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On Fibonacci Numbers of Order r Which Are Expressible as Sum of Consecutive Factorial Numbers

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Abstract: Let $(t_n^{(r)})_{n\geq 0}$ be the sequence of the generalized Fibonacci number of order r, which is defined by the recurrence $t_n^{(r)} = t_{n-1}^{(r)} + \cdots + t_{n-r}^{(r)}$ for $n \geq r$, with initial values $t_0^{(r)} = 0$ and $t_i^{(r)} = 1$, for all $1 \leq i \leq r$. In 2002, Grossman and Luca searched for terms of the sequence $(t_n^{(2)})_n$, which are expressible as a sum of factorials. In this paper, we continue this program by proving that, for any $\ell \geq 1$, there exists an effectively computable constant $C = C(\ell) > 0$ (only depending on ℓ), such that, if (m, n, r) is a solution of $t_m^{(r)} = n! + (n+1)! + \cdots + (n+\ell)!$, with r even, then $\max\{m, n, r\} < C$. As an application, we solve the previous equation for all $1 \leq \ell \leq 5$.

Keywords: diophantine equation; factorial; fibonacci r-numbers; 2-adic valuation

MSC: 11Dxx; 11B39

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1. Introduction

We recall that the factorial of an integer $n \geq 1$, denoted by $n!$, is the product $\prod_{j=1}^{n} j$. Along the years, several authors have considered Diophantine problems involving factorial numbers. For instance, Erdős and Selfridge [1] proved that 1 is the only perfect power in the sequence of factorials. However, the most famous and calssical among such problems was raised by Brocard [2], in 1876, and, independently, by Ramanujan [3,4] (p. 327 in ref. [4]), in 1913. The Diophantine equation

$$n! + 1 = m^2,$$

in positive integers m and n, is known as Brocard–Ramanujan Diophantine equation.

There are three solutions, namely, $(n, m) = (4, 5), (5, 11)$, and $(7, 71)$, and no solution was found for $7 < n < 10^9$ (as can be see in [5]). In fact, the Brocard–Ramanujan equation remains still as an open problem.

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence that is given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$. There are also several interesting problems related to Fibonacci numbers (for recent results, we refer the reader to [6,7] and references therein). For instance, the problem of the perfect powers in the Fibonacci sequence attracted much attention during some past decades. In 2003, Bugeaud et al. ([8] Theorem 1) confirmed the expectation, that $0, 1, 8$, and 144 are the only perfect powers among $(F_n)_{n\geq 0}$. A generalization (for Fibonomial coefficients) of this result can be found in [9]. We still refer the reader to [8] for additional references and history.

Many mathematicians have been interested in Diophantine problems that involve both Fibonacci and factorial numbers. For instance, in 1999, Luca [10] proved that $F_{12} = 2!2!3!3!$ is the largest Fibonacci number which can be written as a product of factorials. Additionally, $F_1F_2F_3F_4F_5F_6F_8F_{10}F_{12} = 11!$ is the largest product of distinct Fibonacci numbers, which is a product of factorials (see [11]).

Moreover, Grossman and Luca [12] showed that, for any given $\ell \geq 1$, there are only finitely many positive integers n, such that

$$F_n = m_1! + m_2! + \cdots + m_\ell!$$

holds for some $m_1, m_2, \ldots, m_\ell \in \mathbb{Z}_{\geq 1}$. Moreover, they determined all of the solutions for $\ell \in \{1, 2\}$. In 2010, the case $\ell = 3$ was completely solved in [13] (for the inverse problem, i.e., factorials that are written as sum of a given number of Fibonacci numbers, we refer the reader to [14] and the references therein).

As any very well-studied object in mathematics, the Fibonacci sequence possesses many kinds of generalizations. The most well-known generalization is probably the so-called k-generalized Fibonacci sequence (or the sequence of the k-bonacci numbers), which is defined for $n \geq 2$ by the kth order recurrence

$$F_n^{(k)} = F_{n-1}^{(k)} + \cdots + F_{n-k}^{(k)}, \quad \text{with} \quad F_{-(k-2)}^{(k)} = \cdots = F_0^{(k)} = 0 \text{ and } F_1^{(k)} = 1.$$

Some interesting features of the sequence $(F_n^{(k)})_n$ we can find, e.g., in [15–19]. Here, we are interested in a sequence $(t_n^{(r)})_{n\geq 0}$ with the same recurrence relation, but with modified initial conditions. This sequence is called generalized Fibonacci numbers of order r and it is defined by

$$t_n^{(r)} = \begin{cases} 0, & \text{if } n = 0; \\ 1, & \text{if } 1 \leq n \leq r-1; \\ \sum_{i=1}^{r} t_{n-i}^{(r)}, & \text{if } n \geq r. \end{cases}$$

For $r = 2$, we have the sequence of Fibonacci numbers and, for $r = 3$, we have the Tribonacci numbers.

We remark that the equation $F_m = n!$ may be solved by a direct application of the Carmichael Theorem about primitive divisors in the Fibonacci sequence (which asserts that for any $n > 12$, there exists a prime number p, such that $p \mid F_n$, but $p \nmid \prod_{i=1}^{n-1} F_i$). However, there is no such a result for Tribonacci numbers. Thus, Lengyel and Marques [20] provided a complete description for the 2-adic valuation $\nu_2(t_n^{(3)})$ (where $\nu_p(s) := \max\{k \geq 0 : p^k \mid s\}$) in order to solve the equation $t_n^{(3)} = m!$. After that, some authors generalized their results for $\nu_2(t_n^{(r)})$ (see [21,22]). In particular, Sobolewski [21] completely characterized $\nu_2(t_n^{(r)})$, for all $n \geq 1$ and $r \geq 4$ with r even.

In this paper, we search for terms of a generalized Fibonacci sequence of order r, which can be written as a sum of consecutive factorial numbers. In particular, we provide an explicit constant (depending only on the number of factorials), which is an upper bound for the number of the possible solutions. More precisely, we have

Theorem 1. *Let ℓ be a positive integer and let $(n, m, r) \in \mathbb{Z}_{\geq 1}^3$ be a solution of the Diophantine equation*

$$t_m^{(r)} = n! + (n+1)! + \cdots + (n+\ell)!, \tag{1}$$

with $m > r \geq 2$ even. We have

(a) For $n \leq 3$, it holds that

(i) If $m \leq 2r$, then either $(n, m, r, \ell) \in \{(1, 4, 2, 1),\ (1, 60, 56, 5)\}$ or

$$(n, m, r, \ell) = \left(3, 2 + \frac{1}{2}\sum_{j=3}^{\ell+3} j!, 1 + \frac{1}{2}\sum_{j=3}^{\ell+3} j!, \ell\right), \tag{2}$$

where ℓ is any positive integer.

(ii) If $m > 2r$, then

$$m < 7.1(\ell + 4)\log(\ell + 3).$$

(b) *For $n \geq 4$, it holds that*

$$n < 2\log((\ell+1)\log(\ell+1)) + 105,\ m < 6(n+\ell+1)\log(n+\ell) \text{ and } r \leq \frac{m-2}{2}.$$

As an application of the previous result, we found all solutions of (1) for some cases of ℓ. Actually, we prove that

Theorem 2. *All of the solutions of the Equation (1) for positive integers n, m, r and ℓ, with $r \equiv 0 \pmod 2$, $m > r \geq 2$, and $\ell \leq 5$ are*

$$\begin{aligned}(n,m,r,\ell) \in \{&(1,4,2,1),(1,60,56,5),(2,6,2,1),(2,10,4,3),(3,17,16,1),(3,77,76,2),\\ &(3,437,436,3),(3,2957,2956,4),(3,23117,23116,5),(4,12,2,1)\}.\end{aligned}$$

We organize this paper, as follows. In Section 2, we will present some helpful properties of the sequence $(t_n^{(r)})_n$. The third section is devoted to the proof of Theorems 1 and 2. The computations of this paper will be performed by using the Wolfram Mathematica software.

2. Auxiliary Results

Before proceeding further, some considerations will be needed for the convenience of the reader.

The characteristic polynomial of the sequence $(t_n^{(r)})_n$ is $\psi_k(x) = x^r - x^{r-1} - \cdots - x - 1$, which has only one root outside the unit circle, say α, which is located in the interval $(2(1-2^{-r}), 2)$ (see [23]). Furthermore, one can deduce from ([21] Lemma 4) that

Lemma 1. *For all $n \geq 1$, we have*

$$t_n^{(r)} > \alpha^{n-r-1}.$$

In particular, $t_n^{(r)} > (\sqrt{2})^{n-r-1}$, for all $n \geq 1$.

The last inequality follows because $\alpha > 2 - 1/2^{r-1} \geq 2 - 1/2 = 3/2 > \sqrt{2}$ (for all $r \geq 2$).

Another very useful tool is related to the *p-adic order* (recall that the p-adic order, or valuation, of s, $\nu_p(s)$, is the exponent of the highest power of a prime p, which divides s). An explicit formula for $\nu_p(F_n)$ was provided, see [24–27]. In particular, Lengyel [25] showed that

Lemma 2. *We have that*

$$\nu_2(F_n) = \begin{cases} 0, & \text{for } n \equiv 1,2 \pmod 3;\\ 1, & \text{for } n \equiv 3 \pmod 6;\\ \nu_2(n)+2, & \text{for } n \equiv 0 \pmod 6.\end{cases}$$

In 2014, Lengyel and Marques [20] characterized $\nu_2(t_n^{(3)})$ and, recently, Sobolewsky [21] and Young [22] worked on a description of $\nu_2(t_n^{(r)})$, for even and odd r, respectively. In particular, the case in which r is even was completely solved:

Lemma 3. *For $r = 2k \geq 4$, we have that*

$$\nu_2(t_n^{(r)}) = \begin{cases} 0, & \text{for } n \equiv 1,2,\ldots,2k \pmod{2k+1};\\ 1, & \text{for } n \equiv 2k+1 \pmod{2(2k+1)};\\ \nu_2(n)+\nu_2(k-1)+2, & \text{for } n \equiv 0 \pmod{2(2k+1)}.\end{cases}$$

Remark 1. *We remark that the $\nu_2(t_n^{(r)})$ is not completely characterized for $r \geq 5$ odd. In fact, the only missing case happens if $n \equiv r+1 \pmod{2r+2}$ and $\nu_2(n-r-1) = \nu_2(r^2-1)$. Indeed, Young [22] showed that, in this case, $\nu_2(t_n^{(r)}) = \nu_2(z-(n-r-1)/(2r+2))+2$, for some 2-adic integer z. As will be seen after, the proof of Theorem 1 only requires an upper bound for $\nu_2(t_n^{(r)})$, but there is no a direct tool for providing a useful such bound for $\nu_2(z-(n-r-1)/(2r+2))$. Even the deep theory of linear forms in p-adic logarithms is not helpful, since it is conjectured that z is a 2-adic transcendental number.*

We require one last fact about ν_2 in order to complete our proof of Theorem 1.

Lemma 4. *For any integer $n \geq 1$, we have*

$$\nu_2(n!) \geq n - \left\lfloor \frac{\log n}{\log 2} \right\rfloor - 1, \tag{3}$$

where $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x. In particular, $\nu_2(n!) \geq n/4$, for all $n \geq 4$.

We refer the reader to ([28] Lemma 2.4) for a proof of this result.

Now, we are ready to deal with the proof of theorems.

3. The Proofs

3.1. The Proof of Theorem 1

Write

$$t_m^{(r)} = n! + (n+1)! + \cdots + (n+\ell)!$$

as $t_m^{(r)} = n! d_{n,\ell}$, where

$$d_{n,\ell} = 1 + (n+1) + (n+1)(n+2) + \cdots + (n+1)\cdots(n+\ell).$$

Note that $d_{n,\ell}$ is an integer and, moreover, the following estimate holds

$$n! d_{n,\ell} \leq (\ell+1)(n+\ell)! \leq (\ell+1)(n+\ell)^{n+\ell} < (n+\ell)^{n+\ell+1},$$

where we used that $s! \leq s^s$, for all $s \geq 1$ (we decided to use this inequality instead of the sharper $s! \leq 2(s/2)^s$, for $s \geq 3$, in order to leave the bounds notationally simpler and we observe that this choice does not change them in order).

The proof splits into two cases.

3.1.1. The Case $n \leq 3$

If $r < m \leq 2r$, then

$$t_{r+i}^{(r)} = 2^{i-1}(2r-3) + 1,$$

for $i \in [1, r]$ (this can be seen in ([22] p. 4)). If $i = 1$, then the equation becomes

$$2(r-1) = n! + (n+1)! + \cdots + (n+\ell)!$$

and so

$$r - 1 = \frac{n!}{2} + \frac{(n+1)!}{2} + \cdots + \frac{(n+\ell)!}{2}.$$

Because the left-hand side is an odd integer, then $n \in \{2, 3\}$. If $n = 2$, then

$$r - 1 = 1 + \frac{3!}{5} + (\text{sum of even terms})$$

yielding that $r = 5 +$ (sum of even terms) is odd (where we convention that (sum of even terms) is zero, for $\ell = 1$). Thus, there is no solution for $n = 2$ and $r < m \leq 2r$. When $n = 3$, we have that

$$r - 1 = \frac{3!}{2} + \frac{4!}{2} + \cdots + \frac{(\ell+3)!}{2}$$

is even. Thus, we obtain the following family of solutions

$$n = 3,\ r = 1 + \frac{3!}{2} + \frac{4!}{2} + \cdots + \frac{(\ell+3)!}{2},\ m = r+1 \text{ and } \ell.$$

For $i > 1$, one has that $t_m^{(r)} > 1$ is odd, which forces $n = 1$. Thus, we have

$$2^{i-1}(2r-3) + 1 = 1! + 2! + \cdots + (\ell+1)!$$

and, so

$$2^{i-1}(2r-3) = 2! + \cdots + (\ell+1)!. \tag{4}$$

If $r = 2$, then the equation becomes

$$2 = 2! + \cdots + (\ell+1)!,$$

where we used that $i = 2$ (since $1 < i \leq r = 2$). The previous equality only holds for $\ell = 1$, yielding the solution $(n, m, r, \ell) = (1, 4, 2, 1)$.

Supposing that $r \geq 4$, then $2r - 3$ is an odd number larger than 1. On the other hand, $2! + 3!$ and $2! + 3! + 4!$ are powers of 2 and so $\ell \geq 4$. However, if $\ell = 4$, then

$$2^{i-1}(2r-3) = 2! + 3! + 4! + 5! = 152 = 2^3 \cdot 19$$

yielding that $2r - 3 = 19$ and, so, $r = 11$, which is odd (remember that our assumption is that r is even). Thus, we may assume $\ell \geq 5$. Because $\nu_2(s!) \geq 4$, for all $s \geq 6$, then the 2-adic valuation of right-hand side of (4) is 3 yielding that $i = 4$. Therefore, we can rewrite the previous equality as

$$2r - 3 = 19 + \frac{6!}{8} + \cdots + \frac{(\ell+1)!}{8}$$

and, then

$$r = 11 + \frac{6!}{16} + \cdots + \frac{(\ell+1)!}{16}.$$

Now, we observe that the right-hand side is only even for $\ell = 5$ (since $7!/16 = 315$ and $(\ell+1)!/16$ is even, for all $\ell \geq 7$). Therefore, $r = 11 + 6!/16 = 56$, which leads to the solution

$$t_{60}^{(56)} = 1! + 2! + 3! + 4! + 5! + 6!.$$

Therefore, let us suppose that $m > 2r$. In this case, by Lemma 1, together with $r < m/2$, one has that

$$(\sqrt{2})^{(m/2)-1} \leq 3! d_{3,\ell} \leq 6(\ell+1)(\ell+3)^{\ell+3} < 6(\ell+3)^{\ell+4}.$$

By applying the log function together with an straightforward calculation, we arrive at

$$m < \frac{4}{\log 2}(\ell+4)\log(\ell+3) + \frac{4\log 6}{\log 2} + 2 < 7.1(\ell+4)\log(\ell+3).$$

In conclusion, we have

$$n \leq 3 \text{ and } r/2 < m < 7.1(\ell+4)\log(\ell+3)$$

as desired.

3.1.2. The Case $n \geq 4$

In this case, from the equation $t_m^{(r)} = n! d_{n,\ell}$, one deduces that 4 divides $t_m^{(r)}$. Accordingly, by combining Lemmas 2 and 3, we have that $2(r+1)$ divides m and

$$\nu_2(t_m^{(r)}) = \nu_2(m) + \nu_2(\widehat{k}-1) + 2,$$

where $\widehat{k} := (1-\delta_{2,r})k$ and $\delta_{i,j}$ is the Kronecker delta (whose value is 1 if $i = j$, and 0 otherwise).

On the other hand, since $d_{n,\ell}$ is an integer,

$$\nu_2(m) + \nu_2(\widehat{k}-1) + 2 = \nu_2(t_m^{(r)}) \geq \nu_2(n!) \geq n/4,$$

where we used Lemma 4, since $n \geq 4$. Because $\nu_2(m) + \nu_2(\widehat{k}-1) = \nu_2(m(\widehat{k}-1))$, we get

$$\nu_2(m(\widehat{k}-1)) \geq \frac{n}{4} - 2.$$

In particular, $2^{\lfloor (n/4)-2 \rfloor}$ divides $m(\widehat{k}-1)$ and so

$$2^{(n/4)-3} < 2^{\lfloor (n/4)-2 \rfloor} \leq m(\widehat{k}-1),$$

where we used that $\lfloor x \rfloor > x - 1$. After some computations, we obtain

$$n < \frac{4}{\log 2} \log(m(\widehat{k}-1)) + 12.$$

Because $2(2k+1)$ divides m, then $\widehat{k} - 1 \leq k - 1 < 2(2k+1) \leq m$ and, hence

$$n < \frac{8}{\log 2} \log m + 12 < 11.6 \log m + 12. \tag{5}$$

On the other hand, it follows again from Lemma 1 that

$$(\sqrt{2})^{m-r-1} \leq t_m^{(r)} \leq n! d_{n,\ell} < (n+\ell)^{n+\ell+1}.$$

Because $2(r+1) \leq m$, then $r + 1 \leq m/2$ and so

$$(\sqrt{2})^{m/2} \leq n! d_{n,\ell} < (n+\ell)^{n+\ell+1}.$$

After some manipulations, we get

$$m \leq \frac{4}{\log 2}(n+\ell+1)\log(n+\ell).$$

Now, the goal is to write the previous right-hand side in a better (product) form. For that, we shall use that $x + y \leq xy$, for all $x, y \in \mathbb{R}_{\geq 2}$. From this inequality, we have $n + \ell + 1 \leq n(\ell+1)$ (since $n \geq 4$ and $\ell + 1 \geq 2$). For $\log(n+\ell)$, we have a more delicate issue (since ℓ may be equal to 1). However, we use the following trick

$$\begin{aligned} \log(n+\ell) &= \log((n-1)+(\ell+1)) \leq \log((n-1)(\ell+1)) \\ &= \log(n-1) + \log(\ell+1) \leq 1.9\log(n-1) + 2.9\log(\ell+1) \\ &\leq 5.6\log(n-1)\log(\ell+1) < 5.6(\log n)\log(\ell+1), \end{aligned}$$

where we used that $1.9\log(n-1) \geq 1.9\log 3 > 2$ and $2.9\log(\ell+1) \geq 2.9\log 2 > 2$. Therefore, we have

$$\begin{aligned} m &\leq \frac{4}{\log 2}(n+\ell+1)\log(n+\ell) < 32.4n(\ell+1)(\log n)\log(\ell+1) \\ &< 33n(\ell+1)(\log n)\log(\ell+1). \end{aligned} \tag{6}$$

Summarizing, we obtained that

$$m < 33n(\ell+1)(\log n)\log(\ell+1). \tag{7}$$

By combining (5) and (7), we get

$$n - \log(n\log n) < 52.6 + \log((\ell+1)\log(\ell+1)). \tag{8}$$

We claim that $\log(n\log n) < n/2$, for all $n \geq 4$. Indeed, let us consider the function $g : (1, +\infty) \to \mathbb{R}$, as defined by $g(x) := xe^{-x/2}\log x$. Thus,

$$g'(x) = e^{-x/2}\left(\log x - \frac{x\log x}{2} + 1\right).$$

However, $(x\log x)/2 > \log x + 1 = \log(ex)$, if and only if $x^{x-2} > e^2$, which is true for all $x \geq 4$. Subsequently, $g'(x) < 0$, for all $x \geq 4$ and, so, g is a decreasing function in the interval $[4, +\infty)$. In particular, for $n \geq 4$, we have

$$ne^{-n/2}\log n = g(n) \leq g(4) = \frac{4\log 4}{e^2} < 0.75\ldots < 1,$$

which yields $n\log n < e^{n/2}$ and, finally, $\log(n\log n) < n/2$ as claimed (we point out that to use the easier inequality $e^x > 1 + x + x^2/2$, for $x > 0$, is not satisfactory, since $1 + n/2 + n^2/8 > n\log n$, only for $n > 19$).

By returning to (8), we deduce that

$$n < 2\log((\ell+1)\log(\ell+1)) + 105$$

as desired. The proof is then complete (by considering the inequality in (6)).

3.2. The Proof of Theorem 2

If $n \leq 3$ and $m \leq 2r$, then we have the following solutions that arise from (2):

$$(n, m, r, \ell) \in \{(3, 17, 16, 1), (3, 77, 76, 2), (3, 437, 436, 3), (3, 2957, 2956, 4), (3, 23117, 23116, 5)\}.$$

Furthermore, the solutions $(n, m, r, \ell) = (1, 4, 2, 1)$ and $(n, m, r, \ell) = (1, 60, 56, 5)$ were detected in the proof of Theorem 1.

For the case in which either $n \in \mathbb{Z}_{\geq 4}$ or $(n, m) \in \{1, 2, 3\} \times \mathbb{Z}_{>2r}$, we use the estimates that are provided in Theorem 1 (for $\ell \leq 5$) to infer that

$$n \leq 109,\ m \leq 3276 \text{ and } r \leq 1638.$$

For dealing with these remaining cases, we wrote two simple routines in Wolfram Mathematica software. First, the nth term of the sequence $(t_n^{(r)})_n$ can be defined as

```
t[n_, r_] :=
 t[n, r] =
  Which[n == 0, 0, 0 < n < r, 1, n >= r,
   Sum[t[n - i, r], {i, 1, r}]];
```

Afterwards, we shall use the following command to search for all solutions of

$$t_m^{(r)} = n! + (n+1)! + \cdots + (n+\ell)!$$

in the range $1 \leq n \leq 109, r < m \leq 3276$ and $2 \leq r \leq 1638$ (r even) when either $n \in \mathbb{Z}_{\geq 4}$ or $(n, m) \in \{1, 2, 3\} \times \mathbb{Z}_{>2r}$.

For the case $n \geq 4$, the routine

```
Catch[Do[{ n, m, r,l};
 If[t[m,r] == Sum[Factorial[n+i], {i,0,l}],
  Print[{n,m,r,l}]], {l,1,5}, {n, 4, 109},{r, 2, 1638,2}, {m,   r+1, 3276}]]
```

returns $\{4, 12, 2, 1\}$ as the only solution.

For the case $n \leq 3$ and $m > 2r$, the routine

```
Catch[Do[{ n, m, r,l};
 If[t[m,r] == Sum[Factorial[n+i], {i,0,l}],
  Print[{n,m,r,l}]], {l,1,5}, {n, 1, 3}, {r, 2, 1638,2}, {m,   2r+1, 3276}]]
```

returns $\{2, 6, 2, 1\}$ and $\{2, 10, 4, 3\}$ as solutions. This finishes the proof.

The calculation took roughly 2 h on 2.5 GHz Intel Core i5 4 GB Mac OSX. The proof is then complete.

4. Conclusions

In this paper, we work on searching for the terms of the Fibonacci sequence of order r, $(t_n^{(r)})_n$, which can be written as sum of consecutive factorials, where $t_n^{(r)} = \sum_{j=1}^{r} t_{n-j}^{(r)}$ with $t_0^{(r)} = 0$ and $t_1^{(r)} = \cdots = t_{r-1}^{(r)} = 1$. More precisely, we prove that, for any given $\ell \geq 1$, there exists a positive explicit constant C, depending only on ℓ, for which all triples $(n, m, r) \in \mathbb{Z}_{\geq 1}^3$ (with r even) of solutions of the Diophantine equation $t_m^{(r)} = \sum_{j=0}^{\ell} (n+j)!$ must satisfy $\max\{n, m, r\} < C$. The methods that are presented in this work combine upper bounds for the 2-adic valuation of $t_n^{(r)}$ together with some estimates and a computational approach. This may benefit future research concerning similar problems for other linear recurrence sequences (or even for $(t_n^{(r)})$ in the case of an odd r).

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