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Geometric Inequalities for Warped Products in Riemannian Manifolds

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Abstract: Warped products are the most natural and fruitful generalization of Riemannian products. Such products play very important roles in differential geometry and in general relativity. After Bishop and O'Neill's 1969 article, there have been many works done on warped products from *intrinsic* point of view during the last fifty years. In contrast, the study of warped products from *extrinsic* point of view was initiated around the beginning of this century by the first author in a series of his articles. In particular, he established an optimal inequality for an isometric immersion of a warped product $N_1 \times_f N_2$ into any Riemannian manifold $R^m(c)$ of constant sectional curvature c which involves the Laplacian of the warping function f and the squared mean curvature H^2 . Since then, the study of warped product submanifolds became an active research subject, and many papers have been published by various geometers. The purpose of this article is to provide a comprehensive survey on the study of warped product submanifolds which are closely related with this inequality, done during the last two decades.

Keywords: warped products; warped product immersion; inequality; space forms; space of quasi-constant curvature; eigenfunction; Laplacian

MSC: 53A07; 53C40; 53C42; 53B25

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1. Introduction

For two given Riemannian manifolds, B and F, of positive dimensions, endowed with Riemannian metrics, g_B and g_F, respectively, and, for a positive smooth function, f on B, the warped product $N = B \times_f F$ is, by definition, the manifold $B \times F$ equipped with the warped product Riemannian metric $g = g_B + f^2 g_F$ (see Reference [1]). The function f is called the warping function of the warped product.

The warped products play important roles in differential geometry, as well as in physics, especially in general relativity. For instance, the best relativistic model of the Schwarzschild spacetime that describes the out space around a massive star or a black hole can be described as a warped product (see Reference [2,3]). (For recent surveys on warped products as Riemannian submanifolds, we refer to Reference [2,4]).

One of the most fundamental problems in the theory of submanifolds is the immersibility of a Riemannian manifold into a Euclidean m-space $\mathbb{E}^m$ (or more generally, into a real space form $R^m(c)$ of constant sectional curvature c). According to J. F. Nash's embedding theorem [5], every Riemannian manifold can be isometrically immersed into some Euclidean space with sufficiently high codimension. The Nash's theorem was aimed for in the hope that, if Riemannian manifolds could always be regarded as Riemannian submanifolds, this would then yield the opportunity to use help from submanifold theory.

Based on Nash's theorem, one of the first author's research programs posted in Reference [6] is:

"To search for control of extrinsic quantities in relation to intrinsic quantities of Riemannian manifolds via Nash's theorem and to search for their applications".

Since Nash's embedding theorem implies that every warped product $N_1 \times_f N_2$ can always be regarded as a Riemannian submanifold in some Euclidean space, a special case of the research program posted in Reference [6] is to study the two following fundamental problems:

Problem 1.

$$\forall \;\; N_1 \times_f N_2 \xrightarrow[\text{immersion}]{\text{isometric}} \mathbb{E}^m \text{ or } R^m(c) \Longrightarrow \textbf{???}.$$

Problem 2. *Let $N_1 \times_f N_2$ be an arbitrary warped product isometrically immersed into $\mathbb{E}^m$ (or into $R^m(c)$) as a Riemannian submanifold. What are the relationships between the warping function f and the extrinsic structures of $N_1 \times_f N_2$?*

In the beginning of this century, the first author provided several solutions to these two fundamental problems in a series of his articles (see Reference [6–10]). For instance, he established in Reference [6,10] some sharp relationships between the Laplacian of the warping function and the squared mean curvature of warped product submanifolds $N_1 \times_f N_2$ in real space forms. As an immediate application, he proved that, if the warping function f of the warped product $N_1 \times_f N_2$ is harmonic, then there do not exist any isometric minimal immersion from $N_1 \times_f N_2$ into a hyperbolic space. Since then, there are many interesting results in warped products in this respect obtained by many authors.

The main purpose of this article is to provide a comprehensive survey on the study of warped product submanifolds which are closely related with this inequality mentioned in abstract, which have been done during the last two decades.

2. Preliminaries

We follow the notations from the books of References [2,11,12]. Let N be an n-dimensional submanifold of a Riemannian m-manifold $\widetilde{M}$. Denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of N and $\widetilde{M}$, respectively. We choose a local field of orthonormal frame $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ in $\widetilde{M}$ such that, restricted to N, the vectors $e_1, \ldots, e_n$ are tangent to N and $e_{n+1}, \ldots, e_m$ are normal to N.

The Gauss and Weingarten formulas are given, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1}$$

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2}$$

for any vector fields X, Y tangent to N and ξ normal to N, where h denotes the second fundamental form, D the normal connection, and A the shape operator of the submanifold. Let $\{h^r_{ij}\}$, $i, j = 1, \ldots, n$; $r = n+1, \ldots, m$, denote the coefficients of the second fundamental form h with respect to $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$.

The mean curvature vector $\overrightarrow{H}$ is defined by

$$\overrightarrow{H} = \frac{1}{n} \operatorname{trace} h = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i), \tag{3}$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of the tangent bundle TN of N. A submanifold N is said to be *minimal* in $\widetilde{M}$ if the mean curvature vector vanishes identically.

The squared mean curvature is given by $H^2 = \langle \overrightarrow{H}, \overrightarrow{H} \rangle$, where $\langle \;,\; \rangle$ is the inner product.

An isometric immersion $\psi : N \to \widetilde{M}$ between Riemannian manifolds is called *pseudo-umbilical* if its shape operator $A_{\overrightarrow{H}}$ at the mean curvature vector $\overrightarrow{H}$ satisfies $A_{\overrightarrow{H}} X = \lambda X$ for any vector field X tangent to N, where λ is a smooth function on N. Similarly, an immersion $\phi : N_1 \times_f N_2 \to \widetilde{M}$ is called N_2-*pseudo-umbilical* if its shape operator $A_{\overrightarrow{H}}$ satisfies $A_{\overrightarrow{H}} Z = \lambda Z$ for any vector field Z tangent to N_2.

Let R and $\widetilde{R}$ be the Riemann curvature tensor of N and $\widetilde{M}$, respectively. Then, the *equation of Gauss* is given by

$$R(X,Y;Z,W) = \widetilde{R}(X,Y;Z,W) + \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle \tag{4}$$

for vector fields X, Y, Z, W tangent to N. In particular, if the ambient space $\widetilde{M}$ is a Riemannian m-manifold $R^m(c)$ of constant sectional curvature c, then we have

$$\begin{aligned} R(X,Y;Z,W) = & \, c\,\{\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\} \\ & + \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle. \end{aligned} \tag{5}$$

For any n-dimensional submanifold N of a Riemannian manifold $\widetilde{M}$, Equation (4) of Gauss gives

$$2\tau = n^2H^2 - \|h\|^2 + \sum_{1\le i,j\le n} \widetilde{K}(e_i \wedge e_j), \tag{6}$$

where $\tau = \sum_{1\le i<j\le n} K(e_i \wedge e_j)$ is the scalar curvature of M, and K and $\widetilde{K}$ denote the sectional curvature of M and $\widetilde{M}$, respectively.

For a smooth function φ on N, the Laplacian of φ is defined by

$$\Delta\varphi = \sum_{j=1}^{n} \{(\nabla_{e_j} e_j)\varphi - e_j(e_j\varphi)\}. \tag{7}$$

If N is compact, then every eigenvalue of Δ is non-negative.

The ordinary warped product $N_1 \times_f N_2$ has been extended to multiply warped product $N_1 \times_{f_2} N_2 \times \cdots \times_{f_\ell} N_\ell$ in a natural way with the warping functions $f_2, \ldots, f_\ell$, $\ell \ge 2$, equipped with the multiply warped metric

$$g = g_1 + f_2^2 g_2 + \cdots + f_\ell^2 g_\ell, \tag{8}$$

where $f_2, \ldots, f_\ell$ are positive smooth functions on N_1, and $g_1, \ldots, g_\ell$ denote the Riemannian metrics of $N_1, \ldots, N_\ell$, respectively.

For a multiply warped product $N_1 \times_{f_2} N_2 \times \cdots \times_{f_\ell} N_\ell$, we denote by $\mathcal{D}_1, \ldots, \mathcal{D}_\ell$ the distributions given by the vector fields tangent to $N_1, \ldots, N_\ell$, respectively.

Remark 1. *Throughout this paper, for a warped product $N_1 \times_f N_2$, we denote the dimensions of N_1 and N_2 by n_1 and n_2, respectively, and the tangent bundles of N_1 and N_2 by $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.*

3. δ-Invariants and Basic Inequalities

Let N be an n-dimensional Riemannian manifold. Denote by $K(\pi)$ the sectional curvature associated with a 2-plane section $\pi \subset T_pN$, $p \in N$. For an r-dimensional subspace $L \subset T_pN$ with $r \ge 2$, the scalar curvature $\tau(L)$ of L is defined by

$$\tau(L) = \sum_{1\le\alpha<\beta\le r} K(e_\alpha \wedge e_\beta),$$

where $\{e_1, \ldots, e_r\}$ is an orthonormal basis of L. In particular, $\tau(p) = \tau(T_pN)$ is the *scalar curvature* of N at the point $p \in N$.

For an integer $k \ge 0$, we denote by $\mathcal{S}(n,k)$ the set consisting of unordered k-tuples $(n_1, \ldots, n_k)$ of integers ≥ 2 satisfying $n > n_1$ and $n_1 + \cdots + n_k \le n$. Let $\mathcal{S}(n)$ denote the set of unordered k-tuples with $k \ge 0$.

For each k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, the first author introduced the notion of δ-invariant $\delta(n_1, \ldots, n_k)(p)$ which is defined by (see Reference [13–15])

$$\delta(n_1, \ldots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \tag{9}$$

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_pN such that $\dim L_j = n_j$, $j = 1, \ldots, k$. In particular, we have

$$\delta(\emptyset) = \tau \quad \text{(the trivial } \delta\text{-invariant)},$$
$$\delta(2) = \tau - \inf K,$$
$$\delta(n-1)(p) = \max \mathrm{Ric}(p),$$

where K is the sectional curvature.

The non-trivial δ-invariants defined above are very different in nature from the "classical" scalar and Ricci curvatures, since scalar and Ricci curvatures are "total sum" of sectional curvatures on a Riemannian manifold. In contrast, the δ-invariants are obtained from the scalar curvature by deleting a certain amount of sectional curvatures.

Some other invariants of similar nature, i.e., invariants obtained from the scalar curvature by removing a certain amount of sectional curvatures, are also known as *δ-invariants*. For instance, one also has affine δ-invariants, warped product δ-invariant, submersion δ-invariant, etc. (see Reference [12]).

For δ-invariants, we have the following optimal universal inequalities for any Riemannian submanifold.

Theorem 1. *Refs [12,15]: For any isometric immersion of a Riemannian n-manifold N into a Riemannian m-manifold $\widetilde{M}$, we have:*

$$\delta(n_1, \ldots, n_k) \leq \frac{n^2(n + k - 1 - \sum_{j=1}^k n_j)}{2(n + k - \sum_{j=1}^k n_j)} H^2 + \frac{1}{2}\left\{ n(n-1) - \sum_{j=1}^k n_j(n_j - 1) \right\} \max \widetilde{K} \tag{10}$$

for each k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, where $\max \widetilde{K}(p)$ denotes the maximum of the sectional curvatures of $\widetilde{M}$ restricted to 2-plane sections of T_pN.

The equality case of (10) holds at a point $p \in N$ if and only if the following two conditions hold:

(1) *there exists an orthonormal basis $\{e_1, \ldots, e_m\}$ such that the shape operator A at p takes the form:*

$$A_{e_r} = \begin{pmatrix} \begin{matrix} A_1^r & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & A_k^r \end{matrix} & 0 \\ 0 & \mu_r I \end{pmatrix}, \quad r = n+1, \ldots, m, \tag{11}$$

where I is an identity matrix, and A_j^r is a symmetric $n_j \times n_j$ submatrix satisfying $\mathrm{trace}\,(A_1^r) = \cdots = \mathrm{trace}\,(A_k^r) = \mu_r$;

(2) *for any k mutual orthogonal subspaces $L_1, \ldots, L_k$ of T_pN satisfying*

$$\delta(n_1, \ldots, n_k) = \tau - \textstyle\sum_{j=1}^k \tau(L_j)$$

at p, we have $\widetilde{K}(e_{\alpha_i}, e_{\alpha_j}) = \max \widetilde{K}(p)$ for any $\alpha_i \in \Delta_i, \alpha_j \in \Delta_j$ with $1 \leq i \neq j \leq k+1$, where $\Delta_1, \ldots, \Delta_{k+1}$ are given by

$$\Delta_1 = \{1, \ldots, n_1\}, \ \ldots$$
$$\Delta_k = \{n_1 + \cdots + n_{k-1} + 1, \ldots, n_1 + \cdots + n_k\},$$
$$\Delta_{k+1} = \{n_1 + \cdots + n_k + 1, \ldots, n\}.$$

If the ambient space $\widetilde{M}^m$ is a Riemannian manifold $R^m(c)$ of constant sectional curvature c, then Theorem 1 reduces to:

Theorem 2. *Ref [15]: For any isometric immersion of a Riemannian n-manifold N into a Riemannian m-manifold $R^m(c)$ of constant sectional curvature c, we have:*

$$\delta(n_1,\ldots,n_k) \leq \frac{n^2(n+k-1-\sum_{j=1}^k n_j)}{2(n+k-\sum_{j=1}^k n_j)}H^2 + \frac{1}{2}\left\{n(n-1) - \sum_{j=1}^{k} n_j(n_j-1)\right\}c. \tag{12}$$

The equality case of (12) holds at a point $p \in N$ if and only if there exists an orthonormal basis $\{e_1,\ldots,e_m\}$ such that the shape operator A at p takes the form as in statement (1) of Theorem 1.

Remark 2. *For Lagrangian version of Theorem 2, see Reference [16,17].*

4. Warped Product Immersions

Let $\psi : N \to \widetilde{M}$ be an isometric immersion between two Riemannian manifolds and let f be a smooth function on $\widetilde{M}$. Denote by ∇f the gradient of f and by Df the normal component of ∇f restricted on N. Assume that $\widetilde{M} = M_1 \times_\rho M_2$ is a warped product and $\phi_i : N_i \to M_i$, $i = 1,2$, are isometric immersions between Riemannian manifolds. We define a positive function f on N_1 by $f = \rho \circ \phi_1$. Then, the map

$$\phi : N_1 \times_f N_2 \to M_1 \times_\rho M_2 \tag{13}$$

given by $\phi(x_1,x_2) = (\phi_1(x_1),\phi_2(x_2))$ is an isometric immersion, which is called a *warped product immersion* (see Reference [18,19]).

The first author proved the following results on warped product immersions in Reference [20].

Theorem 3. *Let $\phi = (\phi_1,\phi_2) : N_1 \times_f N_2 \to M_1 \times_\rho M_2$ be a warped product immersion between two warped product manifolds. Then, we have:*

(1) *ϕ is a mixed totally geodesic immersion;*
(2) *the squared norm of the second fundamental form of ϕ satisfies*

$$\|h\|^2 \geq n_2||D(\ln\rho)||^2 \tag{14}$$

with the equality holding if and only if $\phi_1 : N_1 \to M_1$ and $\phi_2 : N_2 \to M_2$ are both totally geodesic immersions;
(3) *ϕ is N_1-totally geodesic if and only if $\phi_1 : N_1 \to M_1$ is totally geodesic;*
(4) *ϕ is N_2-totally geodesic if and only if $\phi_2 : N_2 \to M_2$ is totally geodesic and $(\nabla(\ln\rho))|_{N_1} = \nabla(\ln f)$ holds, i.e., the restriction of the gradient of $\ln\rho$ to N_1 is the gradient of $\ln f$, or equivalently, $D(\ln\rho) = 0$;*
(5) *ϕ is a totally geodesic immersion if and only if ϕ is both N_1-totally geodesic and N_2-totally geodesic.*

Theorem 4. *A warped product immersion $\phi = (\phi_1,\phi_2) : N_1 \times_f N_2 \to M_1 \times_\rho M_2$ between two warped product manifolds is totally umbilical if and only if we have:*

(1) *$\phi_1 : N_1 \to M_1$ is a totally umbilical immersion with mean curvature vector given by $-D(\ln\rho)$, and*
(2) *$\phi_2 : N_2 \to M_2$ is a totally geodesic immersion.*

Theorem 5. *Let $\phi = (\phi_1,\phi_2) : N_1 \times_f N_2 \to M_1 \times_\rho M_2$ be a warped product immersion between two warped product manifolds. Then, we have:*

(1) *the partial mean curvature vector $\overrightarrow{H}_1$ is equal to the mean curvature vector of $\phi_1 : N_1 \to M_1$; thus, ϕ is N_1-minimal if and only if $\phi_1 : N_1 \to M_1$ is a minimal immersion;*

(2) *ϕ is N_2-minimal if and only if $\phi_2 : N_2 \to M_2$ is a minimal immersion and $(\nabla(\ln \rho))|_{N_1} = \nabla(\ln f)$ holds;*
(3) *ϕ is a minimal immersion if and only if $\phi_2 : N_2 \to M_2$ is a minimal immersion and the mean curvature vector of $\phi_1 : N_1 \to M_1$ is given by $(n_2/n_1)D(\ln \rho)$.*

Theorem 6. *Let $\phi = (\phi_1, \phi_2) : N_1 \times_f N_2 \to M_1 \times_\rho M_2$ be a warped product immersion from a warped product $N_1 \times_f N_2$ into a warped product representation $M_1 \times_\rho M_2$ of a real space form $R^m(c)$. Then, we have:*

(1) *the shape operator of ϕ satisfies*

$$A_{\overrightarrow{H}_1} Z = \left(\frac{\Delta f}{n_1 f} - c \right) Z \tag{15}$$

for Z in $\mathcal{D}_2$, where Δ is the Laplacian on N_1;
(2) *for any $X, Y \in \mathcal{D}_1$ and $Z \in \mathcal{D}_2$, $D_Z h(X, Y) = 0$ holds, where D is the normal connection of ϕ. In particular, we have $D_Z \overrightarrow{H}_1 = 0$;*
(3) *the two partial mean curvature vectors $\overrightarrow{H}_1$ and $\overrightarrow{H}_2$ are orthogonal to each other if and only if the warping function f is an eigenfunction of the Laplacian operator Δ with eigenvalue $n_1 c$;*
(4) *the warping function f is an eigenfunction of Δ with eigenvalue $n_1 c$ if and only if either $\phi_1 : N_1 \to M_1$ is a minimal immersion or $(\nabla(\ln \rho))|_{N_1} = \nabla(\ln f)$ holds;*
(5) *when $c = 0$, the two partial mean curvature vectors $\overrightarrow{H}_1$ and $\overrightarrow{H}_2$ are orthogonal to each other if and only if the warping function f is a harmonic function;*
(6) *if $\phi_1 : N_1 \to M_1$ is a non-minimal immersion and the two partial mean curvature vectors $\overrightarrow{H}_1$ and $\overrightarrow{H}_2$ are parallel at each point, then ϕ is N_2-pseudo-umbilical and $\phi_2 : N_2 \to M_2$ is a minimal immersion.*

5. The First Solutions to Problems 1 and 2

An isometric immersion of a warped product manifold $N_1 \times_f N_2$ into a Riemannian manifold is called *mixed totally geodesic* if its second fundamental form h satisfies $h(X, Z) = 0$ for any vector fields X tangent to N_1 and Z tangent to N_2.

For orthonormal bases $\{e_1, \ldots, e_{n_1}\}$ and $\{e_{n_1+1}, \ldots, e_{n_1+n_2}\}$ of N_1 and N_2, the *partial traces of h* restricted to N_1 and N_2 are defined, respectively, by

$$\operatorname{trace} h_1 = \sum_{i=1}^{n_1} h(e_i, e_i), \quad \operatorname{trace} h_2 = \sum_{j=n_1+1}^{n_1+n_2} h(e_j, e_j).$$

The notions of mixed totally geodesic warped product submanifolds and partial traces of the second fundamental form can be extended to multiply warped product submanifolds $N_1 \times_{f_2} N_2 \times \cdots \times_{f_\ell} N_\ell$ in a Riemannian manifold in a natural way.

5.1. The First Solutions

The first solution to Problems 1 and 2 is given by the following.

Theorem 7. *Ref [6]: Let $\phi : N_1 \times_f N_2 \to R^m(c)$ be an isometric immersion of a warped product into a Riemannian m-manifold of constant sectional curvature c. Then, we have:*

$$\frac{\Delta f}{f} \leq \frac{(n_1 + n_2)^2}{4n_2} H^2 + n_1 c, \tag{16}$$

where H^2 is the squared mean curvature of ϕ and Δ denotes the Laplacian on N_1.

The equality case of (16) holds identically if and only if $\phi : N_1 \times_f N_2 \to R^m(c)$ is a mixed totally geodesic immersion satisfying trace h_1 = trace h_2, *where h_1 and h_2 denote the restrictions of h to N_1 and N_2, respectively.*

Remark 3. *The proof of Theorem 7 given in Reference [6] relied on detailed investigation of the warped product δ-invariant $\delta_{N_1 \times_f N_2}$ defined by*

$$\delta_{N_1 \times_f N_2} = \tau(N_1 \times_f N_2) - \tau(N_1) - \tau(N_2)$$

for the warped product $N_1 \times_f N_2$.

In terms of warped product immersions, Theorem 7 can be restated as the following.

Theorem 8. *Ref [20]: Let $\phi : N_1 \times_f N_2 \to R^m(c)$ be an isometric immersion of a warped product into a Riemannian m-manifold of constant sectional curvature c. Then, we have:*

$$\frac{\Delta f}{f} \leq \frac{(n_1 + n_2)^2}{4n_2} H^2 + n_1 c. \tag{17}$$

The equality case of (17) holds identically if and only if exactly one of the following two cases occurs:

(1) *the warping function f is an eigenfunction of the Laplacian operator Δ with eigenvalue $n_1 c$ and ϕ is a minimal immersion;*
(2) *$\Delta f \neq (n_1 c) f$ and locally ϕ is a non-minimal warped product immersion $(\phi_1, \phi_2) : N_1 \times_f N_2 \to M_1 \times_\rho M_2$ of $N_1 \times_f N_2$ into some warped product representation $M_1 \times_\rho M_2$ of $R^m(c)$ such that $\phi_2 : N_2 \to M_2$ is a minimal immersion and the mean curvature vector of $\phi_1 : N_1 \to M_1$ is given by $-(n_2/n_1) D(\ln \rho)$.*

There are examples which satisfy either case (1) or case (2) of Theorem 8 for $c = 0$, $c > 0$ and $c < 0$. For instance, the following examples are given in Reference [20].

Example 1. There exist many minimal isometric immersions from some warped products $N_1 \times_f N_2$ with harmonic warping function f into a Euclidean space. For instance, if N_2 is a minimal submanifold of the unit $(m-1)$-hypersphere S^{m-1} in $\mathbb{E}^m$ centered at the origin o, then the minimal cone $C(N_2)$ over N_2 with vertex at $o \in \mathbb{E}^m$ is the warped product $\mathbb{R}_+ \times_s N_2$ with warping function $f = s$, which is a harmonic function. Here, s is the coordinate function of the positive real line $\mathbb{R}_+$. This example provides many isometric immersions of warped products in a real space form which satisfy the case (1) of Theorem 8.

Example 2. Let S^{2n_1} be the unit $2n_1$-sphere equipped with the metric:

$$g = du_1^2 + \cos^2 u_1 du_2^2 + \cdots + \prod_{k=1}^{2n_1 - 1} \cos^2 u_k \, du_{2n_1}^2. \tag{18}$$

If we put

$$g_1 = du_1^2 + \cos^2 u_1 du_2^2 + \cdots + \prod_{k=1}^{n_1 - 1} \cos^2 u_k \, du_{n_1}^2,$$
$$g_2 = du_{n_1+1}^2 + \cos^2 u_{n_1+1} du_{n_1+2}^2 + \cdots + \prod_{k=n_1+1}^{2n_1 - 1} \cos^2 u_k \, du_{2n_1}^2,$$

then S^{2n_1} is locally isometric to $N_1 \times_f N_2$, where $f = \cos u_1 \cdots \cos u_{n_1}$, $N_1 = (S^{n_1}, g_1)$ and $N_2 = (S^{n_1}, g_2)$. Further, the warping function f satisfies $\Delta f = n_1 f$.

Let $\phi : N_1 \times_f N_2 \to \mathbb{E}^{2n_1+1}$ be the inclusion of S^{2n_1} in $\mathbb{E}^{2n_1+1}$. Then, we have $H^2 = 1$. Thus, we obtain the equality case of (17). Since ϕ is non-minimal, Theorem 8 shows that ϕ satisfies case (2) of Theorem 8.

Example 3. Let $N_1 \times_f N_2$ denote the warped product representation of the unit $2n_1$-sphere S^{2n_1} with $f = \cos u_1 \cdots \cos u_{n_1}$, $N_1 = (S^{n_1}, g_1)$ and $N_2 = (S^{n_1}, g_2)$ given as in Example 2. Let us consider a totally umbilical immersion:

$$\phi : N_1 \times_f N_2 \to H^{2n_1+1}(c), \quad c < 0.$$

Then, $H^2 = 1 - c$. Since $\Delta f = n_1 f$, the equality case of (17) holds. Since ϕ is a non-minimal immersion, $\phi : N_1 \times_f N_2 \to H^{2n_1+1}(c)$ satisfies the case (2) of Theorem 8.

Example 4. Let $N_1 \times_f N_2$ denote the same warped product representation of S^{2n_1} as given in Examples 3 and 4. Let us consider a totally umbilical immersion:

$$\phi : N_1 \times_f N_2 \to S^{2n_1+1}(c), \quad c < 1.$$

Then, $H^2 = 1 - c$. Since $\Delta f = n_1 f$, the equality case of (17) holds. Now, it is easy to verify that $\phi : N_1 \times_f N_2 \to S^{2n_1+1}(c)$ satisfies the case (2) for $0 < c < 1$.

5.2. Some Early Extensions of Theorem 7

Theorem 7 was extended to the following.

Theorem 9. *Refs [21–23]: Let $\phi : N_1 \times_{f_2} N_2 \times \cdots \times_{f_\ell} N_\ell \to \widetilde{M}$ be an isometric immersion of a multiply warped product $N = N_1 \times_{f_2} N_2 \times \cdots \times_{f_\ell} N_\ell$ into an arbitrary Riemannian manifold $\widetilde{M}$, where $f_2, \ldots, f_\ell$ are positive smooth functions on N_1. Then, we have:*

$$\sum_{j=2}^{\ell} n_j \frac{\Delta f_j}{f_j} \leq \frac{n^2(\ell-1)}{2\ell} H^2 + n_1(n - n_1) \max \widetilde{K}, \tag{19}$$

where $n = \sum_{j=1}^{\ell} n_j$ and $\max \widetilde{K}(p)$ denotes the maximum of the sectional curvature $\widetilde{K}$ of $\widetilde{M}$ restricted to plane sections in T_pN at $p \in N$.

The equality case of (19) holds identically if and only if the following two conditions hold:

(1) *ϕ is a mixed totally geodesic immersion satisfying $\operatorname{trace} \sigma_1 = \cdots = \operatorname{trace} \sigma_\ell$;*
(2) *at each point $p \in N$, we have $\widetilde{K}(u, v) = \max \widetilde{K}(p)$, for any unit vector $u \in T_{p_1}N_1$ and unit vector $v \in T_{(p_2,\ldots,p_\ell)}(N_2 \times \cdots \times N_\ell)$.*

This theorem was proved by modifying the proof of Theorem 7. In particular, if $\ell = 2$, Theorem 9 reduces to

Theorem 10. *Refs [21–23]: Let $\phi : N_1 \times_f N_2 \to \widetilde{M}$ be an isometric immersion of a warped product $N = N_1 \times_f N_2$ into an arbitrary Riemannian manifold $\widetilde{M}$. Then, we have:*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2} H^2 + n_1 \max \widetilde{K}, \tag{20}$$

where $\max \widetilde{K}(p)$ denotes the maximum of the sectional curvature $\widetilde{K}$ of $\widetilde{M}$ restricted to plane sections in T_pN at $p \in N$.

The equality case of (20) holds identically if and only if the following two conditions hold:

(1) *ϕ is a mixed totally geodesic immersion satisfying $\operatorname{trace} \sigma_1 = \operatorname{trace} \sigma_2$;*
(2) *at each point $p \in N$, we have $\widetilde{K}(u, v) = \max \widetilde{K}(p)$ for any unit vector $u \in T_{p_1}N_1$ and unit vector $v \in T_{p_2}N_2$.*

The next result was obtain by B. D. Suceavă and M. B. Vajiac in Reference [24].

Theorem 11. *Let $\phi : N_1 \times_f N_2 \to \widetilde{M}$ be an isometric immersion of a warped product $N = N_1 \times_f N_2$ into an arbitrary Riemannian manifold $\widetilde{M}$. Then, at each point $p \in N_1 \times_f N_2$, the following inequality holds:*

$$n_2 \frac{\Delta f}{f} + scal(T_p N_1) + scal(T_p N_2) \leq \frac{n(n-1)}{2} \|H\|^2 + \sum_{1 \leq i < j \leq n} \widetilde{K}(e_i \wedge e_j), \quad n = n_1 + n_2,$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $N_1 \times_f N_2$ at p, and scal denotes the scalar curvature corresponding to the indicated tangent space with respect to the warped product metric.

Equality holds at a point p if and only if p is a umbilical point.

The proof of this theorem is based on the method used in Reference [25]. For some further results on warped product submanifolds, also see Reference [26].

5.3. Several Direct Applications of Theorem 7

The following are some very easy applications of Theorems 7 and 9 (see Reference [2,6]).

Corollary 1. *If $N_1 \times_f N_2$ is a warped product of Riemannian manifolds in which warping function f is a harmonic function, then we have:*

(1) *$N_1 \times_f N_2$ admits no isometric minimal immersion into any Riemannian manifold of negative sectional curvature;*
(2) *every isometric minimal immersion from $N_1 \times_f N_2$ into a Euclidean space is a warped product immersion.*

Corollary 2. *Let f be an eigenfunction of the Laplacian Δ on N_1 with positive eigenvalue λ. Then, every Riemannian warped product $N_1 \times_f N_2$ does not admit any isometric minimal immersion into any Riemannian manifold of non-positive sectional curvature.*

Corollary 3. *Let N_1 be a compact manifold. Then,*

(1) *every Riemannian warped product $N_1 \times_f N_2$ does not admit an isometric minimal immersion into any Riemannian manifold of negative sectional curvature;*
(2) *every Riemannian warped product $N_1 \times_f N_2$ does not admit an isometric minimal immersion into a Euclidean space.*

Example 5. *There exist many minimal immersions of a warped product $N_1 \times_f N_2$ with harmonic warping function f into a Euclidean space. For instance, if N_2 is a minimal submanifold of the unit $(m-1)$-hypersphere $S^{m-1}(1) \subset \mathbb{E}^m$ centered at the origin, then the minimal cone $C(N_2)$ over N_2 with vertex at the origin of $\mathbb{E}^m$ is a warped product $\mathbb{R}_+ \times_s N_2$ in which warping function $f = s$ is a harmonic function. Here, s is the coordinate function of the positive real line $\mathbb{R}_+$. This provides many examples of minimal warped products in $\mathbb{E}^m$ which satisfy the equality case of (16).*

Example 5 implies that Theorem 7 and Corollary 1 are optimal. Examples 10.2, 10.3, and 10.4 of Reference [2] showed that Corollaries 2 and 3 are optimal, as well.

5.4. Growth Estimates for Warping Functions of Warped Products

Let N_1 be a complete non-compact Riemannian manifold. A function f on N_1 is called an *L^p-function* if

$$\|f\|_{L^p} := \left(\int_{N_1} |f|^p dv \right)^{1/p}$$

converges.

From Theorem 6.2 and Remark 8 of Reference [23], we know that, if f is an L^p-function on N_1 for some $p > 1$, then, for any Riemannian manifold N_2, the warped product $N_1 \times_f N_2$ does not admit any isometric minimal immersion into any Riemannian manifold with non-positive sectional curvature.

S. W. Wei, J. Li, and L. Wu [27] extended the scope of L^p or p-integrable functions on complete non-compact Riemannian manifolds by generalizing them, for each given $q > 1$, to "p-finite, p-mild, p-obtuse, p-moderate and p-small" functions that depend on p and introducing the concepts of their counterparts "p-infinite, p-severe, p-acute, p-immoderate and p-large" growth.

For instance, if N is a complete non-compact Riemannian manifold and $B(x_0; r)$ is the geodesic ball of radius r centered at $x_0 \in N$, then, for each $q > 1$, a function f on N is said to have *p-finite growth* (or, simply, is *p-finite*) if there exists $x_0 \in N$ such that

$$\lim_{r\to\infty} \frac{1}{r^p} \int_{B(x_0;r)} |f|^q dv < \infty,$$

and f has p-infinite growth (or, simply, is p-infinite) otherwise.

The first author and S. W. Wei discovered in Reference [23] some dichotomy between constancy and "infinity" of the warping functions on complete non-compact Riemannian manifolds for an appropriate isometric immersion. For instance, they have applied Theorem 9 to prove the following result in Reference [23].

Theorem 12. *Suppose $q > 1$ and that the warping function f of $N_1 \times_f N_2$ is one of the following: 2-finite, 2-mild, 2-obtuse, 2-moderate and 2-small. If N_2 is compact, then there does not exist an isometric minimal immersion from $N_1 \times_f N_2$ into any Euclidean space.*

For further results in this respect, see Reference [23,28,29].

6. Another Early Solution to Problems 1 and 2

Besides Theorems 7–10, there is another solution to Problems 1 and 2 obtained in Reference [10] for a warped product in a real space form.

Theorem 13. *For any isometric immersion $\phi : N_1 \times_f N_2 \to R^m(c)$, the scalar curvature τ of the warped product $N_1 \times_f N_2$ satisfies*

$$\tau \le \frac{\Delta f}{n_1 f} + \frac{n^2(n-2)}{2(n-1)} H^2 + \frac{1}{2}(n+1)(n-2)c. \tag{21}$$

If $n = 2$, the equality case of (21) holds automatically.

If $n \ge 3$, the equality case of (21) holds identically if and only if one of the following two statement occurs:

(1) *$N_1 \times_f N_2$ is of constant sectional curvature c, the warping function f is an eigenfunction with eigenvalue c, i.e., $\Delta f = cf$, and $N_1 \times_f N_2$ is immersed as a totally geodesic submanifold in $R^m(c)$;*
(2) *locally, $N_1 \times_f N_2$ is immersed as a rotational hypersurface into a totally geodesic submanifold $R^{n+1}(c)$ of $R^m(c)$ with a geodesic of $R^{n+1}(c)$ as its profile curve.*

By applying the method given in the proof of Theorem 9 and using (6), Theorem 7 was extended in Reference [30] to the following.

Theorem 14. *For any isometric immersion $\phi : N_1 \times_f N_2 \to \widetilde{M}$ of $N_1 \times_f N_2$ into a Riemannian manifold $\widetilde{M}$, we have*

$$\frac{\Delta f}{f} \ge \frac{n_1 n^2}{2(n-1)} H^2 - \frac{n_1}{2} \|h\|^2 + n_1 \min \widetilde{K}. \tag{22}$$

Several applications of Theorem 14 were given in Reference [30].

Example 6. Any Riemannian manifold of constant sectional curvature c can be locally expressed as a warped product in which warping function satisfies $\Delta f = cf$, e.g., the unit n-sphere $S^n(1)$ is locally isometric to $(0,\infty)\times_{\cos x} S^{n-1}(1)$; the Euclidean n-space $\mathbb{E}^n$ is locally isometric to $(0,\infty)\times_x S^{n-1}(1)$; the unit hyperbolic n-space $H^n(-1)$ is locally isometric to $\mathbb{R}\times_{e^x}\mathbb{E}^{n-1}$. Besides these, there exist other warped product decompositions of real space forms $R^n(c)$ of constant sectional curvature c in which warping function satisfies $\Delta f = cf$.

For example, let $\{x_1,\ldots,x_{n_1}\}$ be a Euclidean coordinate system of a Euclidean n_1-space $\mathbb{E}^{n_1}$ and let

$$f=\sum_{j=1}^{n_1} a_j x_j + b,$$

where $a_1,\ldots,a_{n_1},b$ are real numbers satisfying $\sum_{j=1}^{n_1} a_j^2 = 1$. Then, the warped product $\mathbb{E}^{n_1}\times_f S^{n_2}(1)$ is a flat space in which warping function is a harmonic function. In fact, those are the only warped product decompositions of flat spaces in which warping functions are harmonic functions.

7. Geometric Inequalities for Warped Products in Spaces of Quasi-Constant Curvature

In this section, we present some extensions of Theorem 7 to warped product submanifolds in spaces of quasi-constant curvature.

7.1. Spaces of Quasi-Constant Curvature

The notion of Riemannian manifolds of quasi-constant curvature was given in Reference [31]; namely, a Riemannian m-manifold $(\widetilde{M},g)$ is said to be of *quasi-constant curvature* if there exist a unit vector field G, called the *generator*, and two smooth functions κ,μ on $\widetilde{M}$ such that the Riemann curvature tensor $\widetilde{R}$ of $(\widetilde{M},g)$ satisfies

$$\begin{aligned}\widetilde{R}(X,Y)Z = \kappa\{g(Y,Z)X-g(X,Z)Y\}+\mu\{g(Y,Z)\zeta(X)G\\ -g(X,Z)\zeta(Y)G+\zeta(Y)\zeta(Z)X-\zeta(X)\zeta(Z)Y\},\end{aligned} \tag{23}$$

for any vector fields X,Y,Z tangent to $\widetilde{M}$, where ζ is the 1-form dual to G. We simply denote such a Riemannian manifold by $\widetilde{M}^m_{\kappa,\mu}(G)$.

It is known that every Riemannian m-manifold of quasi-constant curvature with $\kappa\neq constant$ is a warped product of the form $I\times_f W^{m-1}$, where W^{m-1} is a space of constant sectional curvature (see Reference [32,33]).

A remarkable class of Riemannian manifolds of quasi-constant curvature is the class of subprojective Riemannian manifolds. By definition, a Riemannian m-manifold $\widetilde{M}$ of dimension $m\geq 4$ is called *subprojective* if it is conformally flat and its Cotton tensor L satisfies (see Reference [34–36]):

$$L=\alpha g+\beta(d\alpha)\otimes(d\alpha) \tag{24}$$

for some functions α and $\beta=\beta(\alpha)$.

It is known from Reference [33] that a Riemannian manifold $(\widetilde{M},g)$ is subprojective if and only if it is a space of quasi-constant curvature such that the 1-form ζ in (23) is closed. For further results on subprojetive spaces, see Reference [33,34,36], among some others.

7.2. Warped Product Submanifolds of Spaces of Quasi-Constant Curvature

S. Sular extended Theorem 7 to warped products in spaces of quasi-constant curvature as follows.

Theorem 15. *Ref [37]: Let $\phi: N_1\times_f N_2\to \widetilde{M}^m_{\kappa,\mu}(G)$ be an isometric immersion of a warped product into a Riemannian manifold $\widetilde{M}^m_{\kappa,\mu}(G)$ of quasi-constant curvature. Then, we have:*

$$\frac{\Delta f}{f}\leq \frac{n^2}{4n_2}H^2+n_1\kappa-\frac{\mu}{n_2}\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n}\{\zeta(e_i)^2+\zeta(e_j)^2\}+\frac{\mu}{n_2}(n-1)\|G\|^2, \tag{25}$$

where $\{e_1, \ldots, e_{n_1}\}$ *and* $\{e_{n_1+1}, \ldots, e_n\}$ *are orthonormal frames of* TN_1 *and* TN_2*, respectively.*

The equality case of (25) *holds if and only if* ϕ *is a mixed totally geodesic immersion satisfying* $\operatorname{trace} h_1 = \operatorname{trace} h_2$*, where* h_1 *and* h_2 *denote the restrictions of* h *to* N_1 *and* N_2*, respectively.*

7.3. Warped Product Submanifolds of Spaces of Nearly Quasi-Constant Curvature

In 2009, U. C. De and A. K. Gazi [38] introduced the notion of a Riemannian manifold $(\widetilde{M}, g)$ of *nearly quasi-constant curvature* as a Riemannian manifold with the curvature tensor satisfying the condition:

$$\begin{aligned} \widetilde{R}(X,Y;Z,W) = &\kappa\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} \\ &+ \mu\{g(Y,Z)B(X,W) - g(Y,W)B(X,Z) \\ &+ g(X,W)B(Y,Z) - g(X,Z)B(Y,W)\} \end{aligned} \tag{26}$$

for vector fields X, Y, Z, W tangent to $\widetilde{M}$, where B is a nonzero symmetric $(0,2)$-tensor field.

A non-flat Riemannian m-manifold $(\widetilde{M}, g)$ $(m \geq 3)$ defines a *nearly quasi-Einstein manifold* if its Ricci tensor satisfies the condition [38]

$$\mathrm{Ric} = cg + dE,$$

where c and d are nonzero scalar functions, and E is a nonzero symmetric $(0,2)$-tensor field.

The following example of spaces of nearly quasi-constant curvature was given by U. C. De and A. K. Gazi in Reference [38].

Example 7. *Let* $(\widetilde{M}^4, g)$ *be a Riemannian manifold endowed with the metric given by*

$$g = (x_4)^{\frac{4}{3}}\left[(dx_1)^2 + (dx_2)^2 + (dx_3)^2\right] + (dx_4)^2.$$

Then, $(\widetilde{M}^4, g)$ *is a Riemannian manifold of nearly quasi-constant curvature with nonzero and non-constant scalar curvature which is not a quasi-Einstein manifold.*

The following result was proved by P. Zhang in Reference [39].

Theorem 16. *Let* $\phi : N_1 \times_f N_2 \to \widetilde{M}$ *be an isometric immersion of a warped product into a Riemannian manifold* $\widetilde{M}$ *of nearly quasi-constant curvature. Then, we have:*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}H^2 + n_1\kappa + \frac{\mu}{n_2}(n_2 \operatorname{trace} B_1 + n_1 \operatorname{trace} B_2), \tag{27}$$

where B_1 *and* B_2 *denote the restrictions of* B *to* N_1 *and* N_2*, respectively.*

The equality case of (27) *holds if and only if* ϕ *is a mixed totally geodesic immersion satisfying* $\operatorname{trace} h_1 = \operatorname{trace} h_2$.

8. Geometric Inequalities for Warped Products in Almost Hermitian Manifolds

An *almost Hermitian manifold* is an even-dimensional Riemannian $2m$-manifold $(\widetilde{M}^{2m}, g)$ such that there exists a $(1,1)$-tensor field J on $\widetilde{M}^{2m}$ which satisfies

$$J^2 = -I, \quad g(JX, JY) = g(X, Y),$$

for any vector fields X, Y tangent to $\widetilde{M}^{2m}$.

8.1. Warped Products in Complex Space Forms

For warped products in complex hyperbolic spaces, we have the following result from Reference [40].

Theorem 17. *Let $\phi : N_1 \times_f N_2 \to CH^m(-4c)$ $(c > 0)$ be an isometric immersion of a warped product $N_1 \times_f N_2$ into the complex hyperbolic m-space $CH^m(-4c)$ of constant holomorphic sectional curvature $-4c$. Then, we have:*

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2} H^2 - n_1 c. \tag{28}$$

The equality case of (28) *holds if and only if ϕ is a mixed totally geodesic immersion satisfying* $\operatorname{trace} h_1 = \operatorname{trace} h_2$, *and $J\mathcal{D}_1 \perp \mathcal{D}_2$, where J is the almost complex structure of $CH^m(-4c)$.*

By applying Theorem 17, we have the next three corollaries from Reference [40].

Corollary 4. *Let $N_1 \times_f N_2$ be a Riemannian warped product in which warping function f is harmonic. Then, $N_1 \times_f N_2$ does not admit any isometric minimal immersion into any complex hyperbolic space.*

Corollary 5. *If f is an eigenfunction of the Laplacian on N_1 with eigenvalue $\lambda > 0$, then $N_1 \times_f N_2$ does not admit an isometric minimal immersion into any complex hyperbolic space.*

Corollary 6. *If N_1 is compact, then every Riemannian warped product $N_1 \times_f N_2$ does not admit an isometric minimal immersion into any complex hyperbolic space.*

For warped product submanifolds in a complex space form, A. Mihai proved the following.

Theorem 18. *Ref [41]: Let $\phi : N_1 \times_f N_2 \to \widetilde{M}(4c)$ be an isometric immersion of a warped product $N_1 \times_f N_2$ into the complex space form $\widetilde{M}(4c)$ of constant holomorphic sectional curvature $4c$ with $J\mathcal{D}_1 \perp \mathcal{D}_2$. Then, we have:*

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2} H^2 + n_1 c. \tag{29}$$

The equality case of (29) *holds identically if and only if ϕ is a mixed totally geodesic immersion satisfying* $\operatorname{trace} h_1 = \operatorname{trace} h_2$.

For warped product submanifolds in the complex projective m-space $CP^m(4)$, we also have the following result.

Theorem 19. *Ref [9]: Let $\phi : N_1 \times_f N_2 \to CP^m(4)$ be an isometric immersion of a warped product into the complex projective m-space $CP^m(4)$. Then, we have:*

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2} H^2 + 3 + n_1. \tag{30}$$

The equality case of (30) *holds identically if and only if the following three conditions hold:*

(1) $n_1 = n_2 = 1$,
(2) *f is an eigenfunction of the Laplacian on N_1 with eigenvalue 4, and*
(3) *ϕ is a totally geodesic and holomorphic immersion.*

Theorem 19 implies the following result.

Corollary 7. *If f is a positive smooth function on a Riemannian n_1-manifold N_1 such that $\Delta f > (3 + n_1) f$ at a point $p \in N_1$, then, for any Riemannian manifold N_2, the warped product $N_1 \times_f N_2$ does not admit any minimal immersion into $CP^m(4)$ for any m.*

A submanifold N^n of an almost Hermitian manifold $(\widetilde{M}^m, J, g)$ is called *totally real* if it satisfies $J(T_p N^n) \subset T_p^\perp N^n$, where $T_p^\perp N^n$ denotes the normal space of N^n at a point $p \in N^n$. In particular, a totally real submanifold N^n in $\widetilde{M}^m$ is called a *Lagrangian submanifold* if $\dim_{\mathbb{R}} N^n = \dim_{\mathbb{C}} \widetilde{M}^m$ (see, e.g., Reference [42,43]).

A submanifold N of an almost Hermitian manifold $\widetilde{M}$ is called a *CR-submanifold* [44,45] if there is a holomorphic distribution $\mathcal{H}$ on N in which orthogonal complement $\mathcal{H}^\perp$ is a totally real distribution, i.e., $J\mathcal{H}_p^\perp \subset T_p^\perp N$. A CR-submanifold N is called *anti-holomorphic* if $J\mathcal{H}_p^\perp = T_p^\perp N$.

For totally real submanifolds in a complex projective m-space $CP^m(4)$, Theorem 16 was sharpen in Reference [2] (Theorem 10.7) as follows.

Theorem 20. *Let $\phi : N_1 \times_f N_2 \to CP^m(4)$ be a totally real immersion of a warped product into $CP^m(4)$. Then, we have:*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2} H^2 + n_1. \tag{31}$$

The equality case of (31) holds identically if and only if ϕ is a mixed totally geodesic immersion satisfying $\operatorname{trace} h_1 = \operatorname{trace} h_2$.

Theorem 20 implies the following.

Corollary 8. *If f is a positive smooth function on a Riemannian n_1-manifold N_1 such that $\Delta f > n_1 f$ at a point $p \in N_1$, then, for any Riemannian manifold N_2, the warped product $N_1 \times_f N_2$ does not admit any totally real minimal immersion into $CP^m(4)$ for any m.*

8.2. Warped Products in Generalized Complex Space Forms

An almost Hermitian manifold $(\widetilde{M}, J, g)$ is called an *RK-manifold* if its curvature tensor $\widetilde{R}$ is invariant under the action of J, i.e.,

$$\widetilde{R}(JX, JY; JZ, JW) = \widetilde{R}(X, Y; Z, W), \tag{32}$$

for any vector fields X, Y, Z, W tangent to $\widetilde{M}$. An almost Hermitian manifold $(\widetilde{M}, J, g)$ is said to be *of pointwise constant type* if, for any $x \in \widetilde{M}$ and $X \in T_x\widetilde{M}$, we have

$$\lambda(X, Y) = \lambda(X, Z), \tag{33}$$

with

$$\lambda(X, Y) = \widetilde{R}(X, Y; JX, JY) - \widetilde{R}(X, Y; X, Y),$$

whenever the planes defined by X, Y and X, Z are totally real and with $g(Y, Y) = g(Z, Z)$.

An almost Hermitian manifold $\widetilde{M}$ is said to be *of constant type* if, for any unit vector fields X, Y on $\widetilde{M}$ with $\langle X, Y\rangle = \langle JX, Y\rangle = 0$, $\lambda(X, Y)$ is a constant function.

A *generalized complex space form* is an RK-manifold of constant holomorphic sectional curvature and of constant type. Every complex space form is obviously a generalized complex space form, but the converse is not true. And the 6-sphere S^6 endowed with the standard nearly Kaehler structure is known to be an example of generalized complex space form which is not a complex space form.

In the following, we denote by $\widetilde{M}(c, \alpha)$ a generalized complex space form of constant holomorphic sectional curvature c and constant type α. The Riemann curvature tensor $\widetilde{R}$ of $\widetilde{M}(c, \alpha)$ has the following expression (see Reference [46]):

$$\begin{aligned} \widetilde{R}(X, Y)Z = &\frac{c + 3\alpha}{4}\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\} \\ &+ \frac{c - \alpha}{4}\{\langle X, JZ\rangle JY - \langle Y, JZ\rangle JX + 2\langle X, JY\rangle JZ\}. \end{aligned} \tag{34}$$

For a submanifold N of an almost Hermitian manifold $(\widetilde{M}, J, g)$ and for a vector $X \in TN$, we put

$$JX = TX + FX, \tag{35}$$

where TX and FX denote the tangential and the normal components of JX.

For warped products in a generalized complex space form, A. Mihai obtained the following result.

Theorem 21. *Ref [47]: Let $\phi : N_1 \times_f N_2 \to \widetilde{M}(c, \alpha)$ be an isometric immersion of a warped product into a generalized complex space form. Then, we have:*

(1) *If $c < \alpha$, then*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2} H^2 + n_1 \frac{c + 3\alpha}{4}. \tag{36}$$

The equality case of (36) holds identically if and only if ϕ is a mixed totally geodesic immersion satisfying $\operatorname{trace} h_1 = \operatorname{trace} h_2$, and $J\mathcal{D}_1 \perp \mathcal{D}_2$.

(2) *If $c = \alpha$, then*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2} H^2 + n_1 \frac{c + 3\alpha}{4}. \tag{37}$$

The equality case of (37) holds identically if and only if ϕ is a mixed totally geodesic immersion satisfying $\operatorname{trace} h_1 = \operatorname{trace} h_2$.

(3) *If $c > \alpha$, then*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2} H^2 + n_1 \frac{c + 3\alpha}{4} + 3\frac{c - \alpha}{8} \|T\|^2, \tag{38}$$

where T is defined by (35).

The equality case of (38) holds identically if and only if ϕ is a mixed totally geodesic immersion satisfying $\operatorname{trace} h_1 = \operatorname{trace} h_2$, and N_1, N_2 are both totally real submanifolds.

As applications of Theorem 21, we have the following non-existence results.

Corollary 9. *Let $\widetilde{M}(c, \alpha)$ be a generalized complex space form, N_1 an n_1-dimensional Riemannian manifold and f a smooth function on N_1. If there is a point $p \in N_1$ such that $(\Delta f)(p) > \frac{c+3\alpha}{4} n_1 f(p)$, then there do not exist any minimal CR-warped product submanifold $N_1 \times_f N_2$ into $\widetilde{M}(c, \alpha)$.*

Corollary 10. *Let $\widetilde{M}(c, \alpha)$ be a generalized complex space form, with $c > \alpha$, N_1 an n_1-dimensional totally real submanifold of $\widetilde{M}(c, \alpha)$ and f a smooth function on N_1. If there is a point $p \in N_1$ such that $(\Delta f)(p) > \frac{c+3\alpha}{4} n_1 f(p)$, then there do not exist any totally real submanifold N_2 in $\widetilde{M}(c, \alpha)$ such that $N_1 \times_f N_2$ be a minimal warped product submanifold into $\widetilde{M}(c, \alpha)$.*

8.3. Warped Products in Locally Conformal Kaehler Space Forms

A *locally conformally Kaehler manifold* $(\widetilde{M}, J, g)$ is a Hermitian manifold which is locally conformal to a Kaehler manifold. This is equivalently to say that there is an open cover $\{U_i\}_{i \in I}$ of $\widetilde{M}$ and a family $\{f_i\}_{i \in I}$ of smooth functions $f_i : U_i \to \mathbb{R}$ such that $g_i = e^{-f_i} g|_{U_i}$ is a Kaehlerian metric on U_i, i.e., $\widetilde{\nabla} J = 0$, where $\widetilde{\nabla}$ is the covariant differentiation with respect to g (see, e.g., Reference [48]). The fundamental 2-form ω of a locally conformally Kaehler manifold $(\widetilde{M}, J, g)$ is given by

$$\omega(X, Y) = g(JX, Y), \tag{39}$$

for any vector fields X, Y tangent to $\widetilde{M}$.

The next result can be found in Reference [48].

Proposition 1. *A Hermitian manifold $(\widetilde{M}, J, g)$ is a locally conformal Kaehler manifold if and only if there exists a global closed 1-form α satisfying*

$$(\widetilde{\nabla}_Z\omega)(X,Y) = \beta(Y)g(X,Z) - \beta(X)g(Y,Z) + \alpha(Y)\omega(X,Z) - \alpha(X)\omega(Y,Z),$$

for any vector fields X, Y, Z tangent to $\widetilde{M}$, where $\widetilde{\nabla}$ is the Levi-Civita connection and β is the 1-form given by $\beta(X) = -\alpha(JX)$.

A typical example of a compact locally conformally Kaehler manifold is a *Hopf manifold* which is diffeomorphic to $S^1 \times S^{2n-1}$. It is known that a Hopf manifold admits no Kaehler structure (see Reference [49]).

The 1-form α is called the *Lee form* and its dual vector field is called the *Lee vector field*. A locally conformal Kaehler manifold which has parallel Lee form is called a *generalized Hopf manifold*.

On a locally conformal Kaehler manifold $(\widetilde{M}, J, g)$, there exists a symmetric $(0,2)$-tensor field P defined by

$$P(X,Y) = -(\widetilde{\nabla}_X\alpha)Y - \alpha(X)\alpha(Y) + \frac{1}{2}\|\alpha\|^2 g(X,Y),$$

and another $(0,2)$-tensor $\overline{P}$ defined by $\overline{P}(X,Y) = P(JX,Y)$, where $\|\alpha\|^2$ is the squared norm of α with respect to g.

A locally conformal Kaehler manifold with constant holomorphic sectional curvature c, denoted by $\widetilde{M}(c)$, is called a *locally conformal Kaehler space form*. The Riemann curvature tensor $\widetilde{R}$ of $\widetilde{M}(c)$ is given by (see, e.g., Reference [50–52])

$$\begin{aligned}\widetilde{R}(X,Y)Z = &\frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + \omega(Y,Z)JX - \omega(X,Z)JY - 2\omega(X,Y)JZ\}\\ &+\frac{3}{4}\{g(Y,Z)P_1X - g(X,Z)P_1Y + P(Y,Z)X - P(X,Z)Y\}\\ &-\frac{1}{4}\{\omega(Y,Z)\overline{P}_1X - \omega(X,Z)\overline{P}_1Y + \overline{P}(Y,Z)JX - \overline{P}(X,Z)JY\\ &-2\overline{P}(X,Y)JZ - 2\omega(X,Y)\overline{P}_1Z\},\end{aligned}$$

where $g(P_1X,Y) = P(X,Y)$ and $g(\overline{P}_1X,Y) = \overline{P}(X,Y)$.

Y. H. Kim and D. W. Yoon proved the following result for warped product submanifolds in locally conformal Kaehler space forms.

Theorem 22. *Ref [52]: Let $\phi : N_1 \times_f N_2 \to \widetilde{M}(c)$ be an isometric immersion of a warped product into a locally conformal Kaehler space form $\widetilde{M}(c)$. Then,*

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2}H^2 + \frac{n_1}{4}(c+3\sigma), \tag{40}$$

where $\sigma = \frac{\tilde{\rho}_1}{n_1} + \frac{\tilde{\rho}_2}{n_2}$ and $\tilde{\rho}_i$ is the partial trace of P restricted to N_i, $i = 1,2$.

The equality case of (40) holds identically if and only if ϕ is a mixed totally geodesic immersion satisfying trace h_1 = trace h_2.

The following results are immediate consequences of Theorem 22.

Corollary 11. *Let $N_1 \times_f N_2$ be a warped product in which warping function f is harmonic. Then,*

(1) *$N_1 \times_f N_2$ admits no minimal totally real immersion into a locally conformal Kaehler space form $\widetilde{M}(c)$ with $c < -3\sigma$;*

(2) *every minimal totally real immersion of $N_1 \times_f N_2$ into a Euclidean space is a warped product immersion.*

Corollary 12. *If the warping function f of a warped product $N_1 \times_f N_2$ is an eigenfunction of the Laplacian on N_1 with corresponding eigenvalue $\lambda > 0$, then $N_1 \times_f N_2$ does not admit a minimal totally real immersion into a locally conformal Kaehler space form $\widetilde{M}(c)$ with $c < -3\sigma$.*

Corollary 13. *Let $N_1 \times_f N_2$ be a compact minimal totally real warped product submanifold in a locally conformal Kaehler space form $\widetilde{M}(c)$ of holomorphic sectional curvature c satisfying $c \leq -3\sigma$. Then, $N_1 \times_f N_2$ is a Riemannian product.*

9. Warped Products in Quaternionic Space Forms

Let $\widetilde{M}^{4m}$ be a $4m$-dimensional almost quaternionic Hermitian manifold with metric tensor g. Then, there exists a rank 3 vector bundle Σ of tensors of type $(1,1)$ with local basis of almost Hermitian structures J_1, J_2, J_3 such that

(1) $g(J_\alpha X, J_\alpha Y) = g(X,Y)$, and
(2) $J_\alpha^2 = -I, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2}$,

for $\alpha \in \{1,2,3\}$, where I is the identity transformation on $T\widetilde{M}^{4m}$ and the indices are taken from $\{1,2,3\}$ modulo 3. If the bundle Σ is parallel with respect to the Levi-Civita connection of g, then $(\widetilde{M}^{4m}, \Sigma, g)$ is said to be a *quaternionic Kaehler manifold*.

For a quaternionic Kaehler manifold $(\widetilde{M}^{4m}, \Sigma, g)$, let X be a nonzero vector in $T\widetilde{M}$. The 4-plane $\widetilde{Q}(X)$ spanned by $\{X, J_1X, J_2X, J_3X\}$, is called a quaternionic 4-plane. Any 2-plane in $\widetilde{Q}(X)$ is called a quaternionic plane. The sectional curvature of a quaternionic plane is called a quaternionic sectional curvature. A quaternionic Kaehler manifold is said to be a *quaternionic space form* if its quaternionic sectional curvatures are equal to a constant.

A quaternionic space form of constant quaternionic sectional curvature c is denoted by $\widetilde{M}^{4m}(c)$. The curvature tensor $\widetilde{R}$ of $\widetilde{M}^{4m}(c)$ satisfies

$$\begin{aligned}\widetilde{R}(X,Y)Z = \frac{c}{4}\Big\{ & g(Z,Y)X - g(X,Z)Y \\ & + \sum_{\alpha=1}^{3} [g(Z, J_\alpha Y) J_\alpha X - g(Z, J_\alpha X) J_\alpha Y + 2g(X, J_\alpha Y) J_\alpha Z] \Big\}.\end{aligned}$$

For warped product submanifolds in quaternionic space forms, A. Mihai proved the following results in Reference [53].

Theorem 23. *Let $\phi : N_1 \times_f N_2 \to \widetilde{M}^{4m}(c)$ be an isometric immersion of an n-dimensional warped product into a $4m$-dimensional quaternionic space form $\widetilde{M}^{4m}(c)$ with $c > 0$. Then,*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2} \|H\|^2 + \frac{n_1}{4} c. \tag{41}$$

The equality case of (41) holds identically if and only if ϕ is a mixed totally geodesic immersion satisfying $\operatorname{trace} h_1 = \operatorname{trace} h_2$, *and $J_\alpha \mathcal{D}_1 \perp \mathcal{D}_2$, for any $\alpha = 1,2,3$.*

Theorem 24. *Let $\phi : N_1 \times_f N_2 \to \widetilde{M}^{4m}(c)$ be an isometric immersion of an n-dimensional warped product into a $4m$-dimensional quaternionic space form $\widetilde{M}^{4m}(c)$ with $c < 0$. Then,*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2} \|H\|^2 + \frac{n_1}{4} c + \frac{3c}{4} \min\left\{ \frac{n_1}{n_2}, 1 \right\}. \tag{42}$$

The equality case of (42) holds identically if and only if ϕ is a mixed totally geodesic immersion satisfying $\operatorname{trace} h_1 = \operatorname{trace} h_2$, *and $J_\alpha \mathcal{D}_1 \perp \mathcal{D}_2$, for any $\alpha = 1,2,3$.*

Theorem 25. *Let $\phi : N_1 \times_f N_2 \to \widetilde{M}^{4m}(c)$ be an isometric immersion of an n-dimensional warped product into a $4m$-dimensional quaternionic space form $\widetilde{M}^{4m}(c)$ with $c > 0$ such that $J_\alpha \mathcal{D}_1 \perp \mathcal{D}_2$ for any $\alpha = 1, 2, 3$. Then,*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + \frac{n_1}{4}c. \tag{43}$$

The equality case of (43) holds identically if and only if ϕ is a mixed totally geodesic immersion satisfying $\operatorname{trace} h_1 = \operatorname{trace} h_2$.

A submanifold N in a quaternionic Kaehler manifold $\widetilde{M}^{4m}$ is called a *quaternionic CR-submanifold* [54] if it admits a smooth quaternionic distribution $\mathcal{D}$ such that its orthogonal complementary distribution $\mathcal{D}^\perp$ is totally real, i.e., $J_\alpha \mathcal{D}_p \subset T_p^\perp N$ for any $p \in N$, where $T_p^\perp N$ denotes the normal space of N at $p \in N$.

A warped product $N_1 \times_f N_2$ in a quaternionic Kaehler manifold $\widetilde{M}^{4m}$ is called a *quaternionic CR-warped product* if it is a quaternionic CR-submanifold with $\mathcal{D} = TN_1$ and $\mathcal{D}^\perp = TN_2$.

Remark 4. *Theorem 25 implies that inequality (43) holds for every quaternionic CR-warped product in a quaternionic space form $\widetilde{M}^{4m}(c)$, $c > 0$.*

10. Geometric Inequalities for Warped Products in Almost Contact Metric Manifolds

An *almost contact metric manifold* is an odd-dimensional Riemannian $(2m+1)$-manifold $(\widetilde{M}^{2m+1}, g)$ such that there exist a $(1,1)$-tensor field φ, a vector field ξ, and a 1-form η on $\widetilde{M}^{2m+1}$ which satisfy (see, e.g., Reference [55])

$$\begin{aligned} &\eta(\xi) = 1, \;\; \phi^2(X) = -X + \eta(X)\xi, \;\; \varphi\xi = 0, \;\; \eta \circ \varphi = 0, \\ &g(\varphi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \;\;\; \eta(X) = g(X, \xi), \end{aligned}$$

for any vector fields X, Y tangent to $\widetilde{M}^{2m+1}$. The vector field ξ is called the *structure vector field* or *Reeb vector field*.

For a submanifold N of an almost contact metric manifold $(\widetilde{M}^{2m+1}, \varphi, \xi, \eta, g)$ and for a vector $X \in TN$, we put

$$\varphi X = TX + FX, \tag{44}$$

where TX and FX denote the tangential and the normal components of φX.

10.1. Warped Products in Sasakian Space Forms

An almost contact metric manifold $(\widetilde{M}^{2m+1}, \varphi, \xi, \eta, g)$ is said to be a *Sasakian manifold* if it satisfies

$$(\widetilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X,$$

for any vector fields X, Y tangent to $\widetilde{M}^{2m+1}$.

A *Sasakian space form* is a Sasakian manifold with constant φ-sectional curvature. It is known that the curvature tensor of a Sasakian space form $\widetilde{M}(c)$ of constant φ-sectional curvature c is given by

$$\begin{aligned} \widetilde{R}(X, Y)Z = &\frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\} \\ &+ \frac{c-1}{4}\{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\} \\ &+ \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}. \end{aligned}$$

Sasakian space forms $\widetilde{M}^{2m+1}(c)$ can be modeled based on $c > -3$, $c = -3$ or $c < -3$. We denote by $\mathbb{R}^{2m+1}$ the Sasakian space form which has constant ϕ-sectional curvature -3, while S^{2m+1} denotes the Sasakian space form of constant ϕ-sectional curvature 1 (see Reference [55]).

A submanifold N^n of an almost contact metric manifold $\widetilde{M}^{2m+1}$ is called *C-totally real* if its structure vector field ξ is normal to N^n. For C-totally real submanifolds N^n of $\widetilde{M}^{2m+1}$, we have $\phi(T_pN^n) \subset T_p^\perp N^n$, for any $p \in N^n$. A C-totally real submanifold is said to be a *Legendrian submanifold* if $n = m$ holds. Therefore, Legendrian submanifolds are C-totally real submanifolds with the smallest possible codimension.

Theorem 7 was extended by K. Matsumoto and I. Mihai [56] to warped product submanifolds in Sasakian space forms as follows.

Theorem 26. *Let $\phi : N_1 \times_f N_2 \to \widetilde{M}^{2m+1}(c)$ be a C-totally real isometric immersion of a warped product into a $(2m+1)$-dimensional Sasakian space form. Then, we have:*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1\frac{c+3}{4}. \tag{45}$$

The equality case of (45) *holds identically if and only if ϕ is a mixed totally geodesic immersion satisfying* $\operatorname{trace} h_1 = \operatorname{trace} h_2$.

Theorem 27. *Let $\widetilde{M}^{2m+1}(c)$ be a Sasakian space form and $N_1 \times_f N_2$ a warped product submanifold such that the Reeb vector field ξ is tangent to N_1. Then, N_2 is a C-totally real submanifold and we have*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1\frac{c+3}{4} - \frac{c-1}{4}. \tag{46}$$

The equality case of (46) *holds identically if and only if $N_1 \times_f N_2$ is a mixed totally geodesic submanifold satisfying* $\operatorname{trace} h_1 = \operatorname{trace} h_2$.

Theorem 28. *Any warped product submanifold $N_1 \times_f N_2$ of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ such that ξ is tangent to N_2 is a Riemannian product. Moreover, N_1 is a C-totally real submanifold.*

The notion of a generalized Sasakian space form was introduced by P. Alegre, D. E. Blair, and A. Carriazo in Reference [57]. An odd-dimensional manifold $\widetilde{M}^{2m+1}$ equipped with an almost contact metric structure (ϕ, ξ, η, g) is called *generalized Sasakian space form* if there exist three functions f_1, f_2, f_3 on $\widetilde{M}^{2m+1}$ such that

$$\begin{aligned}\widetilde{R}(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}.\end{aligned}$$

We denote such a manifold by $\widetilde{M}^{2m+1}(f_1, f_2, f_3)$.

A generalized Sasakian space form $\widetilde{M}^{2m+1}(f_1, f_2, f_3)$ reduces to a Sasakian space form if $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$, where c is a constant. Kenmotsu space forms and cosymplectic space forms are special cases of generalized Sasakian space forms. In fact,

(i) a *Kenmotsu space form* is a generalized Sasakian space form with $f_1 = \frac{c-3}{4}$ and $f_2 = f_3 = \frac{c+1}{4}$, and
(ii) a *cosymplectic space form* is a generalized Sasakian space form with $f_1 = f_2 = f_3 = \frac{c}{4}$.

In Reference [58], D. W. Yoon and K. S. Cho extended Theorem 7 further to warped products in generalized Sasakian space forms.

10.2. Warped Products in Kenmotsu Space Forms

An almost contact metric manifold $(\widetilde{M}^{2m+1}, \varphi, \xi, \eta, g)$ is said to be a *Kenmotsu manifold* if it satisfies

$$(\widetilde{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X,$$

where $\widetilde{\nabla}$ is the Levi-Civita connection of g.

If $\widetilde{M}^{2m+1}$ is a Kenmotsu manifold of dimension ≥ 5, then $\widetilde{M}^{2m+1}$ is called a *pointwise Kenmotsu space form* if the φ-sectional curvature function $\mathfrak{c}(X)$ of φ-holomorphic plane $\mathrm{Span}\{X, \varphi X\}$ depends only on the point $x \in \widetilde{M}^{2m+1}$, not on the choice of X at x. If $\mathfrak{c}$ is globally constant, then $\widetilde{M}^{2m+1}(c)$ is nothing but a *Kenmotsu space form*.

It is known that a Kenmotsu manifold $\widetilde{M}^{2m+1}$ is a pointwise Kenmotsu space form if and only if there exists a function c such that the Riemann curvature tensor $\widetilde{R}$ of $\widetilde{M}^{2m+1}$ satisfies (see Reference [59])

$$\begin{aligned}\widetilde{R}(X,Y)Z = & \frac{c-3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ & + \frac{c+1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi \\ & - \eta(X)g(Y,Z)\xi - g(\varphi X,Z)\varphi Y + g(\varphi Y,Z)\varphi X + 2g(X,\varphi Y)\varphi Z\}.\end{aligned}$$

C. Murathan, K. Arslan, R. Ezentas, and I. Mihai [60] extended Theorem 7 to warped product submanifolds in Kenmotsu space forms to the following.

Theorem 29. *Let $\phi : N_1 \times_f N_2 \to \widetilde{M}^{2m+1}(c)$ be a C-totally real isometric immersion of a warped product into a Kenmotsu space form $\widetilde{M}^{2m+1}(c)$ with $c < -1$. Then, we have:*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1\frac{c-3}{4} - \frac{c+1}{4}. \tag{47}$$

The equality case of (47) holds identically if and only if ϕ is a mixed totally geodesic immersion satisfying $\operatorname{trace} h_1 = \operatorname{trace} h_2$, *and $\varphi(TN_1)$ and TN_2 are orthogonal.*

Theorem 30. *Let $\phi : N_1 \times_f N_2 \to \widetilde{M}^{2m+1}(c)$ be a C-totally real isometric immersion of a warped product into a Kenmotsu space form $\widetilde{M}^{2m+1}(-1)$ such that the Reeb vector field ξ is tangent to N_1. Then, we have:*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 - n_1. \tag{48}$$

The equality case of (48) holds identically if and only if ϕ is a mixed totally geodesic immersion satisfying $\operatorname{trace} h_1 = \operatorname{trace} h_2$.

Theorem 31. *Let $\phi : N_1 \times_f N_2 \to \widetilde{M}^{2m+1}(c)$ be an isometric immersion of a warped product into a Kenmotsu space form $\widetilde{M}^{2m+1}(c)$ with $c > -1$ such that the Reeb vector field ξ is tangent to N_1. Then,*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1\frac{c-3}{4} + \left(\frac{3}{n_2}\|T\|^2 - 1\right)\frac{c+1}{4}, \tag{49}$$

where T is defined by (44).

The equality case of (49) holds identically if and only if ϕ is a mixed totally geodesic immersion satisfying $\operatorname{trace} h_1 = \operatorname{trace} h_2$, *and both N_1 and N_2 are anti-invariant submanifolds of $\widetilde{M}^{2m+1}(c)$.*

Theorem 32. *There do not exist warped product submanifolds $N_1 \times_f N_2$ in a Kenmotsu space form such that the Reeb vector field is tangent to N_2.*

10.3. Warped Products in Cosymplectic Space Forms

An almost contact metric manifold is said to be an *almost cosymplectic manifold* if it satisfies $d\eta = 0$ and $d\varphi = 0$. In particular, an almost cosymplectic manifold is called *cosymplectic* if it satisfies (see Reference [55])

$$\tilde{\nabla}\varphi = 0, \quad \tilde{\nabla}\xi = 0. \tag{50}$$

Theorem 7 was extended by D. W. Yoon [61] to warped product submanifolds in cosymplectic space forms as follows.

Theorem 33. *Let $\phi : N_1 \times_f N_2 \to \tilde{M}^{2m+1}(c)$ be an isometric immersion of a warped product into a cosymplectic space form $\tilde{M}^{2m+1}(c)$ such that the Reeb vector field ξ is tangent to N_1. Then, we have:*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + \frac{c}{4}(n_1 + 2). \tag{51}$$

Theorem 34. *Let $\phi : N_1 \times_f N_2 \to \tilde{M}^{2m+1}(c)$ be an isometric immersion of a warped product into a cosymplectic space form $\tilde{M}^{2m+1}(c)$ such that the Reeb vector field ξ is tangent to N_2. Then, we have:*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + \left(3 + n_1 - \frac{n_1}{n_2}\right)\frac{c}{4}. \tag{52}$$

Several applications of Theorems 33 and 34 were also given in Reference [61].

M. M. Tripathi studied in Reference [62] a similar problem for C-totally real warped product submanifolds in a (κ, μ)-space form.

11. Doubly Warped Product Submanifolds

Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and let $f_1 : N_1 \to (0, \infty)$ and $f_2 : N_2 \to (0, \infty)$ be two smooth functions. Then, the *doubly warped product* ${}_{f_2}N_1 \times_{f_1} N_2$ is the product manifold $N_1 \times N_2$ endowed with the doubly warped product metric

$$g = f_2^2 g_1 + f_1^2 g_2.$$

Obviously, the doubly warped product ${}_{f_2}N_1 \times_{f_1} N_2$ is an ordinary warped product if either f_1 or f_2 is a constant positive function.

The following result of A. Olteanu [63] extended Theorem 4 from ordinary warped product submanifolds to doubly warped product submanifolds in Riemannian manifolds.

Theorem 35. *Let $\phi : {}_{f_2}N_1 \times_{f_1} N_2 \to \tilde{M}^m$ be an isometric immersion of a doubly warped product $N = {}_{f_2}N_1 \times_{f_1} N_2$ into an arbitrary Riemannian m-manifold. Then, we have:*

$$n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2} \leq \frac{n^2}{4} H^2 + n_1 n_2 \max \tilde{K}, \tag{53}$$

where Δ_i denotes the Laplacian on N_i, $i = 1, 2$, and $\tilde{K}$ the sectional curvature of $\tilde{M}^m$.

The equality case of (53) holds identically if and only if the following two statements hold:

(1) ϕ is a mixed totally geodesic immersion satisfying $\operatorname{trace} h_1 = \operatorname{trace} h_2$;

(2) at each point $p = (p_1, p_2) \in N$, the function $\tilde{K}$ satisfies $\tilde{K}(u, v) = \max \tilde{K}(p)$ for each unit vector u in $T_{p_1}N_1$ and unit vector v in $T_{p_2}N_2$.

As an immediate consequence of Theorem 35, one has the following extension of Theorem 7.

Theorem 36. *Let $\phi : {}_{f_2}N_1 \times_{f_1} N_2 \to R^m(c)$ be an isometric immersion of a doubly warped product ${}_{f_2}N_1 \times_{f_1} N_2$ into a Riemannian m-manifold $R^m(c)$ of constant sectional curvature c. Then, we have:*

$$n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2} \leq \frac{n^2}{4} H^2 + n_1 n_2 c, \tag{54}$$

where Δ_i denotes the Laplacian on N_i, $i = 1, 2$.

The equality case of (54) holds identically if and only if ϕ is a mixed totally geodesic immersion satisfying $\operatorname{trace} h_1 = \operatorname{trace} h_2$.

A. Olteanu also showed in Reference [63] that the same result holds for an anti-invariant doubly warped product ${}_{f_2}N_1 \times_{f_1} N_2$ in a generalized Sasakian space form such that the Reeb vector field ξ is normal to ${}_{f_2}N_1 \times_{f_1} N_2$. In Reference [64], she obtained similar inequalities for doubly warped products isometrically immersed into locally conformal almost cosymplectic manifolds. In addition, in Reference [65], she derived similar inequalities for doubly warped products isometrically immersed into S-space forms. Further, A. Olteanu derived similar inequalities for multiply warped products in Kenmotsu space forms.

A contact metric manifold $(\widetilde{M}^{2m+1}, \varphi, \xi, \eta, g)$ is called a (κ, μ)-*manifold* if its Riemann curvature tensor satisfies

$$\widetilde{R}(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\},$$

where $h = \frac{1}{2}\mathcal{L}_\xi \varphi$ and $\mathcal{L}$ denotes the Lie derivative. By definition, a (κ, μ)-space form is a (κ, μ)-manifold which has constant φ-sectional curvature [55].

S. Sular and C. Özgür derived in Reference [66] similar sharp inequalities for C-totally real doubly warped product submanifolds in (κ, μ)-space forms and in non-Sasakian (κ, μ)-contact metric manifolds. In addition, M. Faghfouri and A. Majidi [67] extended the results for warped product immersions given in Section 4 to doubly warped product immersions.

12. Geometric Inequalities for Warped Products in Affine Spaces

12.1. Basics of Affine Differential Geometry

Let N be an n-manifold. Consider a non-degenerate hypersurface $\phi : N \to \mathbb{R}^{n+1}$ of the affine $(n+1)$-space in which position vector field is nowhere tangent to N. Then, ϕ can be consider as a transversal field along N. We call $\xi = -\phi$ the centroaffine normal and the ϕ together with this normalization is called a centroaffine hypersurface.

The centroaffine structure equations are given by (see, e.g., Reference [68])

$$D_X \phi_*(Y) = \phi_*(\nabla_X Y) + \sigma(X, Y)\xi, \tag{55}$$
$$D_X \xi = -\phi_*(X), \tag{56}$$

where D is the canonical flat connection of $\mathbb{R}^{n+1}$, ∇ is a torsion-free connection on N, called the induced centroaffine connection, and σ is a nondegenerate symmetric $(0, 2)$-tensor field, called the *centroaffine metric*.

Let us assume that the centroaffine hypersurface is definite, i.e., σ is definite. In case that σ is negative definite, we shall replace $\xi = -\phi$ by $\xi = \phi$ for the affine normal. In this way, the second fundamental form σ is always positive definite. In both cases, (55) holds. Equation (56) changes the sign. In case $\xi = -\phi$, we call N positive definite; in case $\xi = \phi$, we call N negative definite.

Denote by $\hat{\nabla}$ the Levi-Civita connection of σ. The *difference tensor* K is given by

$$K_X Y = K(X, Y) = \nabla_X Y - \hat{\nabla}_X Y. \tag{57}$$

The difference tensor K and the cubic form C are related by

$$C(X,Y,Z) = -2\sigma(K_X Y, Z).$$

Thus, for each X, K_X is self-adjoint with respect to σ. The *Tchebychev* 1*-form* T and the *Tchebychev vector field* $T^{\#}$ are defined, respectively, by (see, e.g., Reference [68,69])

$$T(X) = \frac{1}{n}\mathrm{trace}\, K_X, \tag{58}$$

$$\sigma(T^{\#}, X) = T(X). \tag{59}$$

If the Tchebychev form vanishes and if we consider N as a hypersurface of the equiaffine space, then N is a so-called *proper affine hypersphere* centered at the origin. If the difference tensor K vanishes, then N is a quadric, centered at the origin, in particular an ellipsoid if N is positive definite and a two-sheeted hyperboloid if N is negative definite.

An affine hypersurface $\phi : N \to \mathbb{R}^{n+1}$ is said to be a *graph hypersurface* if the transversal vector field ξ is a constant vector field. From a result of Reference [70], we know that a graph hypersurface is locally affine equivalent to the graph immersion of a certain function F. In the case that σ is nondegenerate, it defines a pseudo-Riemannian metric, known as the *Calabi metric* of the graph hypersurface. If $T = 0$, a graph hypersurface is called an *improper affine hypersphere*.

Let N_1 and N_2 be two improper affine hyperspheres in $\mathbb{R}^{p+1}$ and $\mathbb{R}^{q+1}$ defined, respectively, by the equations:

$$x_{p+1} = F_1(x_1, \ldots, x_p), \quad y_{q+1} = F_2(y_1, \ldots, y_q).$$

One can define an improper affine hypersphere N in $\mathbb{R}^{p+q+1}$ by

$$z = F_1(x_1, \ldots, x_p) + F_2(y_1, \ldots, y_q),$$

where $(x_1, \ldots, x_p, y_1, \ldots, y_q, z)$ are the coordinates on $\mathbb{R}^{p+q+1}$ and the Calabi normal of N is given by $(0, \ldots, 0, 1)$. Clearly, the Calabi metric on N is the direct product metric. This composition is called the *Calabi composition* of N_1 and N_2 (see Reference [71]).

12.2. A Realization Problem in Affine Geometry

For a Riemannian n-manifold (N, g) with Levi-Civita connection ∇, É. Cartan and A. P. Norden studied nondegenerate affine immersions $\phi : (N, \nabla) \to \mathbb{R}^{n+1}$ with a transversal vector field ξ and with ∇ as the induced connection. The Cartan-Norden theorem states that *if f is such an affine immersion, then either ∇ is flat and ϕ is a graph immersion or ∇ is not flat and $\mathbb{R}^{n+1}$ admits a parallel Riemannian metric relative to which ϕ is an isometric immersion and ξ is orthogonal to $\phi(N)$* (see, e.g., Reference [68], p. 159).

The first author investigated in Reference [72,73], from a view point different from Cartan-Norden, Riemannian hypersurfaces in some affine spaces. More precisely, he studied the following.

Realization Problem: *Which Riemannian manifolds (N, g) can be immersed as affine hypersurfaces in an affine space in such a way that the fundamental form σ, induced via the centroaffine normalization or a constant transversal vector field, is the given Riemannian metric g?*

Here, a Riemannian manifold (N, g) said to be *realized as an affine hypersurface* if there exists a codimension one affine immersion of N into some affine space in such a way that the induced affine metric σ is exactly the Riemannian metric g of N (notice that we do not put any assumption on the affine connection). In this respect, we mentioned that the first author proved in Reference [72] that every Robertson-Walker spacetime can be realized as a centroaffine or as a graph hypersurface in some affine space.

For warped products in affine spaces, we have the following results from Reference [73].

Theorem 37. *Ref [73]: If a warped product manifold $N_1 \times_f N_2$ can be realized as a graph hypersurface in $\mathbb{R}^{n+1}$, then the warping function satisfies*

$$\frac{\Delta f}{f} \geq -\frac{(n_1+n_2)^2}{4n_2} h(T^\#, T^\#). \tag{60}$$

The following result characterizes affine hypersurfaces which verify the equality case of inequality (60) identically.

Theorem 38. *Ref [73]: Let $\phi : N_1 \times_f N_2 \to \mathbb{R}^{n+1}$ be a realization of a warped product manifold as a graph hypersurface. If the warping function satisfies the equality case of (60) identically, then we have:*

(1) *the Tchebychev vector field $T^\#$ vanishes identically;*
(2) *the warping function f is a harmonic function;*
(3) *$N_1 \times_f N_2$ is realized as an improper affine hypersphere.*

An immediate application of Theorem 37 is the following.

Corollary 14. *Ref [73]: If N_1 is a compact Riemannian manifold, then every warped product manifold $N_1 \times_f N_2$ cannot be realized as an improper affine hypersphere in $\mathbb{R}^{n+1}$.*

As another application of Theorems 37 and 38, we have:

Theorem 39. *Ref [73]: If the Calabi metric of an improper affine hypersphere in an affine space is the Riemannian product metric of k Riemannian manifolds, then the improper affine hypersphere is locally the Calabi composition of k improper affine spheres.*

Theorem 37 also implies the following.

Corollary 15. *If the warping function f of a warped product manifold $N_1 \times_f N_2$ satisfies $\Delta f < 0$ at some point on N_1, then $N_1 \times_f N_2$ cannot be realized as an improper affine hypersphere in $\mathbb{R}^{n+1}$.*

For centro-affine hypersurfaces we have the following results from Reference [73].

Theorem 40. *If a warped product manifold $N_1 \times_f N_2$ can be realized as a centroaffine hypersurface in $\mathbb{R}^{n+1}$, then the warping function satisfies*

$$\frac{\Delta f}{f} \geq n_1 \varepsilon - \frac{(n_1+n_2)^2}{4n_2} h(T^\#, T^\#), \tag{61}$$

where $\varepsilon = 1$ or -1 according to whether the centroaffine hypersurface is elliptic or hyperbolic.

Theorem 41. *Let $\phi : N_1 \times_f N_2 \to \mathbb{R}^{n+1}$ be a realization of a warped product manifold $N_1 \times_f N_2$ as a centroaffine hypersurface. If the warping function satisfies the equality case of (61) identically, then we have:*

(1) the Tchebychev vector field $T^\#$ vanishes identically;
(2) the warping function f is an eigenfunction of the Laplacian Δ with eigenvalue $n_1\varepsilon$;
(3) $N_1 \times_f N_2$ is realized as a proper affine hypersphere centered at the origin.

Four other consequences of Theorem 40 are the following.

Corollary 16. *If the warping function f of a warped product manifold $N_1 \times_f N_2$ satisfies $\Delta f \leq 0$ at some point on N_1, then $N_1 \times_f N_2$ cannot be realized as an elliptic proper affine hypersphere in $\mathbb{R}^{n+1}$.*

Corollary 17. *If the warping function f of a warped product manifold $N_1 \times_f N_2$ satisfies $(\Delta f)/f < -n_1$ at some point on N_1, then $N_1 \times_f N_2$ cannot be realized as a hyperbolic proper affine hypersphere in $\mathbb{R}^{n+1}$.*

Corollary 18. *If N_1 is a compact Riemannian manifold, then every warped product manifold $N_1 \times_f N_2$ with arbitrary warping function cannot be realized as an elliptic proper affine hypersphere in $\mathbb{R}^{n+1}$.*

Corollary 19. *If N_1 is a compact Riemannian manifold, then every warped product manifold $N_1 \times_f N_2$ cannot be realized as an improper affine hypersphere in an affine space $\mathbb{R}^{n+1}$.*

Several examples were provided in Reference [73] to show that the results given above are all sharp.

13. Some Closely Related Geometric Inequalities

In this section, we briefly present some closely related geometric inequalities for warped product submanifolds.

13.1. CR-Warped Products

In Reference [74], the first author proved that, if $N^\perp \times_f N^\top$ is a warped product submanifold of a Kaehler manifold $\widetilde{M}$ such that $N^\perp$ is a totally real submanifold and $N^\top$ is a complex submanifold of $\widetilde{M}$, then $N^\perp \times_f N^\top$ is always non-proper, i.e., the warping function f must be constant. If the warping f is equal to 1, then the CR-warped product becomes a *CR-product* $N^\perp \times N^\top$ (see Reference [44,75,76]).

On the other hand, he proved that there exist abundant warped product submanifolds of the form $N^\top \times_f N^\perp$ in Kaehler manifolds. He simply called such warped product submanifolds *CR-warped products*.

For any CR-warped product in a Kaehler manifold, we have the following.

Theorem 42. *Refs [74,77]: Let $N = N^\top \times_f N^\perp$ be a CR-warped product in a Kaehler manifold $\widetilde{M}$. Then, we have:*

(1) The squared norm of the second fundamental form h of N satisfies

$$\|h\|^2 \geq 2q\|\nabla(\ln f)\|^2, \tag{62}$$

where $\nabla(\ln f)$ is the gradient of $\ln f$ and $q = \dim N^\perp$.

(2) If the equality case of (62) holds identically, then $N^\top$ is a totally geodesic submanifold and $N^\perp$ is a totally umbilical submanifold of $\widetilde{M}$; moreover, N is a minimal submanifold in $\widetilde{M}$.

(3) When M is anti-holomorphic and $q > 1$, then equality case of (62) holds identically if and only if $N^\perp$ is a totally umbilical submanifold of $\widetilde{M}$.

(4) Let N be anti-holomorphic with $q = 1$. Then, the equality case of (62) holds identically if the characteristic vector field $J\xi$ of M is a principal vector field with zero as its principal curvature. Conversely, if the equality case of (62) holds, then the characteristic vector field $J\xi$ of N is a principal vector field with zero as its principal curvature only if $N = N^\top \times_f N^\perp$ is a trivial CR-warped product immersed in $\widetilde{M}$ as a totally geodesic hypersurface. In addition, when N is anti-holomorphic with $q = 1$, the equality case of (62) holds identically if and only if N is a minimal hypersurface in $\widetilde{M}$.

Many further results concerning CR-warped products in Kaehler manifolds have been obtained in Reference [74,78–83].

13.2. CR-Products in Kaehler Manifolds

For CR-products in Kaehler manifolds, we have the following optimal geometric inequalities.

Theorem 43. *Refs [75,76]: Let $N = N^{\top} \times N^{\perp}$ be a CR-product in a complex projective m-space $CP^m(4)$ of constant holomorphic sectional curvature 4. Then, we have:*

$$\|h\|^2 \geq 4pq, \tag{63}$$

where $p = \dim_{\mathbb{C}} N^{\top}$ and $q = \dim_{\mathbb{R}} N^{\perp}$.

If the equality case of (63) holds identically, then $N^{\top}$ and $N^{\perp}$ are totally geodesic in $CP^m(4)$. Further, the immersion is rigid. Moreover, in this case $N^{\top}$ is a complex space form of constant holomorphic sectional curvature 4, and $N^{\perp}$ is a real space form of constant sectional curvature one.

Theorem 44. *Ref [75]: If $N = N^{\top} \times N^{\perp}$ is a minimal CR-product in a complex projective m-space $CP^m(4)$, then the scalar curvature of N satisfies*

$$\tau \leq 2p(p+1) + \frac{1}{2}q(q-1), \tag{64}$$

with the equality case holding identically if and only if $\|h\|^2 = 4pq$.

In 1891, C. Segre [84] introduced the following embedding:

$$S_{hp} : CP^p(4) \times CP^q(4) \to CP^{p+q+pq}(4), \tag{65}$$

defined by

$$S_{pq}(z_0, \dots, z_p; w_0, \dots, w_q) = (z_j w_t)_{0 \leq j \leq p, 0 \leq t \leq q},$$

where $(z_0, \dots, z_p)$ and $(w_0, \dots, w_q)$ are the homogeneous coordinates of $CP^p(4)$ and $CP^q(4)$, respectively. This embedding S_{pq} is a Kaehlerian embedding which is well-known as the *Segre embedding*.

In 1981, the first author applied Theorem 42 to establish the following "converse of the Segre embedding".

Theorem 45. *Ref [75]: Let $N = N_1 \times N_2$ be the Riemannian product of two Kaehler manifolds with $\dim_{\mathbb{C}} N_1 = p$ and $\dim_{\mathbb{C}} N_2 = q$. If N admits a Kaehlerian immersion into $CP^{p+q+pq}(4)$, then N_1 and N_2 are open submanifolds of totally geodesic $CP^p(4)$ and $CP^q(4)$ in $CP^{p+q+pq}(4)$. Moreover, the immersion is locally a Segre embedding.*

Theorem 45 was later extended by the first author and W. E. Kuan [85,86] for Kaehlerian immersions of Riemannian products $N_1 \times \cdots \times N_k$ of Kaehler manifolds into some complex space forms with $k > 2$.

13.3. Extensions of Theorem 42

Among some others, Theorem 42 was also extended by numerous mathematicians to CR-warped products in several classes of Riemannian manifolds:

1. CR-warped products in Kaehler and para-Kaehler manifolds [83,87–89].
2. CR-warped products of nearly Kaehler manifolds [90–92].
3. CR-warped products in locally conformal Kaehler manifolds [93–97].
4. CR-warped products in Sasakian manifolds [98–102].
5. CR-warped products in Kenmotsu manifolds [103–107].
6. CR-warped products in several other classes of contact metric manifolds [108–127].

13.4. Further Extensions of Theorem 42

Let N be a submanifold of almost Hermitian manifold $(\widetilde{M}, J, g)$. For a nonzero vector $X \in T_pN$ at an arbitrary point $p \in N$, the angle $\theta(X)$ between JX and the tangent space T_pN is called the *Wirtinger angle* of X. The submanifold N is called *slant* if its Wirtinger angle $\theta(X)$ is independent of the choice of $X \in T_pN$ and also of $p \in N$. The Wirtinger angle of a slant submanifold is called the *slant angle* [128]. A slant submanifold with slant angle θ is simply called *θ-slant* (see Reference [11,128]). A slant submanifold is called *proper* if it is either totally real or holomorphic. Similar notions applied to a distribution on N. In 1996, A. Lotta [129] extended the notion of slant submanifolds in the framework of contact geometry.

The first results on slant submanifolds were collected by the first author in his book [11] published in 1990. Later, slant submanifolds have been studied by various authors and since then many results in slant submanifolds have been obtained.

Slant submanifolds were extended to pointwise slant submanifolds in Reference [130,131]. Namely, a submanifold N of an almost Hermitian manifold $\widetilde{M}$ is called *pointwise slant* if, for each given point $p \in N$, the Wirtinger angle $\theta(X)$ is independent of the choice of the nonzero tangent vector $X \in T_pN$. In this case, θ defines a function on N, called the *slant function* of the pointwise slant submanifold.

By applying the notion of slant distributions, CR-warped products have been extended by A. Carriazo [132] to bi-slant warped products.

Definition 1. *A submanifold N of an almost Hermitian manifold $(\widetilde{M}, J, g)$ is called bi-slant if there exists a pair of orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on N such that*

(1) *$TN = \mathcal{D}_1 \oplus \mathcal{D}_2$;*
(2) *$J\mathcal{D}_1 \perp \mathcal{D}_2$ and $J\mathcal{D}_2 \perp \mathcal{D}_1$;*
(3) *the distributions $\mathcal{D}_1$, $\mathcal{D}_2$ are slant with slant angles θ_1, θ_2, respectively.*

The pair $\{\theta_1, \theta_2\}$ of slant angles of a bi-slant submanifold is called the bi-slant angles. In particular, a bi-slant submanifold with bi-slant angles $\{\theta_1, \theta_2\}$ satisfying $\theta_1 = \frac{\pi}{2}$ and $\theta_2 \in (0, \frac{\pi}{2})$ (respectively, $\theta_1 = 0$ and $\theta_2 \in (0, \frac{\pi}{2})$) is called a hemi-slant submanifold (respectively, semi-slant submanifold). A bi-slant submanifold N is called proper if its bi-slant angles satisfy $0 < \theta_1, \theta_2 < \frac{\pi}{2}$. Similar definitions apply to pointwise bi-slant submanifolds. In particular, we have the notions of pointwise hemi-slant and pointwise semi-slant submanifolds.

Definition 2. *A warped product $N_1 \times_f N_2$ of two slant submanifolds N_1 and N_2 of an almost Hermitian manifold $(\widetilde{M}, J, g)$ is called a warped product bi-slant submanifold. A warped product bi-slant submanifold $N_1 \times_f N_2$ is called a warped product hemi-slant submanifold (respectively, warped product semi-slant submanifold) if N_1 is totally real (respectively, holomorphic) in $\widetilde{M}$.*

As extensions of CR-warped products, there are numerous articles which studied pointwise bi-slant warped product submanifolds (in particular, bi-slant warped product submanifolds and contact bi-slant warped product submanifolds) in various ambient spaces during the last two decades. For results in this respect, we refer to References [133–138], among many others.

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