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On the Oscillatory Properties of Solutions of Second-Order Damped Delay Differential Equations

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Abstract: In the work, a new oscillation condition was created for second-order damped delay differential equations with a non-canonical operator. The new criterion is of an iterative nature which helps to apply it even when the previous relevant results fail to apply. An example is presented in order to illustrate the significance of the results.

Keywords: differential equations; second-order; delay; iterative criteria; oscillation

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1. Introduction

In this study, we focus on studying the oscillatory properties of solutions to the delay differential equation (DDE)

$$\left(a_0 \cdot \left(\nu'\right)^{\beta}\right)' + a_1 \cdot \left(\nu'\right)^{\beta} + a_2 \cdot \left(\nu^{\beta} \circ g\right) = 0, \tag{1}$$

where $l \in \mathbf{I}_0 := [l_0, \infty)$, and under the following hypotheses:

Hypothesis 1 (H1). $\beta > 0$ *is a quotient of two odd integers.*

Hypothesis 2 (H2). $a_i \in C(\mathbf{I}_0, [0, \infty))$ *for* $i = 0, 1, 2$, $a_0(l) > 0$, *and* $a_2 \neq 0$ *on any half-line* $[l_*, \infty)$, $l_* \in \mathbf{I}_0$.

Hypothesis 3 (H3). $g \in C(\mathbf{I}_0, \mathbb{R})$, $g(l) \leq l$, $g'(l) \geq 0$, *and* $\lim_{l \to \infty} g(l) = \infty$.

By a solution of (1), we go to a $\nu \in C^1(\mathbf{I}_0)$ with $a_0 \cdot (\nu')^{\beta} \in C^1(\mathbf{I}_0)$ and $\sup\{|\nu(l)| : l \geq l_*\} > 0$, for $l_* \in \mathbf{I}_0$,and ν satisfies (1) on $\mathbf{I}_0$. A solution ν of (1) is called non-oscillatory if it is eventually positive or eventually negative; otherwise, it is called oscillatory.

DDEs, as a subclass of the functional differential equation (FDE), take into account the system's past, allowing for more accurate and efficient future prediction while also describing certain qualitative phenomena such as periodicity, oscillation, and instability. The concept of delay incorporation into systems is now proposed to play an important role in modeling when representing the time it takes to complete certain secret processes, see [1]. DDE theory has improved our understanding of the qualitative behavior of their solutions and has a wide range of applications in mathematical biology and other fields. DDE nonlinearity and sensitivity analysis has been extensively studied in recent years in a variety of fields, see [2–6].

The problem of determining oscillation criteria for specific FDEs has been a very active research field in the recent decades, and many references and summaries of known results can be found in the monographs by Agarwal et al. [7,8] and Gyori and Ladas [9].

The following is a review of the most important results that dealt with the oscillatory behavior of solutions of DDEs with damping term.

By many authors, the oscillation of the ordinary differential equation with damping term

$$\left(r(l)v'(l)\right)' + q_1(l)v'(l) + q_2(l)f(v(l)) = 0, \tag{2}$$

has been investigated. The existence of a damping term in differential equations necessarily requires an improved approach at the study of oscillatory behavior. Among the works that dealt with the oscillation of (2) are, for example, Grace [10,11], Grace and Lalli [12,13], Grace et al. [14], Kirane and Rogovechenkov [15], Li and Agarwal [16], Li and Zhang [17], Rogovechenkov [18], Wong [19], and Yan [20]. However, the common restriction $f'(v) \geq k > 0$ is required in all previous works. Grace [21] studied the oscillation of DDE

$$\left(r(l)v'(l)\right)' + p(l)v(\sigma(l)) + q(l)f(v(g(l))) = 0, \tag{3}$$

with the canonical case.

Theorem 1. *[21] If $\sigma(l) < l$, $\sigma'(l) > 0$, and there is a function $\rho \in C^1([l_0,\infty),\mathbb{R}^+)$ with $\rho'(l) > 0$,*

$$\left(\frac{\rho(l)p(l)}{\sigma'(l)}\right) \leq 0, \tag{4}$$

$$\int_{l_0}^{\infty} \rho(w)q(w)\mathrm{d}w = \infty, \tag{5}$$

$$\liminf_{l\to\infty} \int_{\sigma(l)}^{l} \frac{p(w)}{r(\sigma(w))}\mathrm{d}w > \frac{1}{\mathrm{e}},$$

and

$$\int_{l_0}^{\infty} \frac{1}{r(w)\rho(w)}\left(\int_{l_0}^{w} \rho(u)q(u)\mathrm{d}u\right)\mathrm{d}w = \infty, \tag{6}$$

then every solution of (3) is oscillatory or converges to zero.

In the following Theorem, Saker et al. [22], improved the result of [21].

Theorem 2. *[22] Assume that $g'(l) > 0$, $\sigma'(l) > 0$, $r(l) > 0$, $q(l) > 0$ and $\frac{f(u)}{u} \geq k$ such that (4), (5) and (6) hold. If*

$$\limsup_{l\to\infty} \int_{l_0}^{l} \left(\rho(w)q(w) - \frac{(\rho'(w))^2 r(g(w))}{4k\rho(w)g'(w)}\right)\mathrm{d}w = \infty,$$

then every solution of (3) is oscillatory or converges to zero.

Tunc and Kaymaz [23] established the oscillatory properties of DDE

$$z''(l) + h(l)z'(l) + q(l)v(g(l)) = 0, \tag{7}$$

where $z(l) = v(l) + p(l)v(\tau(l))$, under the condition

$$\int_{l_0}^{\infty} \exp\left(-\int_{l_0}^{l} h(w)\mathrm{d}w\right)\mathrm{d}l = \infty. \tag{8}$$

Theorem 3. [23] *Assume that* $\sigma(l) \leq \tau(l)$ *and* (8) *hold. If there exists a positive function* $\rho \in C^1([l_0,\infty),\mathbb{R}^+)$ *such that*

$$\limsup_{l\to\infty} \int_{l_0}^{l} \left[\rho(w)q(w)\psi(\sigma(w))\frac{\tau^{-1}(\sigma(w))}{w} - \frac{\rho(w)\zeta^2(w)}{4}\right] \mathrm{d}w = \infty,$$

where

$$\zeta(l) = \frac{\rho'(l)}{\rho(l)} - h(l),$$

and

$$\psi(l) := \frac{1}{p(\tau^{-1}(l))}\left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(l)))}\frac{\tau^{-1}\left(\tau^{-1}(l)\right)}{\tau^{-1}(l)}\right),$$

then every solution of (7) *is oscillatory*

In an attempt to reduce the number of possible possibilities for the sign of derivatives of positive solutions, researchers study the DDEs in the canonical case, which often excludes the existence of positive decreasing solutions. On the other hand, in the noncanonical case, one of these possibilities is that the positive solutions are decreasing. The main reason for the difficulty of studying positive decreasing solutions is the probability of their convergence to zero, and this probability prevents the use of one of the most important relationships between derivatives that allows to reduce the order of the equation. It has also been noted that the conditions resulting from the exclusion of positive decreasing solutions have the largest effect on the oscillation criteria. Therefore, the main objective of this work is to study the oscillatory behavior of DDE (1) in the noncanonical case

$$\int_{l_0}^{\infty} \frac{1}{a_0^{1/\beta}(h)} \exp\left(\frac{-1}{\beta}\int_{l_0}^{h}\frac{a_1(w)}{a_0(w)}\mathrm{d}w\right)\mathrm{d}h < \infty. \tag{9}$$

The technique used is based on obtaining criteria of an iterative nature through establishing more sharp estimates for the $a_2(l)a_0^{1/\beta}(l)\eta^{1+1/\beta}(l)\widetilde{\eta}^{\beta+1}(l)$. The iterative nature of the criteria allows us to apply them more than once, even when the other criteria fail.

2. Main Results

For ease of presentation of results, we present the next notations:

$$V^+ := \{x(t):\ x(t) > 0 \text{ is a solutions of } (1)\},$$

$$\eta(l) := \exp\left(\int_{l_0}^{l} a_1(w)a_0^{-1}(w)\mathrm{d}w\right) \text{ and } \widetilde{\eta}(l) := \int_{l}^{\infty}(a_0(w)\eta(w))^{-1/\beta}\mathrm{d}w.$$

Lemma 1. *Assume that* $v \in V^+$ *and*

$$\int_{l_1}^{\infty}\left(\frac{1}{a_0(z)\eta(z)}\int_{l_1}^{z} a_2(w)\eta(w)\mathrm{d}w\right)^{1/\beta}\mathrm{d}z = \infty. \tag{10}$$

Then

($\mathbf{B}_1$) *v is decreasing and converges to zero;*

($\mathbf{B}_2$) *$\frac{v}{\widetilde{\eta}}$ is increasing.*

Proof. Assume that $\nu \in V^+$. Then, we have that $\nu(l)$ and $\nu(g(l))$ are positive for all $l \geq l_1$, for some $l_1 \geq l_0$. Therefore, it follows from (1) that

$$\begin{aligned} 0 &\geq -a_2(l)\eta(l)\left(\nu^\beta(g(l))\right) \\ &= \eta(l)\left(a_0(l)(\nu'(l))^\beta\right)' + a_1(l)\eta(l)(\nu'(l))^\beta \\ &= \left(a_0(l)\eta(l)(\nu'(l))^\beta\right)'. \end{aligned} \tag{11}$$

Hence, $a_0(l)\eta(l)(\nu'(l))^\beta$ is of one sign. Suppose the contrary that $\nu'(l) > 0$ for $l \geq l_2 \geq l_1$, for some l_2. Thus, there is a $c > 0$ such eventually that $\nu(l) > c$. Integrating (11) from l_2 to ∞, we obtain

$$\begin{aligned} a_0(l_2)\eta(l_2)(\nu'(l_2))^\beta &\geq \int_{l_2}^\infty a_2(w)\eta(w)\left(\nu^\beta(g(w))\right)\mathrm{d}w \\ &\geq c^\beta \int_{l_2}^\infty a_2(w)\eta(w)\mathrm{d}w. \end{aligned} \tag{12}$$

From (10) and the fact that $\widetilde{\eta}'(l) < 0$, we have that

$$\int_{l_2}^\infty a_2(w)\eta(w)\mathrm{d}w = \infty,$$

which with (12) gives a contradiction, and so $\nu'(l) < 0$, eventually.
Now, we have that ν is positive decreasing, and then $\lim_{l\to\infty} \nu(l) = k \geq 0$. Suppose that $k > 0$. Then, $\nu(l) \geq k > 0$, eventually. Hence, integrating (11) from l_1 to l, we obtain

$$-a_0(l)\eta(l)(\nu'(l))^\beta \geq \int_{l_1}^l a_2(w)\eta(w)\left(\nu^\beta(g(w))\right)\mathrm{d}w$$

or

$$-\nu'(l) \geq k\left(\frac{1}{a_0(l)\eta(l)}\int_{l_1}^l a_2(w)\eta(w)\mathrm{d}w\right)^{1/\beta}. \tag{13}$$

Integrating (13) from l_1 to ∞, we arrive at

$$\nu(l_1) - k \geq k\int_{l_1}^\infty \left(\frac{1}{a_0(z)\eta(z)}\int_{l_1}^z a_2(w)\eta(w)\mathrm{d}w\right)^{1/\beta}\mathrm{d}z, \tag{14}$$

which contradicts (10). Then, ν converges to zero.
Finally, we have

$$\begin{aligned} -(a_0(s)\eta(s))^{-1/\beta}\nu(s) &= (a_0(s)\eta(s))^{-1/\beta}\int_s^\infty \frac{(a_0(w)\eta(w))^{1/\beta}\nu'(w)}{(a_0(w)\eta(w))^{1/\beta}}\mathrm{d}w \\ &\leq \nu'(s)\widetilde{\eta}(s). \end{aligned} \tag{15}$$

Therefore,

$$\left(\frac{\nu}{\widetilde{\eta}}\right)' = \frac{1}{\widetilde{\eta}^2}\left(\nu'\widetilde{\eta} + (a_0\eta)^{-1/\beta}\nu\right) \geq 0.$$

The proof is complete. □

Lemma 2. *Assume that $\nu \in V^+$, (10) holds, and*

$$\kappa := \liminf_{l\to\infty}\frac{\widetilde{\eta}(g(l))}{\widetilde{\eta}(l)} < \infty. \tag{16}$$

If there is a positive constant $\delta_0 \in (0,1)$ such that

$$a_2(l)a_0^{1/\beta}(l)\eta^{1+1/\beta}(l)\widetilde{\eta}^{\beta+1}(l) \geq \beta\delta_0. \tag{17}$$

Then,

$(\mathbf{B}_{1,m})$ $\dfrac{\nu}{\widetilde{\eta}^{\delta_m}}$ *is decreasing and converges to zero;*

$(\mathbf{B}_{2,m})$ $\dfrac{\nu}{\widetilde{\eta}^{1-\delta_m}}$ *is increasing,*

for $m = 0,1,...,$ where

$$\delta_{m+1} := \delta_0 \frac{\kappa^{\delta_m}}{1-\delta_m}. \tag{18}$$

Proof. Assume that $\nu \in V^+$. Then, we have that $\nu(l)$ and $\nu(g(l))$ are positive for all $l \geq l_1$, for some $l_1 \geq l_0$. From Lemma 1, we have that $(\mathbf{B}_1)$, $(\mathbf{B}_2)$, (11) and (15) hold. Using (15) and the fact that $\nu'(l) < 0$, we get that

$$-\frac{\nu(g(l))}{\widetilde{\eta}(l)} \leq -\frac{\nu(l)}{\widetilde{\eta}(l)} \leq (a_0(l)\eta(l))^{1/\beta}\nu'(l). \tag{19}$$

From (11), (17) and (19), we have

$$\begin{aligned}
\left(a_0^{1/\beta}(l)\eta^{1/\beta}(l)\nu'(l)\right)' &= -\frac{1}{\beta}a_2(l)\eta(l)\left(a_0^{1/\beta}(l)\eta^{1/\beta}(l)\nu'(l)\right)^{1-\beta}\nu^{\beta}(g(l)) \\
&\leq -\frac{1}{\beta}a_2(l)\eta(l)\left(\frac{\nu(g(l))}{\widetilde{\eta}(l)}\right)^{1-\beta}\nu^{\beta}(g(l)) \\
&= -\frac{1}{\beta}a_2(l)\eta(l)\widetilde{\eta}^{\beta-1}(l)\nu(g(l)) &(20)\\
&\leq -\frac{\delta_0}{a_0^{1/\beta}(l)\eta^{1/\beta}(l)\widetilde{\eta}^2(l)}\nu(l). &(21)
\end{aligned}$$

Firstly, at $m = 0$, integrating (21) from l_1 to l, we obtain

$$\begin{aligned}
a_0^{1/\beta}(l)\eta^{1/\beta}(l)\nu'(l) &\leq a_0^{1/\beta}(l_1)\eta^{1/\beta}(l_1)\nu'(l_1) \\
&\quad -\int_{l_1}^{l}\frac{\delta_0}{a_0^{1/\beta}(w)\eta^{1/\beta}(w)\widetilde{\eta}^2(w)}\nu(w)\mathrm{d}w \\
&\leq a_0^{1/\beta}(l_1)\eta^{1/\beta}(l_1)\nu'(l_1) + \delta_0\frac{\nu(l)}{\widetilde{\eta}(l_1)} - \delta_0\frac{\nu(l)}{\widetilde{\eta}(l)} &(22)
\end{aligned}$$

From $(\mathbf{B}_1)$, we have that ν converges to *zero*, and then

$$a_0^{1/\beta}(l_1)\eta^{1/\beta}(l_1)\nu'(l_1) + \delta_0\frac{\nu(l)}{\widetilde{\eta}(l_1)} \leq 0,$$

eventually. Thus, from (22), we obtain

$$a_0^{1/\beta}(l)\eta^{1/\beta}(l)\nu'(l) \leq -\delta_0\frac{\nu(l)}{\widetilde{\eta}(l)}. \tag{23}$$

This implies

$$\left(\frac{\nu}{\widetilde{\eta}^{\delta_0}}\right)' = \frac{1}{\widetilde{\eta}^{\delta_0+1}}\left(\nu'\widetilde{\eta} + \delta_0(a_0\eta)^{-1/\beta}\nu\right) \leq 0.$$

Now, $\nu/\widetilde{\eta}^{\delta_0}$ is positive decreasing, and then $\lim_{l\to\infty}\left(\nu(l)/\widetilde{\eta}^{\delta_0}(l)\right) = k_0 \geq 0$. Suppose that $k_0 > 0$. Then, $\nu(l)/\widetilde{\eta}^{\delta_0}(l) \geq k_0 > 0$, eventually. If we define the function

$$H := \frac{1}{\widetilde{\eta}^{\delta_0}}\left(\nu + \widetilde{\eta}(a_0\eta)^{1/\beta}\nu'\right),$$

then it follows from $(\mathbf{B}_1)$ that $H(l) > 0$, and

$$\begin{aligned} H'(l) &= \frac{1}{\widetilde{\eta}^{\delta_0+1}(l)}\Big[\widetilde{\eta}^2(l)\left((a_0(l)\eta(l))^{1/\beta}\nu'(l)\right)' \\ &\quad +\delta_0(a_0(l)\eta(l))^{-1/\beta}\left(\nu(l) + \widetilde{\eta}(l)(a_0(l)\eta(l))^{1/\beta}\nu'(l)\right)\Big], \end{aligned}$$

which, with (21) and (23) and the fact that $\nu(l)/\widetilde{\eta}^{\delta_0}(l) \geq k_0$, gives

$$H'(l) \leq \frac{\delta_0}{\widetilde{\eta}^{\delta_0}(l)}\nu'(l) \leq -\frac{\delta_0^2}{a_0^{1/\beta}(l)\eta^{1/\beta}(l)\widetilde{\eta}^{\delta_0+1}(l)}\nu(l) \leq -\frac{k_0\delta_0^2}{a_0^{1/\beta}(l)\eta^{1/\beta}(l)\widetilde{\eta}(l)} \leq 0.$$

Integrating this inequality from l_1 to ∞, we obtain

$$H(l_1) \geq k_0\delta_0^2 \ln\frac{\widetilde{\eta}(l_1)}{\widetilde{\eta}(l)}.$$

Letting $l \to \infty$, we arrive at a contradiction, and then $\nu/\widetilde{\eta}^{\delta_0}$ converges to zero.
Next, from (21), we have

$$\frac{1}{\widetilde{\eta}(l)}\left(a_0^{1/\beta}(l)\eta^{1/\beta}(l)\widetilde{\eta}(l)\nu'(l) + \nu(l)\right)' \leq -\frac{\delta_0}{a_0^{1/\beta}(l)\eta^{1/\beta}(l)\widetilde{\eta}^2(l)}\nu(l).$$

Integrating this inequality from l to ∞ and using $(\mathbf{B}_2)$, we obtain

$$\begin{aligned} a_0^{1/\beta}(l)\eta^{1/\beta}(l)\widetilde{\eta}(l)\nu'(l) + \nu(l) &\geq \int_l^\infty \frac{\delta_0}{a_0^{1/\beta}(w)\eta^{1/\beta}(w)}\frac{\nu(w)}{\widetilde{\eta}(w)}\mathrm{d}w \\ &\geq \delta_0\nu(l). \end{aligned}$$

Therefore,

$$\left(\frac{\nu(l)}{\widetilde{\eta}^{1-\delta_0}(l)}\right)' = \frac{1}{\widetilde{\eta}^{2-\delta_0}(l)}\left(\widetilde{\eta}(l)\nu'(l) + (1-\delta_0)a_0^{-1/\beta}(l)\eta^{-1/\beta}(l)\nu(l)\right) \geq 0.$$

That is, $(\mathbf{B}_{1,m})$ and $(\mathbf{B}_{2,m})$ are satisfied for $m = 0$.

Secondly, proceeding to the next induction step, we suppose that $(\mathbf{B}_{1,m})$ and $(\mathbf{B}_{2,m})$ hold for some $m > 0$. Using $(\mathbf{B}_{1,m})$, (20) becomes

$$\left(a_0^{1/\beta}(l)\eta^{1/\beta}(l)\nu'(l)\right)' \leq -\frac{1}{\beta}a_2(l)\eta(l)\widetilde{\eta}^{\beta-1}(l)\widetilde{\eta}^{\delta_m}(g(l))\frac{\nu(l)}{\widetilde{\eta}^{\delta_m}(l)}.$$

Integrating this inequality from l_1 to l and using $(\mathbf{B}_{1,m})$, we find

$$\begin{aligned} a_0^{1/\beta}(l)\eta^{1/\beta}(l)\nu'(l) &\leq a_0^{1/\beta}(l_1)\eta^{1/\beta}(l_1)\nu'(l_1) \\ &\quad -\frac{1}{\beta}\int_{l_1}^{l} a_2(w)\eta(w)\widetilde{\eta}^{\beta-1}(w)\widetilde{\eta}^{\delta_m}(g(w))\frac{\nu(w)}{\widetilde{\eta}^{\delta_m}(w)}\mathrm{d}w \\ &\leq a_0^{1/\beta}(l_1)\eta^{1/\beta}(l_1)\nu'(l_1) \\ &\quad -\frac{1}{\beta}\frac{\nu(l)}{\widetilde{\eta}^{\delta_m}(l)}\int_{l_1}^{l} a_2(w)\eta(w)\widetilde{\eta}^{\beta-1}(w)\widetilde{\eta}^{\delta_m}(g(w))\mathrm{d}w. \end{aligned} \tag{24}$$

From (16), $\widetilde{\eta}(g(l)) \geq \kappa\widetilde{\eta}(l)$, eventually. Thus, (24) turn into

$$\begin{aligned} a_0^{1/\beta}(l)\eta^{1/\beta}(l)\nu'(l) &\leq a_0^{1/\beta}(l_1)\eta^{1/\beta}(l_1)\nu'(l_1) - \frac{\kappa^{\delta_m}}{\beta}\frac{\nu(l)}{\widetilde{\eta}^{\delta_m}(l)}\int_{l_1}^{l} a_2(w)\eta(w)\widetilde{\eta}^{\delta_m+\beta-1}(w)\mathrm{d}w \\ &\leq a_0^{1/\beta}(l_1)\eta^{1/\beta}(l_1)\nu'(l_1) - \kappa^{\delta_m}\delta_0\frac{\nu(l)}{\widetilde{\eta}^{\delta_m}(l)}\int_{l_1}^{l}\frac{\widetilde{\eta}^{\delta_m-2}(w)}{a_0^{1/\beta}(w)\eta^{1/\beta}(w)}\mathrm{d}w \\ &\leq a_0^{1/\beta}(l_1)\eta^{1/\beta}(l_1)\nu'(l_1) + \delta_{m+1}\frac{\nu(l)}{\widetilde{\eta}^{\delta_m}(l)}\widetilde{\eta}^{\delta_m-1}(l_1) - \delta_{m+1}\frac{\nu(l)}{\widetilde{\eta}(l)}. \end{aligned}$$

Since $\frac{\nu}{\widetilde{\eta}^{\delta_m}}$ converges to zero, we have that

$$a_0^{1/\beta}(l_1)\eta^{1/\beta}(l_1)\nu'(l_1) + \delta_{m+1}\frac{\nu(l)}{\widetilde{\eta}^{\delta_m}(l)}\widetilde{\eta}^{\delta_m-1}(l_1) \leq 0,$$

eventually. Thus,

$$a_0^{1/\beta}(l)\eta^{1/\beta}(l)\nu'(l) \leq -\delta_{m+1}\frac{\nu(l)}{\widetilde{\eta}(l)},$$

and hence

$$\left(\frac{\nu(l)}{\widetilde{\eta}^{\delta_{m+1}}(l)}\right)' = \frac{1}{\widetilde{\eta}^{\delta_{m+1}+1}(l)}\left(\widetilde{\eta}(l)\nu'(l) + \delta_{m+1}a_0^{-1/\beta}(l)\eta^{-1/\beta}(l)\nu(l)\right) \leq 0.$$

Proceeding exactly as in the previous step (at $m = 0$), we can verify $(\mathbf{B}_{1,m+1})$ and $(\mathbf{B}_{2,m+1})$. The proof is complete. □

Theorem 4. *Assume that (10) and (16) hold, and that there is a positive constant $\delta_0 \in (0,1)$ such that (17) holds. If $\delta_m > 1/2$ for some $m \in \mathbb{N}$, where δ_m defined as in (18), then all solutions of (1) are oscillatory.*

Proof. If we suppose that (1) has a solution $\nu \in V^+$, then, from Lemma 2, we have $\frac{\nu}{\widetilde{\eta}^{\delta_m}}$ is decreasing and $\frac{\nu}{\widetilde{\eta}^{1-\delta_m}}$ is increasing. Therefore, $\delta_m \leq 1/2$ for all $m = 0, 1, ...$, this is a contradiction. The proof is complete. □

Theorem 5. *Assume that (10) and (16) hold, and that there is a positive constant $\delta_0 \in (0,1)$ such that (17) holds. If, for some $m \in \mathbb{N}$,*

$$\psi'(l) + \frac{1}{\beta(1-\delta_m)}a_2(l)\eta(l)\widetilde{\eta}^{\beta}(l)\psi(g(l)) = 0 \tag{25}$$

is oscillatory, where δ_m defined as in (18), then all solutions of (1) are oscillatory.

Proof. If we suppose that (1) has a solution $\nu \in V^+$, then, from Lemma 2, we have that $(\mathbf{B}_{1,m})$, $(\mathbf{B}_{2,m})$ and (20) hold, for all $m = 0, 1, \ldots$. Now, we define the function

$$\psi := \nu + \widetilde{\eta}(a_0\eta)^{1/\beta}\nu'.$$

As in the proof of Lemma 1, we have that (15) holds, and so $\psi(l) > 0$, eventually. Thus,

$$\psi' = \widetilde{\eta}\left((a_0\eta)^{1/\beta}\nu'\right)'.$$

Hence, from (20), we obtain

$$\psi' \leq -\frac{1}{\beta}a_2\eta\widetilde{\eta}^{\beta}(\nu \circ g). \tag{26}$$

Taking the fact that $\left(v/\widetilde{\eta}^{\delta_m}\right)' < 0$ into account, we obtain

$$\psi(l) - v(l) = \widetilde{\eta}(l)(a_0(l)\eta(l))^{1/\beta}v'(l) < -\delta_m v(l)$$

and then

$$\psi(l) < (1-\delta_m)v(l).$$

Thus, from (26), $\psi(l)$ is a positive solution of the delay differential inequality

$$\psi'(l) + \frac{1}{\beta(1-\delta_m)}a_2(l)\eta(l)\widetilde{\eta}^{\beta}(l)\psi(g(l)) \leq 0.$$

Using Theorem 1 in [24], the associated delay differential (25) has also a positive solution, which contradicts to the assumptions of the theorem. The proof is complete. □

Corollary 1. *Assume that (10) and (16) hold, and that there is a positive constant $\delta_0 \in (0,1)$ such that (17) holds. If, for some $m \in \mathbb{N}$,*

$$\liminf_{l\to\infty} \int_{g(l)}^{l} a_2(w)\eta(w)\widetilde{\eta}^{\beta}(w)\mathrm{d}w > \beta\frac{1-\delta_m}{\mathrm{e}}, \tag{27}$$

where δ_m defined as in (18), then every solution of (1) is oscillatory.

Proof. It follows from Theorem 2 in [25] that condition (27) implies oscillation of (25). □

Example 1. *Consider the differential equation*

$$\left(l^2 \cdot v'\right)' + lv' + a_2^* v(\lambda l) = 0, \tag{28}$$

where $l \geq 1$, $a_2^* > 0$ and $\lambda \in (0,1)$. Note that $\eta(l) = l$, $\widetilde{\eta}(l) := 1/\left(2l^2\right)$, and that (9) and (10) hold. Using Theorem 4, (28) is oscillatory if

$$\delta_{m+1} = \frac{a_2^*}{4\lambda^{2\delta_m}}\frac{1}{1-\delta_m} > \frac{1}{2}, \tag{29}$$

for some $m \in \mathbb{N}$. On the other hand, by Corollary 1, we have that (28) is oscillatory if

$$\frac{a_2^*}{2}\ln\frac{1}{\lambda} > \frac{1-\delta_m}{\mathrm{e}}, \tag{30}$$

for some $m \in \mathbb{N}$.

Remark 1. *Table 1 shows the first value of δ_m which satisfies Condition (29) for different special cases of (28). Note also that, in all cases in Table 1, $\delta_0 < 1/2$, which does not fulfill (29).*

Table 1. The first value of δ_m which satisfies Condition (29).

	a_2^*	λ	δ_0	m	δ_m
(1)	1.00	0.20	0.2500	1	$\delta_1 = 0.745356$
(2)	1.00	0.50	0.2500	2	$\delta_2 = 0.909139$
(3)	0.50	0.20	0.1250	3	$\delta_3 = 0.505743$
(4)	0.15	0.01	0.0375	18	$\delta_{18} = 0.735724$
(5)	0.80	0.50	0.2000	52	$\delta_{52} = 0.728266$

3. Conclusions

We have greatly less results for DDEs with noncanonical operator than for the DDEs with canonical operator. So, in this work, new sufficient conditions for the oscillation of second-order damped DDE with noncanonical operator (1) are established. By inferring and improving some properties of positive solutions, we establish oscillation criteria of an iterative nature. For an overview of the main results, see Figure 1. It would be interesting to extend our results to neutral DDEs.

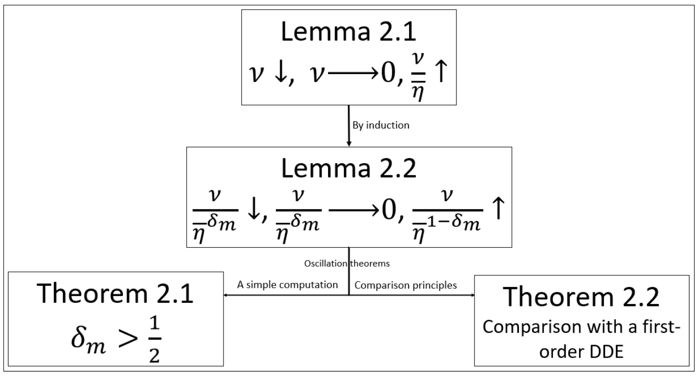

Figure 1. Schematic diagram for main results

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