mathematics

MDPI

Article

Convergence Analysis and Cost Estimate of an MLMC-HDG Method for Elliptic PDEs with Random Coefficients

Meng Li and Xianbing Luo *

School of Mathematics and Statistics, Guizhou University, Guiyang 550025, China; mli2@gzu.edu.cn
* Correspondence: xbluo1@gzu.edu.cn

Abstract: We considered an hybridizable discontinuous Galerkin (HDG) method for discrete elliptic PDEs with random coefficients. By an approach of projection, we obtained the error analysis under the assumption that $a(\omega, x)$ is uniformly bounded. Together with the HDG method, we applied a multilevel Monte Carlo (MLMC) method (MLMC-HDG method) to simulate the random elliptic PDEs. We derived the overall convergence rate and total computation cost estimate. Finally, some numerical experiments are presented to confirm the theoretical results.

Keywords: elliptic PDEs; random coefficients; hybridizable discontinuous Galerkin; multilevel Monte Carlo

Citation: Li, M.; Luo, X. Convergence Analysis and Cost Estimate of an MLMC-HDG Method for Elliptic PDEs with Random Coefficients. *Mathematics* **2021**, *9*, 1072. https://doi.org/10.3390/math9091072

Academic Editor: Carlo Bianca

Received: 12 April 2021
Accepted: 4 May 2021
Published: 10 May 2021

1. Introduction

In this paper, we focus on the elliptic partial differential equations (PDEs) with random coefficients which arise in random vibrations, seismic activity and oil reservoir management (see, e.g., [1–4]). For elliptic stochastic PDEs, there are many numerical methods, such as intrusive and non-intrusive methods. The stochastic collocation method and stochastic Galerkin method are important intrusive methods (see, e.g., [5–9]). The MC method is a non-intrusive method. It is easy to implement, since the deterministic numerical method programs can be used once the samples are given (see, e.g., [10]). However, the MC method has a slow rate of convergence and its computational cost is also very expensive. To improve the computational efficiency, many methods have been proposed. The MLMC method is one of the most important methods. It has been widely used to simulate the stochastic PDEs (see, e.g., [11–14]).

In [15], the HDG method was originally proposed by Cockburn, Gopalakrishnan and Lazarov. It was then used to solve the second-order elliptic problems with remarkable convergence properties in [16]. The error analysis of elliptic PDEs was given by introducing a projection (see, e.g., [17] and references therein). To date, the HDG method has been successfully applied to parabolic, convection diffusion problems, phase flows and optimal control problem (see, e.g., [18–22]). However, we did not find theoretical or numerical analysis works on the HDG method for PDEs with random coefficients.

In this work, we first considered the HDG method to investigate elliptic PDEs with random coefficients under the assumption of a uniform bound. By a projection approach, we derived the convergence analysis of the HDG method. Then, we achieved rigorous bounds of the error and complexity for the MLMC method based on the HDG method.

The outline of the paper is as follows. In Section 2, the problem setting and HDG method are given. In Section 3, we deduce the error analysis of the velocity and the pressure. In Section 4, we use the error results in Section 3 to derive the complexity analysis of the MLMC method for the elliptic PDEs with random coefficients. In Section 5, some numerical experiments are presented to verify the effectiveness of the proposed method.

2. Preliminaries

2.1. Problem Setting

We consider the following elliptic problem with random coefficients: find a random function pair $(\boldsymbol{q}, u) : (\Omega \times D)^2 \to \mathbb{R}$ satisfying almost surely (a.s.):

$$\begin{aligned} a(\omega, x)\boldsymbol{q}(\omega, x) + \nabla u(\omega, x) &= 0, \quad \text{in } \Omega \times D, && (1)\\ \nabla \cdot \boldsymbol{q}(\omega, x) &= f(\omega, x), \quad \text{in } \Omega \times D, && (2)\\ u(\omega, x) &= g(x), \quad \text{on } \partial D, && (3) \end{aligned}$$

where $D \subset \mathbb{R}^2$ is a Lipschitz polyhedral domain, $(\Omega, \mathcal{F}, P)$ is the complete probability space with the set of outcomes Ω, σ-algebra $\mathcal{F}$ and probability measure P. The $a(\omega, x)$ is the random coefficient, $f(\omega, x)$ is the force term, and $g(x)$ is the deterministic Dirichlet boundary condition.

Let Y be a random variable in $(\Omega, \mathcal{F}, P)$. Denote by $\mathbb{E}[Y]$ the expected value which is defined by $\mathbb{E}[Y] = \int_\Omega Y(\omega) dP(\omega)$. We make the following assumptions to the random coefficient $a(\omega, x)$, force term $f(\omega, x)$ and boundary condition $g(x)$.

A1. There exist two constants $0 < \sigma_- < \sigma_+ < \infty$ such that:

$$P\big(\omega \in \Omega : \sigma_- \le \operatorname{essinf}_{x \in D} a(\omega, x) \le \|a(\omega, x)\|_{L^\infty(D)} \le \sigma_+\big) = 1.$$

A2. Let $g(x) \in H^{1/2}(\partial D)$. The source term $f(\omega, x)$ is square integrable with respect to P, i.e., $\int_D \mathbb{E}[|f(\omega, x)|^2] dx < \infty$.

2.2. HDG Method

Let D_h be a shape regular triangulation of the domain D (see, e.g., [23]). Denote by h_K the diameter of simplex $K \in D_h$ and let $h := \max_{k \in D_h} h_K$. Set $\partial D_h := \{\partial K : K \in D_h\}$ and $\mathcal{E}_h := \mathcal{E}_h^i \bigcup \mathcal{E}_h^\partial$, where $\mathcal{E}_h^i$ is the set of the interior faces, $\mathcal{E}_h^\partial$ is the set of the boundary faces. For these stochastic Sobolev spaces $L^2(\Omega, L^2(K))$, $L^2(\Omega, H^m(K))$, the corresponding norms are defined by

$$||u||_{L^2(\Omega, L^2(K))} = \left(\mathbb{E}\left[\int_K u^2 dx\right]\right)^{1/2}, ||u||_{L^2(\Omega, H^m(K))} = \left(\mathbb{E}\left[\sum_{|\alpha| \le m} \int_K |\partial^m u|^2 dx\right]\right)^{1/2}.$$

Define the following finite dimensional spaces $\mathbf{V}$, W, M by

$$\begin{aligned} \boldsymbol{V} : &= \{\boldsymbol{v} \in \boldsymbol{L}^2(D) : \boldsymbol{v}|_K \in [P_k(K)]^2, \quad \forall K \in D_h\},\\ W : &= \{w \in L^2(D) : w|_K \in P_k(K), \quad \forall K \in D_h\},\\ M : &= \{m \in L^2(\partial D_h) : m|_{\partial K} \in P_k(\partial K), \quad \forall \partial K \in \partial D_h\}, \end{aligned}$$

where $P_k(K)$ is the set of polynomials of degree at most k on the element K. We write:

$$(w, v)_{D_h} := \sum_{K \in D_h} (w, v)_K, \qquad \langle \mu, \zeta \rangle_{D_h} := \sum_{K \in D_h} \langle \mu, \zeta \rangle_{\partial K},$$

where $(\cdot, \cdot)$ is the L^2 inner product on K, and $\langle \cdot, \cdot \rangle$ is the L^2 inner product on ∂K. Given a sample $\omega \in \Omega$, the HDG numerical method of (1)–(3) is to find $(\boldsymbol{q}_{\omega,h}, u_{\omega,h}, \hat{u}_{\omega,h}) \in \mathbf{V} \times W \times M$ such that:

$$\begin{aligned} (a(\omega, x)\boldsymbol{q}_{\omega,h}, \boldsymbol{v})_{D_h} - (u_{\omega,h}, \nabla \cdot \boldsymbol{v})_{D_h} + \langle \hat{u}_{\omega,h}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial D_h} &= 0, && (4)\\ -(\boldsymbol{q}_{\omega,h}, \nabla w)_{D_h} + \langle \hat{\boldsymbol{q}}_{\omega,h} \cdot \boldsymbol{n}, w \rangle_{\partial D_h} &= (f, w)_{D_h}, && (5)\\ \langle \hat{u}_{\omega,h}, \mu \rangle_{\partial D} &= \langle g, \mu \rangle_{\partial D}, && (6)\\ \langle \hat{\boldsymbol{q}}_{\omega,h} \cdot \boldsymbol{n}, \mu \rangle_{\partial D_h \setminus \partial D} &= 0, && (7) \end{aligned}$$

for all $(\boldsymbol{v}, w, \mu) \in \mathbf{V} \times W \times M$. The numerical trace $\hat{\boldsymbol{q}}_{\omega,h}$ is defined by

$$\hat{\boldsymbol{q}}_{\omega,h} := \boldsymbol{q}_{\omega,h} + \tau(u_{\omega,h} - \hat{u}_{\omega,h})\boldsymbol{n}, \quad \text{on } \partial D_h,$$

where $\tau > 0$ is a constant on ∂D_h in this paper.

3. Error Analysis

3.1. HDG Projection

Denote by $(\boldsymbol{\Pi_V q}, \Pi_W u)$ the HDG projection function, where $\boldsymbol{\Pi_V q}$ and $\Pi_W u$ are components of $\mathbf{V}$ and W, respectively. On each simplex $K \in D_h$, the projection $(\boldsymbol{\Pi_V q}, \Pi_W u)$ satisfy the following equations:

$$(\boldsymbol{\Pi_V q} - \boldsymbol{q}, \boldsymbol{v})_K = 0, \qquad \forall \boldsymbol{v} \in [P_{k-1}(K)]^2, \tag{8}$$

$$(\Pi_W u - u, w)_K = 0, \qquad \forall w \in P_{k-1}(K), \tag{9}$$

$$\langle (\boldsymbol{\Pi_V q} - \boldsymbol{q}) \cdot \boldsymbol{n} + \tau(\Pi_W u - u), \mu \rangle_{\partial K} = 0, \qquad \forall \mu \in P_k(\partial K). \tag{10}$$

Then, we give an important result for the projection $(\boldsymbol{\Pi_V}, \Pi_W)$ in the following lemma. The proof is similar to Theorem 2.1 in [17] and hence is omitted.

Lemma 1. *Give a sample $\omega \in \Omega$. Suppose the τ is a positive constant on ∂K. Then, the system (8)–(10) is uniquely solvable. Moreover, we have the following estimate for $\ell_{\boldsymbol{q}}, \ell_u \in [0, k]$*

$$\|\boldsymbol{\Pi_V q} - \boldsymbol{q}\|_{L^2(\Omega;L^2(K))} \le C h_K^{\ell_q+1} \|\boldsymbol{q}\|_{L^2(\Omega;\boldsymbol{H}^{\ell_q+1}(K))} + C h_K^{\ell_u+1} \|u\|_{L^2(\Omega;H^{\ell_u+1}(K))},$$

$$\|\Pi_W u - u\|_{L^2(\Omega;L^2(K))} \le C h_K^{\ell_u+1} \|u\|_{L^2(\Omega;H^{\ell_u+1}(K))} + C h_K^{\ell_q+1} \|\nabla \cdot \boldsymbol{q}\|_{L^2(\Omega;\boldsymbol{H}^{\ell_q}(K))},$$

where C is a generic constant independent of h_K.

Let $\boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h} = \boldsymbol{\Pi}_V \boldsymbol{q}_\omega - \boldsymbol{q}_{\omega,h}$, $\rho^u_{\omega,h} = \Pi_W u_\omega - u_{\omega,h}$ and $\rho^{\hat{u}}_{\omega,h} = P_M u_\omega - \hat{u}_{\omega,h}$, where P_M is the L^2-orthogonal projection onto M which satisfies:

$$\langle \tau(P_M u_\omega - u_\omega), \mu \rangle_{\partial D_h} = 0, \quad \forall \mu \in M. \tag{11}$$

Then, we have the following error equations.

Lemma 2. *Give a sample $\omega \in \Omega$. Suppose $(\boldsymbol{q}_\omega, u_\omega)$, $(\boldsymbol{q}_{\omega,h}, u_{\omega,h}, \hat{u}_{\omega,h})$ are the solutions of (1)–(3) and (4)-(7) respectively. We have:*

$$\left(a(\omega,x)\boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}, \boldsymbol{v}\right)_{D_h} - \left(\rho^u_{\omega,h}, \nabla \cdot \boldsymbol{v}\right)_{D_h} + \langle \rho^{\hat{u}}_{\omega,h}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial D_h} = (a(\omega,x)(\boldsymbol{\Pi}_V \boldsymbol{q}_\omega - \boldsymbol{q}_\omega), \boldsymbol{v})_{D_h}, \tag{12}$$

$$-(\boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}, \nabla w)_{D_h} + \langle (\hat{\boldsymbol{\rho}}_{\omega,h} \cdot \boldsymbol{n}, w \rangle_{\partial D_h} = 0, \tag{13}$$

$$\langle \rho^{\hat{u}}_{\omega,h}, \mu \rangle_{\partial D} = 0, \tag{14}$$

$$\langle \hat{\boldsymbol{\rho}}_{\omega,h} \cdot \boldsymbol{n}, \mu \rangle_{\partial D_h \setminus \partial D} = 0, \tag{15}$$

for all $(\boldsymbol{v}, w, \mu) \in \mathbf{V} \times W \times M$, where:

$$\hat{\boldsymbol{\rho}}_{\omega,h} \cdot \boldsymbol{n} := \boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h} \cdot \boldsymbol{n} + \tau(\rho^u_{\omega,h} - \rho^{\hat{u}}_{\omega,h}), \quad \text{on} \quad \partial D_h \setminus \partial D. \tag{16}$$

Proof. Given a sample $\omega \in \Omega$, the exact solution $(\boldsymbol{q}_\omega, u_\omega)$ satisfies:

$$(a(\omega,x)\boldsymbol{q}_\omega, \boldsymbol{v})_{D_h} - (u_\omega, \nabla \cdot \boldsymbol{v})_{D_h} + \langle u_\omega, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial D_h} = 0,$$
$$-(\boldsymbol{q}_\omega, \nabla w)_{D_h} + \langle \boldsymbol{q}_\omega \cdot \boldsymbol{n}, w \rangle_{\partial D_h} = (f, w)_{D_h}.$$

for all $(\boldsymbol{v}, w, \mu) \in \mathbf{V} \times W \times M$. Considering properties (8) and (9) of the projections $(\boldsymbol{\Pi_V}, \Pi_W)$ and L^2-projection P_M of (11), we obtain:

$$(a(\omega,x)\boldsymbol{\Pi_V q}_\omega, \boldsymbol{v})_{D_h} - (\Pi_W u_\omega, \nabla\cdot\boldsymbol{v})_{D_h} + \langle P_M u_\omega, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial D_h} = (a(\omega,x)(\boldsymbol{\Pi_V q}_\omega - \boldsymbol{q}_\omega), \boldsymbol{v})_{D_h},$$
$$-(\boldsymbol{\Pi_V q}_\omega, \nabla w)_{D_h} + \langle \boldsymbol{\Pi_V q}_\omega\cdot\boldsymbol{n} + \tau(\Pi_W u_\omega - u_\omega), w\rangle_{\partial D_h} = (f, w)_{D_h}.$$

Subtracting (4) and (5) from the former equations, we obtain the expected Equations (12) and (13). By the boundary condition (6), we directly have Equation (14). Using the properties of $(\boldsymbol{\Pi_V}, \Pi_W)$ and P_M, we have:

$$\begin{aligned}\langle \hat{\boldsymbol{\rho}}_{\omega,h}\cdot\boldsymbol{n}, \mu\rangle_{\partial D_h\setminus\partial D} &= \langle(\boldsymbol{\Pi_V q}_\omega - \boldsymbol{q}_{\omega,h})\cdot\boldsymbol{n} + \tau(\Pi_W u_\omega - u_{\omega,h} - P_M u_\omega + \hat{u}_{\omega,h}), \mu\rangle_{\partial D_h\setminus\partial D}\\ &= \langle(\boldsymbol{q}_\omega - \boldsymbol{q}_{\omega,h})\cdot\boldsymbol{n} + \tau(u_\omega - u_{\omega,h} - u_\omega + \hat{u}_{\omega,h}), \mu\rangle_{\partial D_h\setminus\partial D}\\ &= \langle \boldsymbol{q}_\omega\cdot\boldsymbol{n}, \mu\rangle_{\partial D_h\setminus\partial D} + \langle \hat{\boldsymbol{q}}_{\omega,h}\cdot\boldsymbol{n}, \mu\rangle_{\partial D_h\setminus\partial D}\\ &= 0,\end{aligned}$$

where last identity is due to the fact that $\boldsymbol{q}_\omega$ is single valued on $\mathcal{E}_h$ and $\hat{\boldsymbol{q}}_{\omega,h}$ satisfies (7). We complete the proof. □

3.2. Flux Estimate

Theorem 1. *Suppose the exact solution* $(\boldsymbol{q}, u)$ *belongs to* $L^2(\Omega; \boldsymbol{H}^{k+1}(D_h)) \times L^2(\Omega; H^{k+1}(D_h))$ $(k > 0)$ *and the assumption A1 holds. We have:*

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(\Omega;L^2(D_h))} \le C\left(1 + \frac{\sigma_+}{\sigma_-}\right) h^{k+1}\left(\|\boldsymbol{q}\|_{L^2(\Omega;\boldsymbol{H}^{k+1}(D_h))} + \|u\|_{L^2(\Omega;H^{k+1}(D_h))}\right),$$

where the constant C is independent of h.

Proof. Give a sample $\omega \in \Omega$. Taking $\boldsymbol{v} = \boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}$ in (12), $w = \rho^u_{\omega,h}$ in (13), $\mu = -\hat{\boldsymbol{\rho}}_{\omega,h}\cdot\boldsymbol{n}$ in (14) and $\mu = -\rho^{\hat{u}}_{\omega,h}$ in (15) and adding these equations together, we obtain:

$$\begin{aligned}&\left(a(\omega,x)\boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}, \boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}\right)_{D_h} - \left(\rho^u_{\omega,h}, \nabla\cdot\boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}\right)_{D_h} + \langle\rho^{\hat{u}}_{\omega,h}, \boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}\cdot\boldsymbol{n}\rangle_{\partial D_h} - (\boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}, \nabla\rho^u_{\omega,h})_{D_h}\\ &\quad + \langle\hat{\boldsymbol{\rho}}_{\omega,h}\cdot\boldsymbol{n}, \rho^u_{\omega,h}\rangle_{\partial D_h} - \langle\hat{\boldsymbol{\rho}}_{\omega,h}\cdot\boldsymbol{n}, \rho^{\hat{u}}_{\omega,h}\rangle_{\partial D_h} = \left(a(\omega,x)(\boldsymbol{\Pi}_V\boldsymbol{q}_\omega - \boldsymbol{q}_\omega), \boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}\right)_{D_h}.\end{aligned}$$

Applying the integration by parts to the left-hand-side (LHS) terms of the above equation, we have

$$\begin{aligned}\text{LHS} &= \left(a(\omega,x)\boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}, \boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}\right)_{D_h} + \left(\nabla\rho^u_{\omega,h}, \boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}\right)_{D_h} - \langle\rho^u_{\omega,h}, \boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}\cdot\boldsymbol{n}\rangle_{\partial D_h} + \langle\rho^{\hat{u}}_{\omega,h}, \boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}\cdot\boldsymbol{n}\rangle_{\partial D_h}\\ &\quad - (\boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}, \nabla\rho^u_{\omega,h})_{D_h} + \langle\hat{\boldsymbol{\rho}}_{\omega,h}\cdot\boldsymbol{n}, \rho^u_{\omega,h}\rangle_{\partial D_h} - \langle\hat{\boldsymbol{\rho}}_{\omega,h}\cdot\boldsymbol{n}, \rho^{\hat{u}}_{\omega,h}\rangle_{\partial D_h}\\ &= \left(a(\omega,x)\boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}, \boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}\right)_{D_h} - \langle\rho^u_{\omega,h} - \rho^{\hat{u}}_{\omega,h}, \boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}\cdot\boldsymbol{n}\rangle_{\partial D_h} + \langle\hat{\boldsymbol{\rho}}_{\omega,h}\cdot\boldsymbol{n}, \rho^u_{\omega,h} - \rho^{\hat{u}}_{\omega,h}\rangle_{\partial D_h}\\ &= \left(a(\omega,x)\boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}, \boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}\right)_{D_h} + \langle\tau(\rho^u_{\omega,h} - \rho^{\hat{u}}_{\omega,h}), \rho^u_{\omega,h} - \rho^{\hat{u}}_{\omega,h}\rangle_{\partial D_h},\end{aligned}$$

where the last identity follows from (16). Hence:

$$\left(a(\omega,x)\boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}, \boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}\right)_{D_h} + \langle\tau(\rho^u_{\omega,h} - \rho^{\hat{u}}_{\omega,h}), \rho^u_{\omega,h} - \rho^{\hat{u}}_{\omega,h}\rangle_{\partial D_h} = \left(a(\omega,x)(\boldsymbol{\Pi}_V\boldsymbol{q}_\omega - \boldsymbol{q}_\omega), \boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}\right)_{D_h}.$$

Considering the assumption **A1** and integrating in the probability space, we obtain:

$$\sigma_-\mathbb{E}\left[\left(\boldsymbol{\rho}^{\boldsymbol{q}}_h, \boldsymbol{\rho}^{\boldsymbol{q}}_h\right)_{D_h}\right] + \mathbb{E}\left[\langle\tau(\rho^u_h - \rho^{\hat{u}}_h), \rho^u_h - \rho^{\hat{u}}_h\rangle_{\partial D_h}\right] \le \sigma_+\mathbb{E}\left[\left(\boldsymbol{\Pi}_V\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{\rho}^{\boldsymbol{q}}_h\right)_{D_h}\right].$$

By Cauchy–Schwarz inequality and the fact $\tau > 0$, we have:

$$\left\|\boldsymbol{\rho}^{\boldsymbol{q}}_h\right\|_{L^2(\Omega;L^2(D_h))} \le \frac{\sigma_+}{\sigma_-}\|\boldsymbol{\Pi}_V\boldsymbol{q} - \boldsymbol{q}\|_{L^2(\Omega;L^2(D_h))}.$$

The triangle inequality implies:

$$\begin{aligned}\|\boldsymbol{q}-\boldsymbol{q}_h\|_{L^2(\Omega;L^2(D_h))} &\le \|\boldsymbol{q}-\boldsymbol{\Pi}_V\boldsymbol{q}\|_{L^2(\Omega;L^2(D_h))}+\|\boldsymbol{\Pi}_V\boldsymbol{q}_\omega-\boldsymbol{q}_h\|_{L^2(\Omega;L^2(D_h))}\\ &\le \left(1+\frac{\sigma_+}{\sigma_-}\right)\|\boldsymbol{\Pi}_V\boldsymbol{q}-\boldsymbol{q}\|_{L^2(\Omega;L^2(D_h))}.\end{aligned}$$

By Lemma 1, we complete the proof. □

3.3. Pressure Estimate

In this subsection, we use the Aubin–Nitsche duality argument to give the error estimate of $\|u-u_h\|_{L^2(\Omega;D_h)}$. We introduce a dual problem for any given $\Theta\in L^2(\Omega;L^2(D))$ by finding $\boldsymbol{\Phi}(\omega,x)$ such that:

$$a(\omega,x)\boldsymbol{\Phi}(\omega,x)+\nabla\varphi(\omega,x) = 0,\quad \text{in } \Omega\times D, \tag{17}$$

$$\nabla\cdot\boldsymbol{\Phi}(\omega,x) = \Theta(\omega,x),\quad \text{in } \Omega\times D, \tag{18}$$

$$\varphi(\omega,x) = 0,\quad \text{on } \partial D. \tag{19}$$

We assume that the boundary value problem admits the regularity estimate:

$$\|\boldsymbol{\Phi}(\omega,x)\|_{L^2(\Omega;\boldsymbol{H}^1(D))}+\|\varphi(\omega,x)\|_{L^2(\Omega;H^2(D))}\le C_{reg}\|\Theta(\omega,x)\|_{L^2(\Omega;L^2(D))}, \tag{20}$$

where C_{reg} depends on σ_- and σ_+. Then, we have the following theorem.

Theorem 2. *Suppose the exact solution $(\boldsymbol{q},u)$ belongs to $L^2(\Omega;\boldsymbol{H}^{k+1}(D_h))\times L^2(\Omega;H^{k+1}(D_h))$ $(k>0)$ and the assumption A1 holds. We have:*

$$\|u-u_h\|_{L^2(\Omega;L^2(D_h))}\le C\left(1+\frac{\sigma_+}{\sigma_-}\right)h^{k+1}\left(|\boldsymbol{q}|_{L^2(\Omega;\boldsymbol{H}^{k+1}(D_h))}+|u|_{L^2(\Omega;H^{k+1}(D_h))}\right),$$

where the constant C is independent of h.

Proof. Give a sample $\omega\in\Omega$. Considering the properties of (8) and (10), we have:

$$\begin{aligned}&\left(\rho^u_{\omega,h},\nabla\cdot\boldsymbol{\Phi}(\omega,x)\right)_{D_h}\\ &=-\left(\nabla\rho^u_{\omega,h},\boldsymbol{\Phi}(\omega,x)\right)_{D_h}+\left\langle\rho^u_{\omega,h},\boldsymbol{\Phi}(\omega,x)\cdot\boldsymbol{n}\right\rangle_{\partial D_h}\\ &=-\left(\nabla\rho^u_{\omega,h},\boldsymbol{\Pi}_V\boldsymbol{\Phi}(\omega,x)\right)_{D_h}+\left\langle\rho^u_{\omega,h},\boldsymbol{\Phi}(\omega,x)\cdot\boldsymbol{n}\right\rangle_{\partial D_h}\\ &=-\left(\nabla\rho^u_{\omega,h},\boldsymbol{\Pi}_V\boldsymbol{\Phi}(\omega,x)\right)_{D_h}+\left\langle\rho^u_{\omega,h},\boldsymbol{\Pi}_V\boldsymbol{\Phi}(\omega,x)\cdot\boldsymbol{n}+\tau(\Pi_W\varphi(\omega,x)-\varphi(\omega,x))\right\rangle_{\partial D_h}\\ &=\left(\rho^u_{\omega,h},\nabla\cdot\boldsymbol{\Pi}_V\boldsymbol{\Phi}(\omega,x)\right)_{D_h}+\left\langle\rho^u_{\omega,h},\tau(\Pi_W\varphi(\omega,x)-\varphi(\omega,x))\right\rangle_{\partial D_h}.\end{aligned}$$

Taking $\boldsymbol{v}=\boldsymbol{\Pi}_V\boldsymbol{\Phi}(\omega,x)$ in (12), we obtain:

$$\begin{aligned}&\left(\rho^u_{\omega,h},\Theta(\omega,x)\right)_{D_h}=\left(\rho^u_{\omega,h},\nabla\cdot\boldsymbol{\Phi}(\omega,x)\right)_{D_h}\\ &=\left(\rho^u_{\omega,h},\nabla\cdot\boldsymbol{\Pi}_V\boldsymbol{\Phi}(\omega,x)\right)_{D_h}+\left\langle\rho^u_{\omega,h},\tau(\Pi_W\varphi(\omega,x)-\varphi(\omega,x))\right\rangle_{\partial D_h}\\ &=\left(a(\omega,x)\rho^{\boldsymbol{q}}_{\omega,h},\boldsymbol{\Pi}_V\boldsymbol{\Phi}(\omega,x)\right)_{D_h}+\langle\rho^{\hat{u}}_{\omega,h},\boldsymbol{\Pi}_V\boldsymbol{\Phi}(\omega,x)\cdot\boldsymbol{n}\rangle_{\partial D_h}\\ &\quad-(a(\omega,x)(\boldsymbol{\Pi}_V\boldsymbol{q}_\omega-\boldsymbol{q}_\omega),\boldsymbol{\Pi}_V\boldsymbol{\Phi}(\omega,x))_{D_h}+\left\langle\rho^u_{\omega,h},\tau(\Pi_W\varphi(\omega,x)-\varphi(\omega,x))\right\rangle_{\partial D_h}\\ &=\left(a(\omega,x)(\boldsymbol{q}_\omega-\boldsymbol{q}_{\omega,h}),\boldsymbol{\Pi}_V\boldsymbol{\Phi}(\omega,x)\right)_{D_h}+\left\langle\rho^{\hat{u}}_{\omega,h},(\boldsymbol{\Pi}_V\boldsymbol{\Phi}(\omega,x)-\boldsymbol{\Phi}(\omega,x))\cdot\boldsymbol{n}\right\rangle_{\partial D_h}\\ &\quad+\left\langle\rho^u_{\omega,h},\tau(\Pi_W\varphi(\omega,x)-\varphi(\omega,x))\right\rangle_{\partial D_h},\end{aligned}$$

where the last equation is due to the fact that $\mathbf{\Phi}(\omega,x)\cdot\boldsymbol{n}$ is continuity and $\rho^{\hat{u}}_{\omega,h}=0$ on ∂D. By (10) and (11), we have:

$$\begin{aligned}
&\left(\rho^{u}_{\omega,h},\Theta(\omega,x)\right)_{D_h}\\
&=\left(a(\omega,x)(\boldsymbol{q}_{\omega}-\boldsymbol{q}_{\omega,h}),\mathbf{\Pi}_V\mathbf{\Phi}(\omega,x)\right)_{D_h}+\left\langle\rho^{u}_{\omega,h}-\rho^{\hat{u}}_{\omega,h},\tau(\Pi_W\varphi(\omega,x)-\varphi(\omega,x))\right\rangle_{\partial D_h}\\
&=\left(a(\omega,x)(\boldsymbol{q}_{\omega}-\boldsymbol{q}_{\omega,h}),\mathbf{\Pi}_V\mathbf{\Phi}(\omega,x)\right)_{D_h}+\left\langle\tau(\rho^{u}_{\omega,h}-\rho^{\hat{u}}_{\omega,h}),\Pi_W\varphi(\omega,x)\right\rangle_{\partial D_h}\\
&\quad-\left\langle\tau(\rho^{u}_{\omega,h}-\rho^{\hat{u}}_{\omega,h}),P_M\varphi(\omega,x)\right\rangle_{\partial D_h}.
\end{aligned}$$

Error Equations (13) and (16) yield:

$$\begin{aligned}
&\left(\rho^{u}_{\omega,h},\Theta(\omega,x)\right)_{D_h}\\
&=\left(a(\omega,x)(\boldsymbol{q}_{\omega}-\boldsymbol{q}_{\omega,h}),\mathbf{\Pi}_V\mathbf{\Phi}(\omega,x)\right)_{D_h}-\left(\nabla\cdot\boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h},\Pi_W\varphi(\omega,x)\right)_{D_h}\\
&\quad+\left\langle\boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}\cdot\boldsymbol{n},P_M\varphi(\omega,x)\right\rangle_{\partial D_h}\\
&=\left(a(\omega,x)(\boldsymbol{q}_{\omega}-\boldsymbol{q}_{\omega,h}),\mathbf{\Pi}_V\mathbf{\Phi}(\omega,x)\right)_{D_h}-\left(\nabla\cdot\boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h},\varphi(\omega,x)\right)_{D_h}+\left\langle\boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h}\cdot\boldsymbol{n},\varphi(\omega,x)\right\rangle_{\partial D_h}\\
&=\left(a(\omega,x)(\boldsymbol{q}_{\omega}-\boldsymbol{q}_{\omega,h}),\mathbf{\Pi}_V\mathbf{\Phi}(\omega,x)\right)_{D_h}+\left(\boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h},\nabla\varphi(\omega,x)\right)_{D_h}.
\end{aligned}$$

Using Equation (17), we derive:

$$\begin{aligned}
&\left(\rho^{u}_{\omega,h},\Theta(\omega,x)\right)_{D_h}\\
&=\left(a(\omega,x)(\boldsymbol{q}_{\omega}-\boldsymbol{q}_{\omega,h}),\mathbf{\Pi}_V\mathbf{\Phi}(\omega,x)-\mathbf{\Phi}(\omega,x)\right)_{D_h}\\
&\quad+\left(a(\omega,x)(\boldsymbol{q}_{\omega}-\boldsymbol{q}_{\omega,h}),\mathbf{\Phi}(\omega,x)\right)_{D_h}+\left(\boldsymbol{\rho}^{\boldsymbol{q}}_{\omega,h},\nabla\varphi(\omega,x)\right)_{D_h}\\
&=\left(a(\omega,x)(\boldsymbol{q}_{\omega}-\boldsymbol{q}_{\omega,h}),\mathbf{\Pi}_V\mathbf{\Phi}(\omega,x)-\mathbf{\Phi}(\omega,x)\right)_{D_h}\\
&\quad+\left(\mathbf{\Pi}_V\boldsymbol{q}_{\omega}-\boldsymbol{q}_{\omega},\nabla\varphi(\omega,x)-\Pi^{k-1}\nabla\varphi(\omega,x)\right)_{D_h},
\end{aligned}$$

where Π^{k-1} is the L^2-projection on $[P_{k-1}(K)]^2$. Considering the assumption A1 and integrating in the probability space, we have:

$$\begin{aligned}
\mathbb{E}\left[(\rho^{u}_{h},\Theta)_{D_h}\right]\leq&\sigma_+\mathbb{E}\left[(\boldsymbol{q}-\boldsymbol{q}_h,\mathbf{\Pi}_V\mathbf{\Phi}(x)-\mathbf{\Phi}(x))_{D_h}\right]\\
&+\mathbb{E}\left[\left(\mathbf{\Pi}_V\boldsymbol{q}-\boldsymbol{q},\nabla\varphi-\Pi^{k-1}\nabla\varphi(\omega,x)\right)_{D_h}\right].
\end{aligned}$$

By the Cauchy–Schwarz inequality, we deduce:

$$\begin{aligned}
\mathbb{E}\left[(\rho^{u}_{h},\Theta)_{D_h}\right]\leq&\,\sigma_+\|\boldsymbol{q}-\boldsymbol{q}_h\|_{L^2(\Omega;L^2(D_h))}\|\mathbf{\Pi}_V\mathbf{\Phi}-\mathbf{\Phi}\|_{L^2(\Omega;L^2(D_h))}\\
&+\|\mathbf{\Pi}_V\boldsymbol{q}-\boldsymbol{q}\|_{L^2(\Omega;L^2(D_h))}\left\|\nabla\varphi-\Pi^{k-1}\nabla\varphi\right\|_{L^2(\Omega;L^2(D_h))}.
\end{aligned}$$

Applying Lemma 1 and Theorem 1, we have:

$$\begin{aligned}
\mathbb{E}\left[(\rho^{u}_{h},\Theta)_{D_h}\right]\leq&\,C\left(1+\frac{\sigma_+}{\sigma_-}\right)h^{k+1}\left(|\boldsymbol{q}|_{L^2(\Omega;\boldsymbol{H}^{k+1}(D_h))}+|u|_{L^2(\Omega;H^{k+1}(D_h))}\right)\cdot\\
&\left(h\sigma_+\|\mathbf{\Phi}\|_{L^2(\Omega;\boldsymbol{H}^1(D_h))}+h^2\sigma_+\|\varphi\|_{L^2(\Omega;H^2(D_h))}+h^{\min\{k,1\}}\sigma_+\|\varphi\|_{L^2(\Omega;H^2(D_h))}\right).
\end{aligned}$$

The regularity assumption (20) implies:

$$\mathbb{E}\Big[(\rho_h^u,\Theta)_{D_h}\Big] \le C\left(1+\frac{\sigma_+}{\sigma_-}\right)h^{k+1}\Big(|\boldsymbol{q}|_{L^2(\Omega;\boldsymbol{H}^{k+1}(D_h))}+|u|_{L^2(\Omega;H^{k+1}(D_h))}\Big) \times \Big(h\sigma_+\|\Theta\|_{L^2(\Omega;L^2(D_h))}+h^2\sigma_+\|\Theta\|_{L^2(\Omega;L^2(D_h))}+h^{\min\{k,1\}}\sigma_+\|\Theta\|_{L^2(\Omega;L^2(D_h))}\Big).$$

Hence:

$$\|\rho_h^u\|_{L^2(\Omega;L^2(D_h))} \le C\left(1+\frac{\sigma_+}{\sigma_-}\right)h^{k+1}\Big(h\sigma_++h^2\sigma_++h^{\min\{k,1\}}\Big) \times \Big(|\boldsymbol{q}|_{L^2(\Omega;\boldsymbol{H}^{k+1}(D_h))}+|u|_{L^2(\Omega;H^{k+1}(D_h))}\Big).$$

Together with the Lemma 1, we obtain:

$$\|u-u_h\|_{L^2(\Omega;L^2(D_h))} \le C\left(1+\frac{\sigma_+}{\sigma_-}\right)h^{k+1}\Big(|\boldsymbol{q}|_{L^2(\Omega;\boldsymbol{H}^{k+1}(D_h))}+|u|_{L^2(\Omega;H^{k+1}(D_h))}\Big).$$

We complete the proof. □

4. Multilevel Monte Carlo HDG Method

4.1. The MC Method

Let $u \in L^2(\Omega;L^2(D_h))$ be a random field. The expectation $\mathbb{E}[u]$ is approximated by the sample average $E_M[u]$, which is defined by

$$E_M[u] := \frac{1}{M}\sum_{i=1}^{M} u_{\omega_i},$$

where $u_{\omega_i} := u(\omega_i,\cdot), i=1,\dots,M$ are independent identically distributed (i.i.d.) realizations of the random field u. We give statistical error of the $E_M[u]$ in the following lemma.

Lemma 3. *Let $u \in L^2(\Omega;L^2(D_h))$. Then, we have for any $M \in \mathbb{N}$:*

$$\|\mathbb{E}[u]-E_M[u]\|_{L^2(\Omega;L^2(D_h))} \le M^{-1/2}\|u\|_{L^2(\Omega;L^2(D_h))}.$$

Proof. Considering that $u_{\omega_i} := u(\omega_i,\cdot), i=1,\dots,M$ are i.i.d. samples of the random field u, we have:

$$\begin{aligned}
&\mathbb{E}\Big[\|\mathbb{E}[u]-E_M[u]\|^2_{L^2(D_h)}\Big] \\
&= \mathbb{E}\left[\left\|\mathbb{E}[u]-\frac{1}{M}\sum_{i=1}^{M}u_{\omega_i}\right\|^2_{L^2(D_h)}\right] = \frac{1}{M^2}\mathbb{E}\left[\left\|\sum_{i=1}^{M}(\mathbb{E}[u]-u_{\omega_i})\right\|^2_{L^2(D_h)}\right] \\
&= \frac{1}{M^2}\sum_{i=1}^{M}\mathbb{E}\Big[\|\mathbb{E}[u]-u_{\omega_i}\|^2_{L^2(D_h)}\Big] = \frac{1}{M}\mathbb{E}\Big[\|\mathbb{E}[u]-u\|^2_{L^2(D_h)}\Big] \\
&= \frac{1}{M}\Big(\mathbb{E}\Big[\|u\|^2_{L^2(D_h)}\Big]-\|\mathbb{E}[u]\|^2_{L^2(D_h)}\Big) \le \frac{1}{M}\mathbb{E}\Big[\|u\|^2_{L^2(D_h)}\Big].
\end{aligned}$$

Taking the square root of both sides of the inequality, we complete the proof. □

In practice, it is difficult to take samples from the random field u, since we do not know it most of time. To overcome this difficulty, we choose samples from the HDG approximation u_{h_L} with $h_L = 1/(2^{2L})$, where $L \in \mathbb{N}$. We define the the classical MC estimator:

$$E_M[u_{h_L}] := \frac{1}{M}\sum_{i=1}^{M} u_{\omega_i,h_L}, \tag{21}$$

where $u_{\omega_i,h_L} := u_{h_L}(\omega_i,\cdot), i = 1,\ldots,M$ are i.i.d. realizations of the random field u_{h_L}.

Theorem 3. *Suppose assumptions A1-A2 hold, then:*

$$\begin{aligned}\left\|\mathbb{E}[u] - E_M[u_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))} &\le C\left(h_L^{k+1} + M^{-1/2}\right),\\ \left\|\mathbb{E}[\boldsymbol{q}] - E_M[\boldsymbol{q}_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))} &\le C\left(h_L^{k+1} + M^{-1/2}\right),\end{aligned}$$

where the constant C is independent of h_L and M.

Proof. By the triangle inequality, we have:

$$\begin{aligned}&\left\|\mathbb{E}[u] - E_M[u_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))}\\ &\le \left\|\mathbb{E}[u] - \mathbb{E}[u_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))} + \left\|\mathbb{E}[u_{h_L}] - E_M[u_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))}.\end{aligned}$$

Using the Cauchy–Schwarz inequality, we have:

$$\begin{aligned}\left\|\mathbb{E}[u] - \mathbb{E}[u_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))} = \left\|\mathbb{E}[u - u_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))} &\le \mathbb{E}\left[\left\|u - u_{h_L}\right\|_{L^2(D_h)}\right]\\ &\le \left\|u - u_{h_L}\right\|_{L^2(\Omega;L^2(D_h))} \le Ch_L^{k+1},\end{aligned}$$

where the third inequality follows from Theorem 2. For term $\left\|\mathbb{E}[u_{h_L}] - E_M[u_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))}$, applying Lemma 3, we obtain:

$$\left\|\mathbb{E}[u_{h_L}] - E_M[u_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))} \le M^{-1/2}\left\|u_{h_L}\right\|_{L^2(\Omega;L^2(D_h))}.$$

Hence:

$$\left\|\mathbb{E}[u] - E_M[u_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))} \le C\left(h_L^{k+1} + M^{-1/2}\right).$$

Similarly, we can obtain:

$$\left\|\mathbb{E}[\boldsymbol{q}] - E_M[\boldsymbol{q}_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))} \le C\left(h_L^{k+1} + M^{-1/2}\right).$$

The proof is complete. □

Theorem 3 implies that the total error result can be decomposed into two parts: statistical error with order $M^{-1/2}$ and discretization error with order h_L^{k+1}. For a fixed error, the optimal number of samples M should be equilibrated with the mesh size h_L, meaning that:

$$M^{-1/2} = \mathcal{O}(h_L^{k+1}) = \mathcal{O}(N_L^{-(k+1)/2}),$$

where $N_L = 2^{2L}$ is the degrees of freedom of the HDG method. Hence, the total computational cost is:

$$\mathcal{O}(M \cdot N_L) = \mathcal{O}(2^{2(k+2)L}).$$

4.2. The MLMC Method

For the MLMC method, the random field u_{h_L} can be written as

$$u_{h_L} = \sum_{l=1}^{L}(u_{h_l} - u_{h_{l-1}}),$$

where $u_{h_0} = 0, l = 0,\ldots,L$. The linearity of the expectation operator yields:

$$\mathbb{E}[u_{h_L}] = \sum_{l=1}^{L}\mathbb{E}[u_{h_l} - u_{h_{l-1}}].$$

We approximate $\mathbb{E}[u_{h_l} - u_{h_{l-1}}]$ by the MC estimator with M_l i.i.d. samples on sub-level l. Hence, we estimate $\mathbb{E}[u]$ by

$$E^L[u_{h_L}] = \sum_{l=1}^{L} E_{M_l}[u_{h_l} - u_{h_{l-1}}],$$

where the samples over all levels are independent of each other.

Theorem 4. *Let assumptions A1-A2 hold, then:*

$$\left\|\mathbb{E}[u] - E^L[u_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))} \leq Ch_L^{k+1} + C\sum_{l=1}^{L} M_l^{-1/2} h_l^{k+1},$$
$$\left\|\mathbb{E}[\boldsymbol{q}] - E^L[\boldsymbol{q}_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))} \leq Ch_L^{k+1} + C\sum_{l=1}^{L} M_l^{-1/2} h_l^{k+1},$$

where the constant C is independent of h_l and M, $l = 1, \ldots, L$.

Proof. By the triangle inequality, we obtain:

$$\begin{aligned}
&\left\|\mathbb{E}[u] - E^L[u_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))} \\
&= \left\|\mathbb{E}[u] - \mathbb{E}[u_L] + \mathbb{E}[u_L] - E^L[u_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))} \\
&\leq \|\mathbb{E}[u] - \mathbb{E}[u_L]\|_{L^2(\Omega;L^2(D_h))} + \left\|\mathbb{E}[u_L] - E^L[u_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))} \\
&\leq \|\mathbb{E}[u] - \mathbb{E}[u_L]\|_{L^2(\Omega;L^2(D_h))} + \left\|\sum_{l=1}^{L}(\mathbb{E}[u_{h_l} - u_{h_{l-1}}] - E_{M_l}[u_{h_l} - u_{h_{l-1}}])\right\|_{L^2(\Omega;L^2(D_h))} \\
&:= I + II.
\end{aligned}$$

For estimating I, similarly to the proof of Theorem 2, we have:

$$\|\mathbb{E}[u] - \mathbb{E}[u_L]\|_{L^2(\Omega;L^2(D_h))} \leq Ch_L^{k+1}.$$

For II, using the triangle inequality, we obtain:

$$\begin{aligned}
&\left\|\sum_{l=1}^{L}\Big(\mathbb{E}[u_{h_l} - u_{h_{l-1}}] - E_{M_l}[u_{h_l} - u_{h_{l-1}}]\Big)\right\|_{L^2(\Omega;L^2(D_h))} \\
&\leq \sum_{l=1}^{L}\left\|\mathbb{E}[u_{h_l} - u_{h_{l-1}}] - E_{M_l}[u_{h_l} - u_{h_{l-1}}]\right\|_{L^2(\Omega;L^2(D_h))} \\
&= \sum_{l=1}^{L}\left\|(\mathbb{E} - E_{M_l})[u_{h_l} - u_{h_{l-1}}]\right\|_{L^2(\Omega;L^2(D_h))} \\
&\leq \sum_{l=1}^{L} M_l^{-1/2}\left\|u_{h_l} - u_{h_{l-1}}\right\|_{L^2(\Omega;L^2(D_h))} \\
&\leq \sum_{l=1}^{L} M_l^{-1/2}\left(\|u - u_{h_l}\|_{L^2(\Omega;L^2(D_h))} + \left\|u - u_{h_{l-1}}\right\|_{L^2(\Omega;L^2(D_h))}\right) \\
&\leq \sum_{l=1}^{L} M_l^{-1/2}(h_l^{k+1} + h_{l-1}^{k+1})\|u\|_{L^2(\Omega;L^2(D_h))} \\
&\leq C\sum_{l=1}^{L} M_l^{-1/2} h_l^{k+1}\|u\|_{L^2(\Omega;L^2(D_h))}.
\end{aligned}$$

Combing the estimates of I and II, we obtain the desired result:

$$\left\|\mathbb{E}[u] - E^L[u_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))} \leq Ch_L^{k+1} + C\sum_{l=1}^{L} M_l^{-1/2} h_l^{k+1}.$$

Analogously, we have:

$$\left\|\mathbb{E}[\boldsymbol{q}] - E^L[\boldsymbol{q}_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))} \leq Ch_L^{k+1} + C\sum_{l=1}^{L} M_l^{-1/2} h_l^{k+1}.$$

The proof is complete. □

Then, we give the error bounds of MLMC-HDG approximation for any distribution $\{M_l\}_{l=1}^L$ over the mesh levels. Just like the single level MC-HDG approximation, we are interested in the optimal ratio of sample size versus grid size in every level. To achieve the overall convergence rate, how to select the sampling number M_l is given in the following theorem.

Theorem 5. *Suppose assumptions A1-A2 hold. If the sampling number:*

$$M_l = \mathcal{O}(l^{(2+2\epsilon)} 2^{2(k+1)(L-l)} h_0) \tag{22}$$

on sub-level l, $l = 1, 2, \ldots, L$, then we have the error bounds:

$$\left\|\mathbb{E}[u] - E^L[u_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))} \leq Ch_L^{k+1},$$
$$\left\|\mathbb{E}[\boldsymbol{q}] - E^L[\boldsymbol{q}_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))} \leq Ch_L^{k+1},$$

where the constant C is independent of h_l and M, $\epsilon > 0$ is arbitrarily small. If each sample u_l^i ($\boldsymbol{q}_l^i$) in the estimator $E_{M_l}[u_l]$ ($E_{M_l}[\boldsymbol{q}_l]$) is approximated with accuracy $\mathcal{O}(h_l^{k+1})$ at each sub-level l, the total cost for computing $E^L[u]$ ($E^L[\boldsymbol{q}]$) is:

$$Cost(L) \leq C_\epsilon N_L^{k+1},$$

where the constant C_ϵ depends on ϵ but is independent of level L.

Proof. We only give the detailed proof of: $\left\|\mathbb{E}[u] - E^L[u_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))}$, since the proof of $\left\|\mathbb{E}[\boldsymbol{q}] - E^L[\boldsymbol{q}_{h_L}]\right\|_{L^2(\Omega;L^2(D_h))}$ is similar.

To obtain the overall convergence rate $\mathcal{O}(h_L^{k+1})$, we choose:

$$M_l = O(l^{(2+2\epsilon)}(h_l/h_L)^{2(k+1)}) = \mathcal{O}(l^{(2+2\epsilon)} 2^{2(k+1)(L-l)}), \quad l = 1, 2, \ldots, L,$$

for some $\epsilon > 0$. Using Theorem 4, we obtain:

$$\begin{aligned}\sum_{l=1}^{L} M_l^{-1/2} h_l^{k+1} &\leq \sum_{l=1}^{L} l^{-(1+\epsilon)} 2^{(k+1)(l-L)} (2^{-l} h_0)^{k+1} \\ &\leq 2^{-(k+1)L} h_0^{k+1} \sum_{l=1}^{L} l^{-(1+\epsilon)} \\ &\leq h_L^{k+1}.\end{aligned}$$

Similar to the standard finite element, the computational cost of the HDG method is of linear complexity in the number N_l on every sub-level l (see, e.g., [24]). For M_l samples, the cost is $\mathcal{O}(M_l \cdot N_l)$. Hence, we have the following bound of overall cost at level L:

$$\begin{aligned} Cost(L) &\leq \sum_{l=1}^{L} M_l \cdot N_l \leq \sum_{l=1}^{L} l^{(2+2\epsilon)} 2^{2(k+1)(L-l)} 2^{2l} \\ &= 2^{2L} \sum_{l=1}^{L} l^{(2+2\epsilon)} 2^{2(k+1)(L-l)} 2^{2(l-L)} = 2^{2L} \sum_{l=1}^{L} l^{(2+2\epsilon)} 2^{2k(L-l)} \\ &\leq N_L \sum_{l'=0}^{L-1} (L-l')^{(2+2\epsilon)} 2^{2kl'} \\ &\leq C_\epsilon N_L^{k+1}. \end{aligned}$$

We complete the proof. □

5. Numerical Experiments

To conform the results of the approximation error and computational cost given in Theorem 5, we present a numerical example. The diffusion coefficient and exact solution are selected as follows:

$$\begin{aligned} a(\omega, x) &= (8+\omega)\sin(x_1 x_2) \\ u(\omega, x) &= (8+\omega)\sin(2\pi x_1)\sin(2\pi x_2) \end{aligned}$$

where ω obeys a uniform distribution on $[0,1]$ and $x = (x_1, x_2) \in [0,1]^2$. The homogeneous Dirichlet boundary condition and source term are chosen to match the exact solution. The expectation of the solution is $\mathbb{E}[u] = \frac{17}{2}\sin(2\pi x_1)\sin(2\pi x_2)$. We take $k = 1$ and apply the MLMC-HDG method to simulate the example. All numerical experiments are carried out by using MATLAB R2018b software on an Intel Core i5 machine with 8 GB of memory.

We choose the number of samples with $M_l = \mathcal{O}(l^{(2+2\epsilon)} 2^{4(L-l)})$ on every sub-level l. The convergence rate of the MLMC-HDG approximation depends on the sub-level l, which is displayed in Figure 1. Figure 2 shows the total CPU-time of $E^L(u)$ and $E^L(\boldsymbol{q})$ for different levels L. The CPU-time of sub-level l is plotted in Figure 3, $l = 1, \ldots, L$.

We can see in Figure 1 that $\mathbb{E}\left[\|u - u_h\|_{L^2(D_h)}\right]$ and $\mathbb{E}\left[\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(D_h)}\right]$ converge with h^2. We can observe that the computational cost is $Cost(L) \leq N_L^2$ in Figure 2. It shows that the CPU-time is $\mathcal{O}(l^2 2^{-4l})$ on each level l from Figure 3.

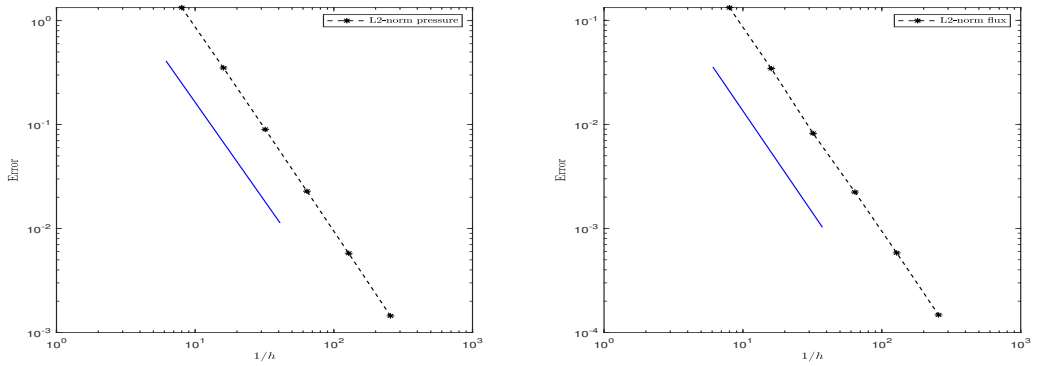

Figure 1. (**Left**): Plot of $\mathbb{E}[\|u - u_h\|_{L^2(D_h)}]$ versus $1/h$. The slope of the line is -2. (**Right**): Plot of $\mathbb{E}[\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(D_h)}]$. The slope of the line is -2.

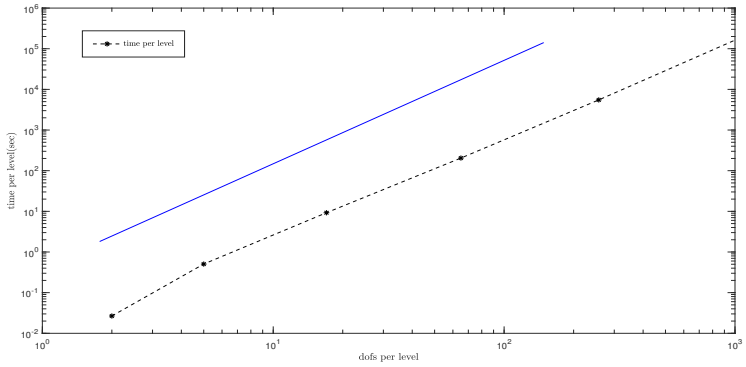

Figure 2. Total CPU time for the MLMC-HDG approximation of $E^L(u)$ and $E^L(\boldsymbol{q})$.

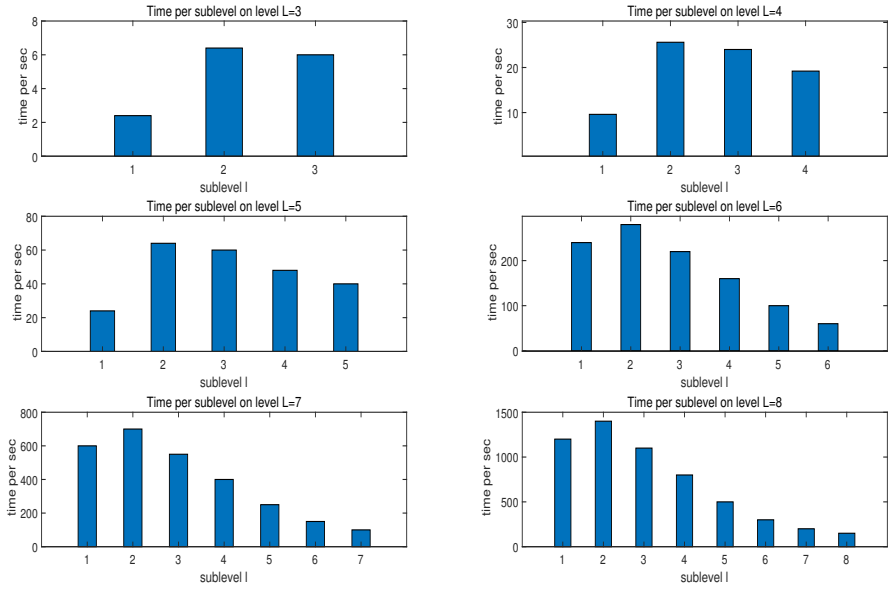

Figure 3. CPU time for different levels for the MLMC-HDG approximation of $E^L(u)$ and $E^L(\boldsymbol{q})$.

6. Discussion

We first consider numerical solutions of stochastic PDEs based on the MLMC-HDG method. Under assumptions A1-A2, we obtain some error and complexity results on the numerical approximation. If assumption A1 is changed to $P\big(\omega \in \Omega : 0 < \text{essinf}_{x\in D} a(\omega, x) \leq \|a(\omega, x)\|_{L^\infty(D)} < \infty\big) = 1$, the regularity of u will only be $L^2(\Omega, H^{1+s}(D))$, where $0 < s < 1$. Consequently, the space discrete error $||u(\omega, \cdot) - u_h(\omega, \cdot)||_{L^2(D)}$ will be no better than Ch^{1+s}.

The study of HDG methods is active, since it can capture the discontinuity property well and keep the conservation property. The MLMC method is a good variance reduction technique. The combination of HDG and MLMC methods is a good choice for numerical approximation on PDEs with random coefficients. The MLMC-HDG method can be used in the numerical simulation of evolution equation with random diffusion coefficients. Furthermore, the combination of HDG and multilevel quasi-Monte Carlo methods is also a good topic for stochastic PDEs.

Author Contributions: Conceptualization, M.L.; methodology, M.L. and X.L.; software, M.L.; validation, M.L. and X.L.; formal analysis, M.L.; investigation, M.L.; writing—original draft preparation, M.L.; writing—review and editing, M.L. and X.L.; visualization, M.L.; supervision, X.L.; project administration, X.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by NSF of China grant number 11961008 and 11461013.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The study did not report any data.

Acknowledgments: The authors thank Sun Shuyu's suggestions, which significantly improved this manuscript.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

References

1. Holden, H.; Oksendal, B.; Uboe, J.; Zhang, T. *Stochastic Partial Differential Equations—A Modeling, White Noise Functional Approach*, 2nd ed.; Springer: New York, NY, USA, 2010; pp. 213–257.
2. Ragusa, M.A. Regularity of solutions of divergence form elliptic equations. *Proc. Am. Math. Soc.* **2000**, *128*, 533–540.
3. Roberts, J.B.; Spanos, P.D. *Random Vibration and Statistical Linearization*; Dover Pubilications: New York, NY, USA, 2003; pp. 1–57. [CrossRef]
4. Zakaria, A. Stochastic system for generalized polytropic filtration. *Math. Appl. Sci.* **2020**, *43*, 134–173. [CrossRef]
5. Babuška, I.; Nobile, F.; Tempone, R. A stochastic collocation method for elliptic partial differential equations with random input data. *SIAM Rev.* **2010**, *52*, 317–335.
6. Babuška, I.; Tempone, R.; Zouraris, G. Galerkin finite element approximations of stochastic elliptic partial differential equations. *SIAM J. Numer. Anal.* **2004**, *42*, 800–825. [CrossRef]
7. Barth, A.; Schwab, C.; Zollinger, N. Multi-level Monte Carlo finite element method for elliptic PDEs with stochastic coefficients. *Numer. Math.* **2011**, *119*, 123–161. [CrossRef]
8. Bespalov, A.; Powell, C.E.; Silvester, D. A priori error analysis of stochastic Galerkin mixed approximations of elliptic PDEs with random data. *SIAM J. Numer. Anal.* **2012**, *50*, 2039–2063. [CrossRef]
9. Lord, G.J.; Powell, C.E.; Shardlow, T. *An Introduction to Computational Stochastic PDEs*; Cambridge University Press: New York, NY, USA, 2014; pp. 396–421. [CrossRef]
10. Caflish, R.E. Monte Carlo and quasi-Monte Carlo methods. *Acta Numer.* **2007**, *7*, 1–49. [CrossRef]
11. Giles, M.B. Multilevel Monte Carlo methods. *Acta Numer.* **2015**, *24*, 259–328.
12. Gunzburger, M.D.; Webster, C.G.; Zhang, G. Stochastic finite element methods for partial differential equations with random input data. *Acta Numer.* **2014**, *23*, 521–650. [CrossRef]
13. Koley, U.; Ray, D.; Sarkar, T. Multilevel Monte Carlo Finite Difference Methods for Fractional Conservation Laws with Random Data. *SIAM/ASA J. Uncertain. Quantif.* **2021**, *9*, 65–105. [CrossRef]
14. Teckentrup, A.L.; Scheichl, R.; Giles, M.B.; Ullmann, E. Further analysis of multilevel Monte Carlo methods for elliptic PDEs with random coefficients. *Numer. Math.* **2013**, *125*, 569–600. [CrossRef]
15. Cockburn, B.; Gopalakrishnan, J.; Lazarov, R. Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems. *SIAM J. Numer. Anal.* **2009**, *47*, 1319–1365. [CrossRef]
16. Cockburn, B.; Dong, B.; Guzmán, J. A superconvergent LDG-hybridizable Galerkin method for second-order elliptic problems. *Math. Comp.* **2008**, *77*, 1887–1916. [CrossRef]
17. Cockburn, B.; Gopalakrishnan, J.; Sayas, F.J. A projection-based error analysis of HDG methods. *Math. Comput.* **2010**, *79*, 1351–1367. [CrossRef]
18. Chen, G.; Monk, P.B.; Zhang, Y.W. L^∞ norm error estimates for HDG methods applied to the Poisson equation with an application to the Dirichlet boundary control problem. *SIAM J. Numer. Anal.* **2021**, *59*, 720–745. [CrossRef]
19. Chen, G.; Pi, L.Y.; Xu, L.W.; Zhang, Y.W. A superconvergence ensemble HDG method for parameterized convection diffusion equations. *SIAM J. Numer. Anal.* **2019**, *57*, 2551–2578. [CrossRef]
20. Cockburn, B.; Singler, J.R.; Zhang, Y. Interpolatory HDG Method for Parabolic Semilinear PDEs. *J. Sci. Comput.* **2019**, *79*, 1777–1800. [CrossRef]
21. Costa-Solé, A.; Ruiz-Girones, E.; Sarrate, J. High-Order Hybridizable Discontinuous Galerkin Formulation for One-Phase Flow Through Porous Media. *J. Sci. Comput.* **2021**, *87*. [CrossRef]
22. Yu, Y.; Chen, G.; Pi, L.Y.; Zhang, Y.W. A new ensemble HDG method for parameterized convection diffusion PDEs. *Numer. Math. Theory Methods Appl.* **2021**, *14*, 144–175. [CrossRef]
23. Ciarlet, P. *The Finite Element Method for Elliptic Problems*; North-Holland: Amsterdam, The Netherlands, 1978; pp. 123–126. [CrossRef]
24. Brenner, S.C.; Scott, L.R. *The Mathematical Theory of Finite Element Methods*, 3rd ed.; Springer: Berlin, Germany, 2007; pp. 155–170. [CrossRef]