9.



(**a**) r_1^E (**b**) r_2^E

(**c**) r_3^E (**d**) r_4^E

Figure 9. Plot of r_i^E, $i = 1, \dots, 4$, in $[-1,1] \times [-1,1]$.

Moreover, since $\beta_0 = 1$, $\beta_k = \dfrac{1}{2}$, $k = 1, \dots, n$, the first recurrence relation is

$$r_0^E(x,y) = 1, \qquad r_n^E(x,y) = \sum_{j=0}^{n} \frac{n!}{j!} x^j y^{n-j} - \frac{1}{2} \sum_{j=0}^{n-1} \binom{n}{j} r_j^E(x,y), \qquad n \geq 1,$$

and the conjugate sequence is

$$\hat{r}_n^E(x,y) = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} p_k(x,y).$$

As in the case (3) of the previous Examples, $b_0 = -\dfrac{1}{2}$, $b_k = \dfrac{E_k(0)}{2}$, $k \geq 1$. Hence the third recurrence relation is

$$r_{n+1}^E(x,y) = \left(x + y - \frac{1}{2}\right) r_n^E(x,y) + \sum_{k=0}^{n-1} \binom{n}{k} \left((n-k)! y^{n-k+1} + \frac{E_{n-k}(0)}{2}\right) r_k^E(x,y).$$

The related determinant form for $n > 0$ is

$$r_{n+1}^E(x,y) = \begin{vmatrix} x+y-\frac{1}{2} & -1 & 0 & \cdots & 0 \\ y^2 + \frac{E_1(0)}{2} & x+y-\frac{1}{2} & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -1 \\ n! y^{n+1} + \frac{E_n(0)}{2} & \binom{n}{1}\left((n-1)! y^n + \frac{E_{n-1}(0)}{2}\right) & \cdots & \binom{n}{n-1}\left(y^2 + \frac{E_1(0)}{2}\right) & x+y-\frac{1}{2} \end{vmatrix}.$$

Remark 12. *In [29,30] the authors introduced the functions $\phi(y,t) = \cos yt$, $\phi(y,t) = \sin yt$. They studied the related elementary sequences and respectively the Bernoulli and Genocchi sequences but matricial and determinant forms are not considered. Most of their results are a consequence of our general theory.*

10. Concluding Remarks

In this work, an approach to general bivariate Appell polynomial sequences based on elementary matrix calculus has been proposed.

This approach, which is new in the literature [3,22,24,27,28], generated a systematic, simple theory. It is in perfect analogy with the theory in the univariate case (see [19] and the references therein). Moreover, our approach provided new results such as recurrence formulas and related differential equations and determinant forms. The latter are useful both for numerical calculations and for theoretical results, such as combinatorial identities and biorthogonal systems of linear functionals and polynomials. In particular, after some definitions, the generating function for a general bivariate Appell sequence is given. Then matricial forms are considered, based on the so called elementary bivariate Appell polynomial sequences. These forms provide three recurrence relations and the related determinant forms. Differential definitions and recurrence relations generate differential equations. For completeness of discussion the multiplicative and derivatives differential operators are hinted. A linear functional on $\mathcal{S}_n = \text{span}\{p_0, \ldots, p_n \mid n \in \mathbb{N}\}$ is considered. It generates a general bivariate Appell sequence. Hence, an interesting theorem on representation for any polynomial belonging to $\mathcal{S}_n$ is established. Finally, some examples of general bivariate Appell sequences are given.

Further developments are possible. In particular, the extension of the considered linear functional to a suitable class of bivariate real functions and the related Appell interpolant polynomial. These interpolant polynomials can be applied not only as an approximant of a function, but also to generate new cubature and summation formulas. It would also be interesting to consider the bivariate generating functions for polynomials.

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