mathematics

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A Study of Spaces of Sequences in Fuzzy Normed Spaces

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Abstract: In this paper, spaces of sequences in fuzzy normed spaces are considered. These spaces are a new concept in fuzzy normed spaces. We develop fuzzy norms for spaces of sequences in fuzzy normed spaces. Especially, we study the representation of the dual of a space of sequences in a fuzzy normed space. The approximation property in our context is investigated.

Keywords: bounded sequences; null sequences; strong fuzzy dual; approximation property; bounded approximation property

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Citation: Kim, J.-M.; Lee, K.-Y. A Study of Spaces of Sequences in Fuzzy Normed Spaces. *Mathematics* **2021**, *9*, 1040. https://doi.org/10.3390/math9091040

Academic Editor: Gustavo Santos-García

Received: 20 January 2021
Accepted: 28 April 2021
Published: 4 May 2021

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1. Introduction

Since Katsaras first introduced the notion of fuzzy norm on a vector space [1], a study for fuzzy normed spaces has actively progressed. In 1992, Felbin introduced a new definition of a fuzzy norm (namely, Felbin-fuzzy norm, in this paper, fuzzy norm) related to a specific fuzzy metric [2,3]. Because of his pioneering researches, topological properties have been studied according to Felbin type's fuzzy norms [4,5]. Especially, the authors recently introduced the approximation properties in fuzzy normed spaces [6,7].

A study of spaces of sequences in vector spaces is a very important concept to research functional analysis, because these spaces have been investigated for Shauder basis and operator theory [8]. In 1989, Nanda introduced sequences of fuzzy numbers [9]. In 2000, Savaş introduced summable sequences of fuzzy numbers [10–12]. Felbin investigated sequences and their limits in the sense of fuzzy normed spaces [2]. In Felbin's sense, Sencimen and Pehlivan [13] introduced the notions of a statistically convergent sequence and statistically Cauchy sequence in a fuzzy normed linear space. Hazarika [14,15] provided the concepts of I -convergence, I -convergence, and I -Cauchy sequence in a fuzzy normed linear space in terms of general framework. For more researches of sequences in fuzzy normed spaces, we refer to [16–18].

In previous studies, many researchers concentrated on investigating the convergency of sequences in fuzzy normed spaces and their topological properties. However, until now, no research for spaces of sequences in fuzzy normed spaces itself has been done, because it is difficult to define fuzzy norm for spaces of sequences in fuzzy normed space. Because the space of sequences in fuzzy normed spaces itself is very important in fuzzy functional analysis, we need to investigate that space in terms of fuzzy norms.

In this paper, we study spaces of sequences in fuzzy normed spaces. We establish a well-defined fuzzy norm for spaces of sequences in fuzzy normed spaces and its completeness. This is an important contribution of our paper. Moreover, we investigate the fuzzy dual of spaces of sequences in fuzzy normed spaces. We characterize the approximation property for spaces of sequences in fuzzy normed spaces. The contribution of our paper is to make tools for fuzzy analysis, because we characterize the duality and approximation property in the sense of sequences in fuzzy normed spaces.

Our paper is organized, as follows. Section 2 gives some preliminary results. In Section 3, we define fuzzy norms for spaces of sequences in fuzzy normed spaces. Furthermore, we provide their completeness and several examples. In Section 4, we give the representation of the dual space of the space of sequences in a fuzzy normed space. In Section 5, we provide the results of the approximation property for spaces of sequences in fuzzy normed spaces (for reference, see [19]).

2. Preliminaries

Definition 1 (See [2]). *A mapping $\eta : \mathbb{R} \to [0,1]$ is called a fuzzy real number with α-level set $[\eta]_\alpha = \{t : \eta(t) \geqslant \alpha\}$, if it satisfies the following conditions:*

(i) *there exists a $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$*
(ii) *for each $\alpha \in (0,1]$, there exist real numbers $\eta_\alpha^- \leqslant \eta_\alpha^+$, such that the α-level set $[\eta]_\alpha$ is equal to the closed interval $[\eta_\alpha^-, \eta_\alpha^+]$*

The set of all fuzzy real numbers is denoted by $F(\mathbb{R})$. If $\eta \in F(\mathbb{R})$ and $\eta(t) = 0$ whenever $t < 0$, then η is called a non-negative fuzzy real number and $F^*(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers. We note that the real number $\eta_\alpha^- \geqslant 0$ for all $\eta \in F^*(\mathbb{R})$ and all $\alpha \in (0,1]$. Because each $r \in \mathbb{R}$ can be considered as the fuzzy real number $\tilde{r} \in F(\mathbb{R})$ denoted by

$$\tilde{r}(t) = \begin{cases} 1, & t = r \\ 0, & t \neq r, \end{cases}$$

hence, it follows that $\mathbb{R}$ can be embedded in $F(\mathbb{R})$.

Lemma 1 (See [20]). *Let $\eta, \gamma \in F(\mathbb{R})$ and $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+], [\gamma_\alpha] = [\gamma_\alpha^-, \gamma_\alpha^+]$. Then for all $\alpha \in (0,1]$,*

$$[\eta \oplus \gamma]_\alpha = [\eta_\alpha^- + \gamma_\alpha^-, \eta_\alpha^+ + \gamma_\alpha^+],$$
$$[\eta \ominus \gamma]_\alpha = [\eta_\alpha^- - \gamma_\alpha^+, \eta_\alpha^+ - \gamma_\alpha^-],$$
$$[\eta \otimes \gamma]_\alpha = [\eta_\alpha^- \gamma_\alpha^-, \eta_\alpha^+ \gamma_\alpha^+], \forall \eta, \gamma \in F^+(\mathbb{R}),$$
$$[\tilde{1}/\eta]_\alpha = [\frac{1}{\eta_\alpha^+}, \frac{1}{\eta_\alpha^-}], \forall \eta_\alpha^- > 0,$$
$$[|\eta|]_\alpha = [\max(0, \eta_\alpha^-, -\eta_\alpha^+), \max(|\eta_\alpha^-|, |\eta_\alpha^+|)].$$

Definition 2 (See [20]). *Let $\eta, \gamma \in F(\mathbb{R})$ and $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+], [\gamma_\alpha] = [\gamma_\alpha^-, \gamma_\alpha^+]$, for all $\alpha \in (0,1]$. Define a partial ordering by $\eta \preceq \gamma$ in $F(\mathbb{R})$ if and only if $\eta_\alpha^- \leqslant \gamma_\alpha^-, \eta_\alpha^+ \leqslant \gamma_\alpha^+$, for all $\alpha \in (0,1]$. The strict inequality in $F(\mathbb{R})$ is defined by $\eta \prec \gamma$ if and only if $\eta_\alpha^- < \gamma_\alpha^-, \eta_\alpha^+ < \gamma_\alpha^+$, for all $\alpha \in (0,1]$.*

Definition 3 (See [21]). *Let X be a vector space over $\mathbb{R}$. Assume that the mappings $L, R : [0,1] \times [0,1] \to [0,1]$ are symmetric and non-decreasing in both arguments, and that $L(0,0) = 0$ and $R(1,1) = 1$. Let $\|\cdot\| : X \to F^*(\mathbb{R})$. The quadruple $(X, \|\cdot\|, L, R)$ is called a fuzzy normed space with the fuzzy norm $\|\cdot\|$, if the following conditions are satisfied:*

(F1) *if $x \neq 0$, then $\inf_{0<\alpha\leqslant 1} \|x\|_\alpha^- > 0$,*
(F2) *$\|x\| = \tilde{0}$ if and only if $x = 0$,*
(F3) *$\|rx\| = |\tilde{r}|\|x\|$ for $x \in X$ and $r \in \mathbb{R}$,*
(F4) *for all $x, y \in X$,*
(F4L) *$\|x+y\|(s+t) \geqslant L(\|x\|(s), \|y\|(t))$ whenever $s \leqslant \|x\|_1^-, t \leqslant \|y\|^-$ and $s+t \leqslant \|x+y\|_1^-$,*
(F4R) *$\|x+y\|(s+t) \leqslant R(\|x\|(s), \|y\|(t))$ whenever $s \geqslant \|x\|_1^-, t \geqslant \|y\|^-$ and $s+t \geqslant \|x+y\|_1^-$.*

Remark 1 (See [2]). *If $L(s,t) = \min(s,t)$ and $R(s,t) = \max(s,t)$ for all $s, t \in [0,1]$, then the triangle inequality (F4) shown in Definition 4 is equivalent to*

$$\|x+y\| \preceq \|x\| \oplus \|y\|.$$

In this paper, we fix $L(s,t) = \min(s,t)$ and $R(s,t) = \max(s,t)$ for all $s,t \in [0,1]$ because of the triangle inequality, and we write $(X, \|\cdot\|)$.

Definition 4 (See [2]). *Let $(X, \|\cdot\|)$ be a fuzzy normed space. A sequence $\{x_n\}$ of X is said to converge to $x \in X$ ($\lim_{n\to\infty} x_n = x$) if*

$$\lim_{n\to\infty} \|x_n - x\|_\alpha^+ = 0$$

for all $\alpha \in (0,1]$. A subset A of X is called compact in $(X, \|\cdot\|)$ if each sequence of elements of A has a convergent subsequence in $(X, \|\cdot\|)$.

Remark 2. *Let $(X, \|\cdot\|)$ be a fuzzy normed space. Subsequently, for all $\alpha \in (0,1]$, $(X, \|\cdot\|_\alpha^-)$ and $(X, \|\cdot\|_\alpha^+)$ are normed spaces.*

Definition 5 (See [21]). *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|^\sim)$ be fuzzy normed spaces. The linear operator $T : X \to Y$ is said to be a strongly fuzzy bounded if there is a real number $M > 0$, such that $\|Tx\|^\sim \preceq \tilde{M} \otimes \|x\|$ for all $x \in X$. We will denote the set of all strongly fuzzy bounded operators from $(X, \|\cdot\|)$ to $(Y, \|\cdot\|^\sim)$ by $F(X,Y)$. Afterwards, $F(X,Y)$ is a vector space. For all $M > 0$, we denote $F(X,Y,M)$ by*

$$\{T \in F(X,Y) : \|Tx\|^\sim \preceq \tilde{M} \otimes \|x\|, \forall x \in X\},$$

where M is a positive real number.

$\mathcal{A}$ is called a bounded in $F(X,Y)$ if $\mathcal{A} = F(X,Y,M)$ for some $M > 0$. Moreover, we denote the set of all finite rank strongly fuzzy bounded operators from $(X, \|\cdot\|)$ to $(Y, \|\cdot\|^\sim)$ by $\mathcal{F}(X,Y)$. Subsequenty, $\mathcal{F}(X,Y)$ is a subspace of $F(X,Y)$. Similarly, we define $\mathcal{F}(X,Y,M)$ for some $M > 0$.

3. Spaces of Bounded Sequences and Null Sequences in Fuzzy Normed Spaces

In this section, we provide definitions of fuzzy norms for spaces of bounded sequences and null sequences in fuzzy normed spaces. Recall that a sequence $\{x_n\}$ of a fuzzy normed space X is said to be bounded if there exists a fuzzy real number η, such that

$$\|x_n\| \preceq \eta, \forall n \in \mathbf{N}.$$

([9]).

Definition 6. *Let $(X, \|\cdot\|)$ be a fuzzy normed space. We denote, by $\ell_\infty(X, \|\cdot\|)$, the set of all bounded sequences in $(X, \|\cdot\|)$. A set $\ell_\infty(X, \|\cdot\|)$ is clearly vector space with respect to componentwise summation and componentwise multiplication by a scalar. We define $\|(x_n)\|_{\ell_\infty}$ by,*

$$[\|(x_n)\|_{\ell_\infty}]_\alpha = \left[\sup_n \|x_n\|_\alpha^-, \inf\{\eta_\alpha^+ : \|x_n\| \preceq \eta, \forall n\}\right].$$

We shall write $\|(x_n)\|_{\ell_\infty\alpha}^- = \sup_n \|x_n\|_\alpha^-$ and $\|(x_n)\|_{\ell_\infty\alpha}^+ = \inf\{\eta_\alpha^+ : \|x_n\| \preceq \eta, \forall n\}$.

Lemma 2 (See [3]). *Let $[a_\alpha, b_\alpha], 0 < \alpha \leqslant 1$ be a given family of non-empty intervals. Assume*

(a) $[a_{\alpha_1}, b_{\alpha_1}] \supseteq [a_{\alpha_2}, b_{\alpha_2}]$ *for all* $0 < \alpha_1 \leqslant \alpha_2$.

(b) $[\lim_{k\to\infty} a_{\alpha_k}, \lim_{k\to\infty} b_{\alpha_k}] = [a_\alpha, b_\alpha]$ *whenever $\{a_k\}$ is an increasing sequence in $(0,1]$ converging to α.*

Subsequently, the family $[a_\alpha, b_\alpha]$ represents the α-level sets of a fuzzy number. Conversely, if $[a_\alpha, b_\alpha], 0 < \alpha \leqslant 1$, are the α-level sets of a fuzzy number the conditions (a) *and* (b) *are satisfied.*

Theorem 1. *Let $(X, \|\cdot\|)$ be a fuzzy normed space and (x_n) a bounded sequence in $(X, \|\cdot\|)$. Afterwards, $\|(x_n)\|_{\ell_\infty}$ is a fuzzy real number.*

Proof. Let $\alpha \in (0,1]$ be given. There exists a fuzzy real number η such that $\|x_n\| \preceq \eta, \forall n$. Subsequently, we have
$$\sup_n \|x_n\|_\alpha^- \leqslant \eta_\alpha^- \leqslant \eta_\alpha^+,$$
thus we obtain
$$\sup_n \|x_n\|_\alpha^- \leqslant \inf\{\eta_\alpha^+ : \|x_n\| \leqslant \eta, \forall n\}.$$

Hence, $[\|(x_n)\|_{\ell_\infty}]_\alpha = \left[\|(x_n)\|_{\ell_\infty\alpha}^-, \|(x_n)\|_{\ell_\infty\alpha}^+\right]$ is a nonempty interval for all $\alpha \in (0,1]$. We show that the interval $[\|(x_n)\|_{\ell_\infty}]_\alpha$ satisfies conditions (a) and (b) in Lemma 2.

(a) Let $0 < \alpha_1 \leqslant \alpha_2$. Afterwards,
$$\sup_n \|x_n\|_{\alpha_1}^- \leqslant \sup_n \|x_n\|_{\alpha_2}^-.$$

Because $\eta_{\alpha_2}^+ \leqslant \eta_{\alpha_1}^+$, we have
$$\inf\{\eta_{\alpha_2}^+ : \|x_n\| \preceq \eta, \forall n\} \leqslant \inf\{\eta_{\alpha_1}^+ : \|x_n\| \preceq \eta, \forall n\},$$
hence $[\|(x_n)\|_{\ell_\infty}]_{\alpha_2} \subseteq [\|(x_n)\|_{\ell_\infty}]_{\alpha_1}$.

(b) Let (α_k) be a increasing sequence in (0,1] converging to α. Subsequently, $\alpha_k \leqslant \alpha_{k+1} \leqslant \alpha$ and, thus,
$$\sup_k \sup_n \|x_n\|_{\alpha_k}^- \leqslant \sup_n \|x_n\|_\alpha^-.$$

Let $\epsilon > 0$. Afterwards, there exists $n_0 \in \mathbf{N}$ such that $\|x_{n_0}\|_\alpha^- > \sup_n \|x_n\|_\alpha^- - \epsilon$. By Lemma 2 (b), we have $\|x_{n_0}\|_\alpha^- = \lim_k \|x_{n_0}\|_{\alpha_k}^-$. Subsequently, there exists k_0, such that
$$\|x_{n_0}\|_{\alpha_{k_0}}^- + \epsilon > \|x_{n_0}\|_\alpha^-.$$

Hence, we have
$$\sup_k \sup_n \|x_n\|_{\alpha_k}^- + 2\epsilon \geqslant \|x_{n_0}\|_{\alpha_{k_0}}^- + 2\epsilon > \sup_n \|x_n\|_\alpha^-.$$

Because $\epsilon \to 0$,
$$\sup_k \sup_n \|x_n\|_{\alpha_k}^- \geqslant \sup_n \|x_n\|_\alpha^-$$

On the other hand, by the proof of [5] (Theorem 5.4), we can show that
$$\lim_k \inf\{\eta_{\alpha_k}^+ : \|x_n\| \preceq \eta, \forall n\} = \inf\{\eta_\alpha^+ : \|x_n\| \preceq \eta, \forall n\},$$
hence the interval $[\|(x_n)\|_{\ell_\infty}]_\alpha$ satisfies conditions (a) and (b) in Lemma 2.
□

Remark 3. *Let $(X, \|\cdot\|)$ be a fuzzy normed space and (x_n) in $\ell_\infty(X, \|\cdot\|)$. Let*
$$A_\eta = \{\eta : \|x_n\| \preceq \eta, \eta \in F^*(\mathbf{R})\}.$$
Afterwards, the fuzzy real number $\|(x_n)\|_{\ell_\infty}$ is the element of A_η, such that $\|(x_n)\|_{\ell_\infty} \preceq \eta$ for every $\eta \in A_\eta$, i.e., $\|(x_n)\|_{\ell_\infty}$ is the smallest fuzzy real number of A_η. Indeed, take any $\eta \in A_\eta$ and $\alpha \in (0,1]$. Subsequently, we have
$$\|(x_n)\|_{\ell_\infty\alpha}^+ = \inf\{\eta_\alpha^+ : \|x_n\| \preceq \eta, \forall n\} \leqslant \eta_\alpha^+,$$

$$\|(x_n)\|^-_{\ell_\infty\alpha} = \sup_n \|x_n\|^-_\alpha \leqslant \eta^-_\alpha.$$

Additionally, since $\|x_n\|^+_\alpha \leqslant \eta^+_\alpha$, *we have*

$$\|x_n\|^+_\alpha \leqslant \inf\{\eta^+_\alpha : \|x_n\| \preceq \eta, \forall n\},$$

$$\|x_n\|^-_\alpha \leqslant \sup_n \|x_n\|^-_\alpha,$$

hence it follows that $\|(x_n)\|_{\ell_\infty} \in A_\eta$ *and* $\|(x_n)\|_{\ell_\infty} \preceq \eta$.

Theorem 2. *The vector space* $\ell_\infty(X, \|\cdot\|)$ *that is equipped with the norm defined in Definition 9 is a fuzzy normed space.*

Proof. We shall show that $\|\cdot\|_{\ell_\infty}$ satisfies the conditions (F1)–(F4) of Definition 4.

(F1) Let $(x_n) \neq 0$ in $\ell_\infty(X, \|\cdot\|)$. Afterwards, there exists $n_0 \in \mathbf{N}$ such that $x_{n_0} \neq 0$. We have for all $\alpha \in (0, 1]$

$$0 < \inf_{0<\beta\leqslant 1} \|x_{n_0}\|^-_\beta \leqslant \|x_{n_0}\|^-_\alpha \leqslant \sup_n \|x_n\|^-_\alpha,$$

so we obtain $0 < \inf_{0<\beta\leqslant 1} \|x_{n_0}\|^-_\beta \leqslant \inf_{0<\alpha\leqslant 1} \sup_n \|x_n\|^-_\alpha = \inf_{0<\alpha\leqslant 1} \|(x_n)\|^-_{\ell_\infty\alpha}$.

(F2) Let $(x_n) \in \ell_\infty(X, \|\cdot\|)$. It is clear that if $(x_n) = 0$, then $\|(x_n)\|_{\ell_\infty} = \widetilde{0}$. Conversely, let $\|(x_n)\|_{\ell_\infty} = \widetilde{0}$. Subsequently, we have for all $\alpha \in (0, 1]$

$$\|(x_n)\|^-_{\ell_\infty\alpha} = \|(x_n)\|^+_{\ell_\infty\alpha} = 0,$$

so we obtain $\sup_n \|x_n\|^-_\alpha = 0$. Afterwards, it is clear that $x_n = 0$ for all n.

(F3) Let $r \neq 0 \in \mathbf{R}$ and $(x_n) \in \ell_\infty(X, \|\cdot\|)$. For all $\alpha \in (0, 1]$, we have

$$\begin{aligned}
[\|r(x_n)\|_{\ell_\infty}]_\alpha &= \left[\sup_n \|rx_n\|^-_\alpha, \inf\{\eta^+_\alpha : \|rx_n\| \preceq \eta, \forall n\}\right] \\
&= \left[|r| \sup_n \|x_n\|^-_\alpha, \inf\{\eta^+_\alpha : \|x_n\| \preceq \frac{1}{|r|}\eta, \forall n\}\right] \\
&= \left[|r| \sup_n \|x_n\|^-_\alpha, \inf\{|r|\gamma^+_\alpha : \|x_n\| \preceq \gamma, \forall n\}\right] \\
&= \left[|r| \sup_n \|x_n\|^-_\alpha, |r| \inf\{\gamma^+_\alpha : \|x_n\| \preceq \gamma, \forall n\}\right] \\
&= [|r| \|(x_n)\|_{\ell_\infty}]_\alpha
\end{aligned} \tag{1}$$

where $\gamma = \frac{1}{|r|}\eta$.

(F4) Let $\alpha \in (0, 1]$ be given. It is clear that

$$\|(x_n) + (y_n)\|^-_{\ell_\infty\alpha} = \sup_n \|x_n + y_n\|^-_\alpha \leqslant \sup_n (\|x_n\|^-_\alpha + \|y_n\|^-_\alpha) \leqslant \|(x_n)\|^-_{\ell_\infty\alpha} + \|(y_n)\|^-_{\ell_\infty\alpha}.$$

Now, let $\epsilon > 0$ be given. There are fuzzy real numbers η, γ, such that

$$\|(x_n)\|^+_{\ell_\infty\alpha} + \epsilon > \eta^+_\alpha,$$

$$\|(y_n)\|^+_{\ell_\infty\alpha} + \epsilon > \gamma^+_\alpha$$

where $\|x_n\| \preceq \eta$ and $\|y_n\| \preceq \gamma$ for all n. Subsequently, we have

$$\|x_n + y_n\| \preceq \|x_n\| \oplus \|y_n\| \preceq \eta \oplus \gamma.$$

Therefore,

$$\|(x_n) + (y_n)\|^+_{\ell_\infty\alpha} \leqslant \eta^+_\alpha + \gamma^+_\alpha < \|(x_n)\|^+_{\ell_\infty\alpha} + \|(y_n)\|^+_{\ell_\infty\alpha} + 2\epsilon.$$

Hence,

$$\|(x_n) + (y_n)\|^+_{\ell_\infty\alpha} \leqslant \|(x_n)\|^+_{\ell_\infty\alpha} + \|(y_n)\|^+_{\ell_\infty\alpha}$$

□

Recall that a sequence (x_n) is called Cauchy if, for a given $\epsilon > 0$, there exists $N \in \mathbf{N}$, such that, for all $m, n > N$

$$\lim_{n,m\to\infty} \|x_m - x_n\| < \epsilon.$$

Additionally, a fuzzy normed space $(X, \|\cdot\|)$ is said to be complete if every Cauchy sequence converges in $(X, \|\cdot\|)$[5].

Theorem 3. *Let $(X, \|\cdot\|)$ be a complete fuzzy normed space. Subsequently, $\ell_\infty(X, \|\cdot\|)$ is a complete fuzzy normed space.*

Proof. Let $X_n = (x_1^{(n)}, x_2^{(n)}, \cdots)$ be a Cauchy sequence. First, we claim that, for each j, there exists $z_j \in X$, such that $x_j^{(n)} \longrightarrow z_j$ as $n \to \infty$ in $(X, \|\cdot\|)$. Indeed, take any $\epsilon > 0$ and $\alpha \in (0, 1]$. Subsequently, there exists $n_0 \in \mathbf{N}$, such that $n, \ell \geqslant n_0$,

$$\|X_n - X_\ell\|^+_{\ell_\infty\alpha} < \epsilon,$$

i.e., we have $\inf\{\eta^+_\alpha : \|x_j^{(n)} - x_j^{(\ell)}\| \preceq \eta, \forall j\} < \epsilon$ for all $n, \ell \geqslant n_0$. Fix $n, \ell \geqslant n_0$. Afterwards, there exists a fuzzy real number η_0 such that

$$\eta^+_{0\alpha} < \inf\{\eta^+_\alpha : \|x_j^{(n)} - x_j^{(\ell)}\| \preceq \eta, \forall j\} + \epsilon < 2\epsilon,$$

$$\|x_j^{(n)} - x_j^{(\ell)}\| \preceq \eta_0, \forall j.$$

Therefore,

$$\sup_j \|x_j^{(n)} - x_j^{(\ell)}\|^+_\alpha < \eta^+_{0\alpha} < 2\epsilon.$$

Hence,

$$\sup_j \|x_j^{(n)} - x_j^{(\ell)}\|^+_\alpha < 2\epsilon, \forall n, \ell \geqslant n_0.$$

Thus, $j = 1$, we obtain $\|x_1^{(n)} - x_1^{(\ell)}\|^+_\alpha < 2\epsilon, \forall n, \ell \geqslant n_0$, so it follows that $(x_1^{(n)})$ is a Cauchy sequence in $(X, \|\cdot\|)$. Because $(X, \|\cdot\|)$ is complete, there exists $z_1 \in X$, such that $x_1^{(n)} \longrightarrow z_1$ as $n \to \infty$. Continuing this process, it proves our claim. Now, we put $Z = (z_1, z_2, \cdots)$.

Second, we claim that $X_n \xrightarrow{\|\|_{\ell_\infty}} Z$ as $n \to \infty$. Let $\epsilon > 0$ and $\alpha \in (0, 1]$ be given. By results of the first claim, if $n, \ell \geqslant n_0$, then there exists a fuzzy real number η_0, such that we have

$$\eta^+_{0\alpha} < 2\epsilon,$$

$$\|x_j^{(n)} - x_j^{(\ell)}\| \preceq \eta_0, \forall j.$$

Now, we fix $j \in \mathbf{N}$. There exists $\ell_0 \in \mathbf{N}$ with $\ell_0 \geq n_0$, such that $\ell \geq \ell_0$,

$$\|x_j^{(\ell)} - z_j\|^+_\alpha < \epsilon.$$

Therefore, if $n \geqslant n_0$, we have

$$\|x_j^{(n)} - z_j\| \preceq \|x_j^{(n)} - x_j^{(\ell_0)}\| \oplus \|x_j^{(\ell_0)} - z_j\| \preceq \eta_0 \oplus \widetilde{\epsilon}.$$

Hence, if $n \geqslant n_0$, we obtain for all j,

$$\|x_j^{(n)} - z_j\| \preceq \eta_0 + \widetilde{\epsilon},$$

so if $n \geqslant n_0$,

$$\|X_n - Z\|^+_{\ell_\infty \alpha} = \inf\{\eta_\alpha^+ : \|x_j^{(n)} - z_j\| \preceq \eta, \forall j\} \leqslant \eta_{0\alpha}^+ + \epsilon < 3\epsilon.$$

Finally, we show that $Z \in \ell_\infty(X, \|\cdot\|)$. By the second claim, there exist a fuzzy real number η_0 and $n_0 \in \mathbf{N}$, such that $n \geqslant n_0$, we obtain for all j,

$$\|x_j^{(n)} - z_j\| \preceq \eta_0 \oplus \widetilde{\epsilon}$$

Now, we fix $n \geqslant n_0$. Since $X_n \in \ell_\infty(X, \|\cdot\|)$, there exists a fuzzy real number η_1 such that

$$\|x_j^{(n)}\| \preceq \eta_1$$

Hence, for all j, we obtain

$$\|z_j\| \preceq \|x_j^{(n)} - z_j\| \oplus \|x_j^{(n)}\| \preceq \eta_0 \oplus \widetilde{\epsilon} \oplus \eta_1.$$

□

Definition 7. *Let $(X, \|\cdot\|)$ be a fuzzy normed space. A bounded sequence (x_n) in $(X, \|\cdot\|)$ is null if*

$$x_n \longrightarrow 0.$$

We denote, by $c_0(X, \|\cdot\|)$, the set of all null sequences in $(X, \|\cdot\|)$ with the same vector space operations and fuzzy norm as $\ell_\infty(X, \|\cdot\|)$, i.e.,

$$\|(x_n)\|_{c_0} = \|(x_n)\|_{\ell_\infty}, \forall (x_n) \in c_0(X, \|\cdot\|).$$

Corollary 1. *Let $(X, \|\cdot\|)$ be a complete fuzzy normed space. Subsequently, $c_0(X, \|\cdot\|)$ is a complete fuzzy normed space.*

Proof. We are enough to show that $c_0(X, \|\cdot\|)$ is a closed subspace of $\ell_\infty(X, \|\cdot\|)$. Let us take a sequence $X_n \in c_0(X, \|\cdot\|)$ converging to Z in $\ell_\infty(X, \|\cdot\|)$ and $\alpha \in (0, 1]$ and $\epsilon > 0$. Put $X_n = (x_1^{(n)}, x_2^{(n)}, \cdots)$ and $Z = (z_1, z_2, \cdots)$. Afterwards, there exists $n_0 \in \mathbf{N}$, such that if $n \geqslant n_0$,

$$\|X_n - Z\|^+_{\ell_\infty \alpha} = \inf\{\eta_\alpha^+ : \|x_j^{(n)} - z_j\| \preceq \eta, \forall j\} < \epsilon.$$

We fix $n \geqslant n_0$. Subsequently, there exists a fuzzy real number η_0, such that

$$\eta_{0\alpha}^+ < \inf\{\eta_\alpha^+ : \|x_j^{(n)} - z_j\| \preceq \eta, \forall j\} + \epsilon < 2\epsilon,$$

$$\|x_j^{(n)} - z_j\| \preceq \eta_0, \forall j.$$

Because $X_n \in c_0(X, \|\cdot\|)$, there exists $j_0 \in \mathbf{N}$, such that if $j \geqslant j_0$,

$$\|x_j^{(n)}\|_\alpha^+ < \epsilon.$$

Hence, if $j \geqslant j_0$, we obtain

$$\|z_j\|_\alpha^+ < \|x_j^{(n)} - z_j\|_\alpha^+ + \|x_j^{(n)}\|_\alpha^+ < \eta_{0\alpha}^+ + \epsilon < 3\epsilon.$$

□

Example 1. *Consider the linear space* $\mathbf{R}$ *of all real numbers. Let us define*

$$\|x\|(t) = \begin{cases} 1, & t = |x| \\ 0, & \textit{otherwise.} \end{cases}$$

It is clear that $\|\cdot\|$ *is a fuzzy normed space and*

$$[\|x\|]_\alpha = [|x|, |x|]$$

for all $\alpha \in (0,1]$*. Consider* $(1,0,1,0,\cdots) \in \ell_\infty(\mathbf{R}, \|\|)$*. Afterwards, for all* $\alpha \in (0,1]$*, we have*

$$\|(1,0,1,0,\cdots)\|_{\ell_\infty \alpha}^- = \sup_{n, a_n \in \{0,1\}} \|a_n\|_\alpha^- = 1$$

$$\|(1,0,1,0,\cdots)\|_{\ell_\infty \alpha}^+ = \inf\{\eta_\alpha^+ : \|a_n\| \preceq \eta, a_n \in \{0,1\}, \forall n\} = 1.$$

Hence, we have

$$\|(1,0,1,0,\cdots)\|_{\ell_\infty} = \widetilde{1}$$

Example 2. *Consider the linear space* $\mathbf{R}$ *of all real numbers. Let us define*

$$\|x\|(t) = \begin{cases} 1, & t = |x| \\ 0, & \textit{otherwise.} \end{cases}$$

It is clear that $\|\cdot\|$ *is a fuzzy normed space and*

$$[\|x\|]_\alpha = [|x|, |x|]$$

for all $\alpha \in (0,1]$*. Consider* $\{\frac{1}{n} : n \in \mathbf{N}\} \in c_0(\mathbf{R}, \|\|)$*. Subsequently, for all* $\alpha \in (0,1]$*, we have*

$$\|(\frac{1}{n})\|_{c_0 \alpha}^- = \sup_n \|\frac{1}{n}\|_\alpha^- = 1$$

$$\|(\frac{1}{n})\|_{c_0 \alpha}^+ = \inf\{\eta_\alpha^+ : \|\frac{1}{n}\| \preceq \eta, \forall n\} = 1.$$

Hence, we have

$$\|(\frac{1}{n})\|_{c_0} = \widetilde{1}$$

4. The Dual Space of $c_0((X, \|\cdot\|))$

We note that, if $Y = \mathbf{R}$, we define a function $\|r\|^\sim : \mathbf{R} \to [0,1]$ by

$$\|r\|^\sim = \begin{cases} 1, & t = |r| \\ 0, & t \neq |r| \end{cases}$$

Afterwards, $\|\cdot\|^\sim$ is a fuzzy norm on $\mathbf{R}$ and α-level sets of $\|r\|^\sim$ are given by

$$[\|r\|^\sim]_\alpha = [|r|, |r|], 0 < \alpha \leqslant 1.$$

Definition 8 (See [4]). *Let* $(X, \|\cdot\|)$ *be a fuzzy normed space. A strongly fuzzy bounded operator from* $(X, \|\cdot\|)$ *to* $(\mathbf{R}, \|\cdot\|^\sim)$ *is called a strongly fuzzy bounded functional. Denote by* $(X, \|\cdot\|)^*$ *the set of all strongly fuzzy bounded functionals over* $(X, \|\cdot\|)^*$*. Let* $f \in (X, \|\cdot\|)^*$*. Subsequently,*

$\left\{\left[\sup_{x\neq 0}\frac{|f(x)|}{\|x\|_\alpha^+},\sup_{x\neq 0}\frac{|f(x)|}{\|x\|_\alpha^-}\right];\alpha\in(0,1]\right\}$ *is a nested bounded and closed intervals of real numbers and, thus, it generates a fuzzy interval say* $\|f\|^*$*, By [4] (Definition 6.3,* $\|\cdot\|^*$ *is a fuzzy norm of* $(X,\|\cdot\|)^*$*. We call* $(X,\|\cdot\|)^*$ *the strong fuzzy dual space of* $(X,\|\cdot\|)$*.*

Remark 4. *The definition of a strong fuzzy bounded operator on* $(X,\|\cdot\|)$ *introduced by T. Bag and S. K. Samanta [4] is slightly different from Definition 8. However, they have the same strong fuzzy dual space.*

Subsequently, it is natural to consider the following question.

Q. Let $(X,\|\cdot\|)$ be a fuzzy normed space. What is the representation of $f\in c_0(X,\|\cdot\|)^*$?

The answer is positive. First, we consider the following lemma.

Lemma 3. *If* $a_{\ell n}$ *is monotonically increasing in* ℓ *and the sequence* $c_{\ell n}=a_{\ell n}-a_{\ell-1n}$ *is monotonically increasing in* n*, then we have*

$$\lim_{\ell\to\infty}\lim_{n\to\infty}a_{\ell n}=\lim_{n\to\infty}\lim_{\ell\to\infty}a_{\ell n}$$

Proof. This lemma is a well-known result in real analysis. For completeness, we give a proof. By the assumption, we obtain that each $c_{\ell n}\geqslant 0$ and that the $c_{\ell n}$ are monotonically increasing in n. Afterwards, by the Monotone Convergence theorem with respect to the counting measure, we have

$$\lim_{n\to\infty}\int c_{\ell n}=\int\lim_{n\to\infty}c_{\ell n}.$$

By the property of the counting measure, we have,

$$\lim_{n\to\infty}\sum_{\ell=1}^{\infty}c_{\ell n}=\sum_{\ell=1}^{\infty}\lim_{n\to\infty}c_{\ell n},$$

so we obtain

$$\lim_{n\to\infty}\lim_{L\to\infty}\sum_{\ell=1}^{L}c_{\ell n}=\lim_{L\to\infty}\lim_{n\to\infty}\sum_{\ell=1}^{L}c_{\ell n}.$$

Hence, we have

$$\lim_{n\to\infty}\lim_{\ell\to\infty}a_{\ell n}=\lim_{\ell\to\infty}\lim_{n\to\infty}a_{\ell n}.$$

□

Now, we provide the representation of strong fuzzy dual space of space of null sequence in fuzzy normed spaces.

Theorem 4. *Let* $(X,\|\cdot\|)$ *be a fuzzy normed space. If* $f\in(c_0(X,\|\cdot\|))^*$*, then there is only one the sequence* (f_n) *in* $(X,\|\cdot\|)^*$*, such that* $\sum_{n=1}^{\infty}\|f_n\|_\alpha^{*+}=\|f\|_\alpha^{*+},\sum_{n=1}^{\infty}\|f_n\|_\alpha^{*-}=\|f\|_\alpha^{*-},\forall\alpha\in(0,1)$ *and*

$$f((x_n))=\sum_{n=1}^{\infty}f_n(x_n),\forall(x_n)\in c_0(X,\|\cdot\|).$$

Proof. Let f be in $(c_0(X,\|\cdot\|))^*$. Subsequently, there exists $M>0$, such that

$$|f((x_n))|\leq M\|(x_n)\|_{c_0\alpha}^-,\forall(x_n)\in c_0(X,\|\cdot\|),\alpha\in(0,1).$$

For each $n \in \mathbf{N}$, we define

$$f_n(x) := f((0, \cdots, 0, x, 0, \cdots)),$$

where x is the n-th component. Clearly, f_n is linear functional on $(X, \|\cdot\|)$. Additionally, we have $f_n \in (X, \|\cdot\|)^*$ for all $n \in \mathbf{N}$ since

$$|f_n(x)| = |f((0, \cdots, 0, x, 0, \cdots))| \leq M\|(0, \cdots, 0, x, 0, \cdots)\|^-_{c_0\alpha} = M \sup_n \|x\|^-_\alpha = M\|x\|^-_\alpha$$

for all $x \in X, \alpha \in (0, 1)$. First, we claim that

$$f((x_n)) = \sum_{n=1}^{\infty} f_n(x_n), \forall (x_n) \in c_0((X, \|\cdot\|)).$$

Fix $(x_n) \in c_0(X, \|\cdot\|)$. We can easily observe that

$$\sum_{n=1}^{\ell} f_n(x_n) = f((x_1, \cdots, x_\ell, 0, \cdots)), \forall \ell \in \mathbf{N}.$$

Additionally, we have

$$f((x_1, \cdots, x_\ell, 0, \cdots)) \to f((x_n))$$

as $\ell \to \infty$, since, for each $\alpha \in (0, 1)$

$$|f((x_n)) - f((x_1, \cdots, x_\ell, 0, \cdots))| = |f((0, \cdots, 0, x_{\ell+1}, x_{\ell+2}, \cdots)| \leqslant M \sup_{\ell < n} \|x_n\|^-_\alpha$$

and $\sup_{\ell<n} \|x_n\|^-_\alpha \to 0$ as $\ell \to \infty$, hence it proves the first claim.

Second, we claim that $\sum_{n=1}^{\infty} \|f_n\|^{*+}_\alpha = \|f\|^{*+}_\alpha, \sum_{n=1}^{\infty} \|f_n\|^{*-}_\alpha = \|f\|^{*-}_\alpha, \forall \alpha \in (0, 1)$. Indeed, take any $\alpha \in (0, 1)$. Afterwards, we obtain that for each $(x_n) \in c_0((X, \|\cdot\|))$,

$$\begin{aligned} |f((x_n))| = |\sum_{n=1}^{\infty} f_n(x_n)| &\leq \sum_{n=1}^{\infty} |f_n(x_n)| \\ &\leq \sum_{n=1}^{\infty} \|f_n\|^{*+}_\alpha \|x_n\|^-_\alpha \\ &\leq (\sum_{n=1}^{\infty} \|f_n\|^{*+}_\alpha) \sup_n \|x_n\|^-_\alpha \\ &= (\sum_{n=1}^{\infty} \|f_n\|^{*+}_\alpha) \|(x_n)\|^-_{c_0\alpha}, \end{aligned} \tag{2}$$

hence, we have $\|f\|^{*+}_\alpha \leq \sum_{n=1}^{\infty} \|f_n\|^{*+}_\alpha$. Now, let us show the converse. By definition of $\|\cdot\|^{*+}_\alpha$, for each $n \in \mathbf{N}$, there exists a sequence $(x_n^{k,\alpha})_{k=1}^{\infty}$ in $B_{(X, \|\cdot\|^-_\alpha)}$, such that

$$\lim_{k\to\infty} |f_n(x_n^{k,\alpha})| = \|f_n\|^{*+}_\alpha.$$

Because, for each n, k, there is a scalar $\gamma_n^{k,\alpha}$, such that $|\gamma_n^{k,\alpha}| = 1$ and $f_n(\gamma_n^{k,\alpha} x_n^{k,\alpha}) = |f_n(x_n^{k,\alpha})|$, we may assume that $f_n(x_n^{k,\alpha}) \geq 0$ for each n, k and, so, that

$$\lim_{k\to\infty} f_n(x_n^{k,\alpha}) = \|f_n\|^{*+}_\alpha$$

and $f_n(x_n^{k,\alpha})$ is monotonically increasing in k. Put $a_{\ell k} = \sum_{n=1}^{\ell} f_n(x_n^{k,\alpha})$. Because $a_{\ell k}$ is monotonically increasing in ℓ and the sequence $c_{\ell k} = a_{\ell k} - a_{\ell-1k}$ is monotonically increasing in k, by Lemma 3, we have

$$\lim_{\ell\to\infty}\lim_{k\to\infty}\sum_{n=1}^{\ell} f_n(x_n^{k,\alpha}) = \lim_{k\to\infty}\lim_{\ell\to\infty}\sum_{n=1}^{\ell} f_n(x_n^{k,\alpha}).$$

Because

$$f((x_1^{k,\alpha},\cdots,x_\ell^{k,\alpha},0,\cdots)) \le \|f\|_\alpha^{*+}, \forall k,\ell \in \mathbf{N},$$

we have

$$\begin{aligned}\sum_{n=1}^{\infty}\|f_n\|_\alpha^{*+} &= \sum_{n=1}^{\infty}\lim_{k\to\infty} f_n(x_n^{k,\alpha}) = \lim_{\ell\to\infty}\lim_{k\to\infty}\sum_{n=1}^{\ell} f_n(x_n^{k,\alpha}) \\ &= \lim_{k\to\infty}\lim_{\ell\to\infty}\sum_{n=1}^{\ell} f_n(x_n^{k,\alpha}) \\ &= \lim_{k\to\infty}\lim_{\ell\to\infty} f((x_1^{k,\alpha},x_2^{k,\alpha},\cdots,x_\ell^{k,\alpha},0,\cdots)) \\ &\le \|f\|_\alpha^{*+},\end{aligned} \tag{3}$$

it proves our claim. Additionally, we shall show that $\sum_{n=1}^{\infty}\|f_n\|_\alpha^{*-} = \|f\|_\alpha^{*-}, \forall\alpha \in (0,1)$ Indeed, for any $\alpha \in (0,1]$ and $(x_n) \in c_0((X,\|\cdot\|))$, we have

$$|f((x_n))| = |\sum_{n=1}^{\infty} f_n(x_n)| \le \sum_{n=1}^{\infty}|f_n(x_n)| \le \sum_{n=1}^{\infty}\|f_n\|_\alpha^{*-}\|x_n\|_\alpha^{+} \le (\sum_{n=1}^{\infty}\|f_n\|_\alpha^{*-})\sup_n\|x_n\|_\alpha^{+},$$

so we have

$$|f((x_n))| \le (\sum_{n=1}^{\infty}\|f_n\|_\alpha^{*-})\inf\{\eta_\alpha^{+} : \|x_n\| \preceq \eta, \forall n\},$$

so,

$$\|f\|_\alpha^{*-} = \frac{|f((x_n))|}{\|(x_n)\|_{c_0,\alpha}^{+}} = \frac{|f((x_n))|}{\inf\{\eta_\alpha^{+} : \|x_n\| \preceq \eta, \forall n\}} \le \sum_{n=1}^{\infty}\|f_n\|_\alpha^{*-},$$

hence, it follows $\|f\|_\alpha^{*-} \le \sum_{n=1}^{\infty}\|f_n\|_\alpha^{*-}$. For the converse part, we can use the method for showing "$\sum_{n=1}^{\infty}\|f_n\|_\alpha^{*+} \le \|f\|_\alpha^{*+}$". Indeed, by the definition of $\|\cdot\|_\alpha^{*-}$, for each $n \in \mathbf{N}$, there exists a sequence $(x_n^{k,\alpha})_{k=1}^{\infty}$ in $B_{(X,\|\cdot\|_\alpha^{+})}$, such that

$$\lim_{k\to\infty}|f_n(x_n^{k,\alpha})| = \|f_n\|_\alpha^{*-}.$$

Subsequently, the remain part is similar to the proof of "$\sum_{n=1}^{\infty}\|f_n\|_\alpha^{*+} \le \|f\|_\alpha^{*+}$".

Finally, to show the uniqueness, suppose that there exist such two (f_n) and (g_n) in $(X,\|\cdot\|)^*$. Because

$$f((x_n)) = \sum_{n=1}^{\infty} f_n(x_n) = \sum_{n=1}^{\infty} g_n(x_n), \forall (x_n) \in c_0((X,\|\cdot\|)),$$

we obtain that, for each $n \in \mathbf{N}$ and $(0,\cdots,0,x,0,\cdots)$,

$$f_n(x) = g_n(x),$$

hence we have $(f_n) = (g_n)$.

□

5. Approximation Properties of $c_0((X, \|\cdot\|))$

In this section, we show that, given a fuzzy normed space $(X, \|\cdot\|)$, $c_0((X, \|\cdot\|))$ has the approximation property, even the bounded approximation property. The authors gave the definitions of approximation properties in fuzzy normed spaces [7]. Recently, the author also provided the definitions of approximation properties in Felbin-fuzzy normed spaces (see [6]). The definitions are as follows.

Definition 9. *A fuzzy normed space $(X, \|\cdot\|)$ is said to have the approximation property, briefly AP, if, for every compact set K in $(X, \|\cdot\|)$ and for each $\alpha \in (0,1]$ and $\varepsilon > 0$, there exists an operator $T \in \mathcal{F}(X, X)$, such that*

$$\|T(x) - x\|_\alpha^+ \leqslant \varepsilon$$

for every $x \in K$.

Definition 10. *Let λ be a positive real number. A fuzzy normed space $(X, \|\cdot\|)$ is said to have the λ-approximation property, briefly λ-BAP, if for every compact set K in $(X, \|\cdot\|)$ and for each $\alpha \in (0,1]$ and $\varepsilon > 0$, there exists an operator $T \in \mathcal{F}(X, X, \lambda)$, such that*

$$\|T(x) - x\|_\alpha^+ \leqslant \varepsilon$$

for every $x \in K$. Additionally, we say that $(X, \|\|)$ has the BAP if $(X, \|\|)$ has the λ-BAP for some $\lambda > 0$.

Lemma 4. *Let $(X, \|\cdot\|)$ be a complete fuzzy normed space. If a subset K in $c_0((X, \|\cdot\|))$ is compact, then for every $\alpha \in (0,1]$ and $\epsilon > 0$, there exists $N \in \mathbf{N}$, such that*

$$\|(0, 0, \cdots, k_{N+1}, k_{N+2}, \cdots)\|_{c_{0,\alpha}}^+ < \epsilon$$

for all $(k_n) \in K$.

Proof. Let $\alpha \in (0,1]$ and $\epsilon > 0$ be given. Because K is compact in $c_0((X, \|\cdot\|))$, by [22] (Theorem 4.2), there exists a finite subset $F = \{x_1, x_2, \cdots, x_r\}$ of K such that $\forall x \in K$,

$$\|x - x_i\|_{c_{0,\alpha}}^+ < \frac{\epsilon}{4}$$

for some $x_i \in F$. Put

$$\begin{aligned} x_1 &= (k_1^1, k_2^1, k_3^1, \cdots), \\ x_2 &= (k_1^2, k_2^2, k_3^2 \cdots), \\ &\vdots \\ x_r &= (k_1^r, k_2^r, k_3^r, \cdots), \end{aligned}$$

for all $k_n^i \in X$.

First, we claim that there exists $N \in \mathbf{N}$, such that

$$\|(0, 0, \cdots, k_{N+1}^i, k_{N+2}^i, \cdots)\|_{c_{0,\alpha}}^+ < \frac{\epsilon}{4}$$

for all $i = 1, 2, \cdots, r$. Consider x_1. Subsequently, we can generate the following sequence (y_n^1) in $c_0((X, \|\cdot\|))$:

$$\begin{aligned} y_1^1 &= (k_1^1, k_2^1, k_3^1, \cdots), \\ y_2^1 &= (0, k_2^1, k_3^1 \cdots), \\ y_3^1 &= (0, 0, k_3^1 \cdots), \\ &\vdots \end{aligned}$$

We shall show that (y_n^1) is Cauchy sequence in $c_0((X, \|\cdot\|))$. Indeed, let $\beta \in (0,1]$ and $\delta > 0$. Because $x_1 = (k_1^1, k_2^1, k_3^1, \cdots)$ in $c_0((X, \|\cdot\|))$, there exists $N_1 \in \mathbf{N}$, such that, if $n \geqslant N_1$, then $\|k_n^1\|_\beta^+ < \delta$. Now, take any $m > n \geqslant N_1$. Subsequently, we have

$$y_n^1 = (0, 0, \cdots, 0, k_n^1, k_{n+1}^1, \cdots, k_{m-1}^1, k_m^1, \cdots)$$

$$y_m^1 = (0, 0, \cdots, 0, k_m^1, k_{m+1}^1 \cdots).$$

Afterwards, we have

$$y_m^1 - y_n^1 = (0, 0, \cdots, 0, k_n^1, k_{n+1}^1, \cdots, k_{m-1}^1, 0, \cdots).$$

By [23] (Theorem 2), we can put

$$\omega = \max\{\|k_n^1\|, \|k_{n+1}^1\|, \cdots, \|k_{m-1}^1\|\}.$$

and ω is a fuzzy real number and $\omega_\beta^+ = \max\{\|k_n^1\|_\beta^+, \|k_{n+1}^1\|_\beta^+, \cdots, \|k_{m-1}^1\|_\beta^+\}$.

Subsequently, we have

$$\|y_m^1 - y_n^1\|_{c_0,\beta}^+ \leqslant \omega_\beta^+ < \delta,$$

so it follows that (y_n^1) is a Cauchy sequence in $c_0((X, \|\cdot\|))$. Because $c_0((X, \|\cdot\|))$ is complete, we have

$$(y_n^1) \to 0.$$

Afterwards, there exists $n_1 \in \mathbf{N}$ such that if $n \geqslant n_1$, then

$$\|y_n^1\|_{c_0,\alpha}^+ < \frac{\epsilon}{4},$$

i.e.,

$$\|(0, 0, , \cdots, k_n^1, k_{n+1}^1, \cdots)\|_{c_0,\alpha}^+ < \frac{\epsilon}{4}.$$

Continuing this process, for each $i = 1, 2, \cdots, r$, there exists $n_i \in \mathbf{N}$, such that, if $n \geqslant n_i$, then

$$\|(0, 0, , \cdots, k_n^i, k_{n+1}^i, \cdots)\|_{c_0,\alpha}^+ < \frac{\epsilon}{4}.$$

Now, we put $N = \max\{n_1, n_2, \cdots, n_r\}$. Afterwards, we prove our claim.

Now, we shall show our lemma. By the above claim, there exists a fuzzy real number γ with $\gamma_\alpha^+ < \frac{\epsilon}{2}$ such that

$$\|k_j^i\| \preceq \gamma$$

for all $i = 1, 2, \cdots, r$ and $j = N+1, N+2, \cdots$. Now, take any $(k_j) \in K$. Subsequently, there exists $(k_j^i)_{j=1}^\infty \in F$, such that

$$\|(k_j - k_j^i)_j\|_{c_0,\alpha}^+ < \frac{\epsilon}{4}.$$

Afterwards, there exists a fuzzy real number η with $\eta_\alpha^+ < \frac{\epsilon}{2}$ such that

$$\|k_j - k_j^i\| \preceq \eta, \forall j.$$

Subsequenty, it follows that for all $j \geqslant N+1$,

$$\|k_j\| \preceq \|k_j - k_j^i\| \oplus \|k_j^i\| \preceq \eta \oplus \gamma,$$

$$(\eta + \gamma)_\alpha^+ = \eta_\alpha^+ + \gamma_\alpha^+ < \epsilon.$$

Hence, we have

$$\|(0, 0, \cdots, k_{N+1}, k_{N+2}, \cdots)\|_{c_0,\alpha}^+ < \epsilon.$$

□

Theorem 5. *Let $(X, \|\cdot\|)$ be a complete fuzzy normed space. If $(X, \|\cdot\|)$ has the AP, then $c_0((X, \|\cdot\|))$ also has the AP.*

Proof. First, we put $Y = \{(x_1, x_2, 0, \cdots) : x_1, x_2 \in X\}$. Afterwards, Y is a subspace of $c_0((X, \|\cdot\|))$. We simply denote

$$Y = \{(x_1, x_2) : x_1, x_2 \in X\}.$$

We shall show that Y has the AP. Let $K \subset Y$ be compact and $\alpha \in (0,1]$ and $\epsilon > 0$. We consider i_1 (resp. i_2) is the operator from $(X, \|\cdot\|)$ into Y defined by $i_1(x) = (x, 0, 0, \cdots)$ (resp. $i_2(x) = (0, x, 0, \cdots)$). Afterwards, i_1, i_2 are strong fuzzy bounded operator, since, for each $\alpha \in (0,1]$, $n = 1, 2$

$$\|i_n(x)\|^-_{c_0,\alpha} = \|x\|^-_\alpha,$$

$$\|i_n(x)\|^+_{c_0\alpha} = \inf\{\eta^+_\alpha : \|x\| \leqslant \eta\} = \|x\|^+_\alpha.$$

Now, for each $n = 1, 2$, let us consider the projection $P_n : Y \to (X, \|\cdot\|)$ by

$$P_n((x_1, x_2)) = x_n.$$

It follows that P_n is a strong fuzzy bounded operator, because, for each $\alpha \in (0,1]$,

$$\|P_n((x_1, x_2))\|^-_\alpha = \|x_n\|^-_\alpha \leqslant \sup_{1 \leqslant k \leqslant 2} \|x_k\|^-_\alpha = \|(x_1, x_2)\|^-_{c_0,\alpha},$$

$$\|P_n((x_1, x_2))\|^+_\alpha = \|x_n\|^+_\alpha \leqslant \inf\{\eta^+_\alpha : \|x_k\| \leqslant \eta, k = 1, 2\} = \|(x_1, x_2)\|^+_{c_0,\alpha}.$$

We note that $P_n(K)$ is compact in $(X, \|\cdot\|)$ for $n = 1, 2$ because P_n is strong fuzzy bounded operator for $n = 1, 2$. Subsequently, $n = 1, 2$ there is a $T_n \in \mathcal{F}(X, X)$ such that

$$\|T_n P_n(k_1, k_2) - P_n(k_1, k_2)\|^+_\alpha \leqslant \frac{\epsilon}{2}$$

for every $(k_1, k_2) \in K$. Put $T_0 = i_1 T_1 P_1 + i_2 T_2 P_2 \in \mathcal{F}(Y, Y)$. Afterwards, for all $(k_1, k_2) \in K$, we have

$$\begin{aligned} \|T_0(k_1, k_2) - (x_1, x_2)\|^+_{c_0,\alpha} &= \|i_1 T_1 P_1(k_1, k_2) + i_2 T_2 P_2(k_1, k_2) - i_1 P_1(k_1, k_2) - i_2 P_2(k_1, k_2)\|^+_{c_0,\alpha} \\ &\leqslant \|i_1 T_1 P_1(k_1, k_2) - i_1 P_1(k_1, k_2)\|^+_{c_0,\alpha} + \|i_2 T_2 P_2(k_1, k_2) - i_2 P_2(k_1, k_2)\|^+_{c_0,\alpha} \\ &= \|T_1 P_1(k_1, k_2) - P_1(k_1, k_2)\|^+_\alpha + \|T_2 P_2(k_1, k_2) - P_2(k_1, k_2)\|^+_\alpha \\ &\leqslant \epsilon, \end{aligned} \tag{4}$$

hence, Y has the AP. Similarly, we can show that for all $N \in \mathbf{N}$,

$$Y_N = \{(x_1, x_2, \cdots, x_N, 0, \cdots) : x_k \in X\}$$

has the AP.

Finally, we shall show that $c_0((X, \|\cdot\|))$ has the AP. Let $K \subset c_0((X, \|\cdot\|))$ be compact and $\alpha \in (0,1]$ and $\epsilon > 0$. By Lemma 5.3, there exists $N \in \mathbf{N}$, such that

$$\|(0, 0, \cdots, k_{N+1}, k_{N+2}, \cdots)\|^+_{c_0,\alpha} < \frac{\epsilon}{2}$$

for all $(k_n) \in K$. Let P_N be the projection from $c_0((X, \|\cdot\|))$ into Y_N defined by $P_N((x_n)) = (x_1, \cdots, x_N, 0, \cdots)$. Clearly, P_N is a strong fuzzy bounded operator, because, for each $\alpha \in (0,1]$,

$$\|P_N((x_n))\|^-_{c_0,\alpha} = \sup_{1 \leqslant n \leqslant N} \|x_n\|^-_\alpha \leqslant \sup_n \|x_n\|^-_\alpha = \|(x_n)\|^-_{c_0,\alpha},$$

$$\|P_N((x_n))\|^+_{c_0,\alpha} = \inf\{\eta^+_\alpha : \|x_n\| \leqslant \eta, 1 \leqslant n \leqslant N\} \leqslant \inf\{\eta^+_\alpha : \|x_n\| \leqslant \eta, \forall n\} = \|(x_n)\|^+_{c_0,\alpha}.$$

Since $P_N(K)$ is compact in Y_N and Y_N has the AP, there is a $T_0 \in \mathcal{F}(Y_N, Y_N)$, such that

$$\|T_0 P_N((k_n)) - P_N((k_n))\|^+_{c_0,\alpha} \leqslant \frac{\epsilon}{2}$$

for all $(k_n) \in K$. Now, $T_0 P_N \in \mathcal{F}(c_0((X, \|\cdot\|), c_0((X, \|\cdot\|))$ and for all $(k_n) \in K$

$$\begin{aligned} \|T_0 P_N((k_n)) - (k_n)\|^+_{c_0,\alpha} &= \|T_0 P_N((k_n)) - P_N((k_n)) - (0, 0, \cdots, k_{N+1}, k_{N+2}, \cdots)\|^+_{c_0,\alpha} \\ &\leqslant \|T_0 P_N((k_n)) - P_N((k_n))\|^+_{c_0,\alpha} + \|(0, 0, \cdots, k_{N+1}, k_{N+2}, \cdots)\|^+_{c_0,\alpha} \\ &\leqslant \epsilon \end{aligned} \tag{5}$$

Hence, $c_0((X, \|\cdot\|))$ has the AP.
□

Corollary 2. *Let $(X, \|\cdot\|)$ be a complete fuzzy normed space. If $(X, \|\cdot\|)$ has the BAP, then $c_0((X, \|\cdot\|))$ also has the BAP.*

Proof. It comes from the proof of Theorem 5. □

6. Conclusions and Further Works

In this paper, we have introduced spaces of sequences in fuzzy normed spaces and investigated their several examples. We have established a well-defined fuzzy norm for a space of sequences in fuzzy normed spaces. The completeness of the fuzzy nom in our context has been proved. We provided the representation of of the dual of a space of sequences in a fuzzy normed space. Moreover, the results of the approximation property for spaces of sequences in fuzzy normed spaces are given. We hope that our approach may provide a key role in fuzzy analysis by applying to fuzzy function spaces, for example, spaces of fuzzy continuous functions.

Author Contributions: Conceptualization, J.-M.K. and K.-Y.L.; Investigation, K.-Y.L.; Methodology, J.-M.K. and K.-Y.L.; Writing—original draft, J.-M.K. and K.-Y.L. All authors have read and agreed to the published version of the manuscript.

Funding: The first author was supported by NRF-2018R1D1A1B07043566 funded by the Korean Government. The corresponding author was supported by NRF-2017R1C1B5017026 funded by the Korean Government.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Informed consent was obtained from all subjects involved in the study.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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