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Modified Hybrid Method with Four Stages for Second Order Ordinary Differential Equations

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Abstract: A modified explicit hybrid method with four stages is presented, with the first stage exactly integrating $\exp(wx)$, while the remaining stages exactly integrate $\sin(wx)$ and $\cos(wx)$. Special attention is paid to the phase properties of the method during the process of parameter selection. Numerical comparisons of the proposed and existing hybrid methods for several second-order problems show that the proposed method gives high accuracy in solving the Duffing equation and Kramarz's system.

Keywords: hybrid method; variable coefficients; second-order ordinary differential equation

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1. Introduction

Many problems that arise in modelling physical phenomena in engineering and applied sciences are in the form of second ordinary initial value problems

$$y''(t) = f(t, y(t)), y(t_0) = y_0, y'(t_0) = y'_0$$

where the first derivative does not appear explicitly. These problems are often solved by using numerical methods such as Runge–Kutta–Nystrom methods, multistep methods, and hybrid methods (see [1–4]). The numerical methods can be grouped into two categories: (1) methods with constant coefficients and (2) methods with variable coefficients. The methods with variable coefficients require prior knowledge of the frequency of the problem, in contrast to the methods with constant coefficients in which the frequency of the problem is not needed. In this paper, our purpose is to derive a modified hybrid method with variable coefficients for solving the special second-order initial value problems by paying special attention to the phase properties of the methods.

Consider the class of hybrid methods proposed by Kalogiratou et al. [5]:

$$y_{n+1} = 2\sigma_{s+1}y_n - \mu_{s+1}y_{n-1} + h^2\sum_{j=1}^{s} b_j f(t_n + c_j h, g_j) \tag{1}$$

with $g_i = \sigma_i(1 + c_i)y_n - \mu_i c_i y_{n-1} + h^2 \sum_{j=1}^{s} a_{ij} f(t_n + c_j h, g_j), i = 1, \dots, s$. It is noted that if $\sigma_i = 1$ and $\mu_i = 1$ for $i = 1, \dots, s+1$ then the above class of hybrid methods is reduced to the class of hybrid methods as stated in [6]:

$$y_{n+1} = 2y_n - y_{n-1} + h^2\sum_{j=1}^{s} b_j f(t_n + c_j h, g_j) \tag{2}$$

with $g_i = (1 + c_i)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{s} a_{ij} f(t_n + c_j h, g_j)$.

Coefficients of this class of methods are as shown in Butcher tableau notation below:

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

with $c^T = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \end{pmatrix}$, $b^T = \begin{pmatrix} b_1 & b_2 & \cdots & b_s \end{pmatrix}$, and $A = [a_{ij}]_{s\times s}$.

2. Phase Lag and Stability Analysis

The standard equation

$$y''(t) = -\lambda^2 y(t),\ \lambda > 0 \tag{3}$$

with exact solution $y(t) = C_1 \exp(i\lambda t) + C_2 \exp(-i\lambda t)$ is usually used to study the stability of numerical methods in solving second-order ordinary differential equations. Applying the hybrid methods defined in (1) with coefficients depending on $v = wh$, where w is the frequency of the problem and h is the step-size, to the differential Equation (3) gives us

$$y_{n+1} - S(H^2, v)y_n + P(H^2, v)y_{n-1} = 0 \tag{4}$$

where $H = \lambda h$, $e = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^T$, $\sigma(v) = (\sigma_1\sigma_2\cdots\sigma_s)^T$, $\mu(v) = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_s \end{pmatrix}^T$, $S(H^2, v) = 2\sigma_{s+1} - H^2 b^T (I + H^2 A)^{-1} \sigma(v) \times (e + c)$, $P(H^2, v) = \mu_{s+1} - H^2 b^T (I + H^2 A)^{-1}$ $\mu(v) \times c$ and the symbol "$\times$" denotes component-wise multiplication. The characteristic polynomial associated with the difference Equation (4) is given by

$$\pi(\varsigma) = \varsigma^2 - S(H^2, v)\varsigma + P(H^2, v) \tag{5}$$

The following definition gives a condition to be satisfied by the region of absolute stability of hybrid methods (refer to [5]).

Definition 1. *For hybrid methods corresponding to Equation (5), a region of absolute stability is the region of the H-v plane throughout which* $|P(H^2, v)| < 1$ *and* $|S(H^2, v)| < 1 + P(H^2, v)$.

The phase properties of hybrid methods are given by these definitions (refer to [7]).

Definition 2. *For hybrid methods corresponding to Equation (5), the phase-lag or dispersion error is given by* $\phi(H^2, v) = H - \arccos\left(S(H^2, v)/2\sqrt{P(H^2, v)}\right)$ *and the phase-lag order is q if* $\phi(H^2, v) = c_\phi H^{q+1} + O(H^{q+3})$.

Definition 3. *For hybrid methods corresponding to Equation (5), the amplification or dissipation error is given by* $d(H^2, v) = 1 - \sqrt{P(H^2, v)}$ *and the dissipation order is u if* $d(H^2, v) = c_d H^{u+1} + O(H^{u+3})$. *The method is called zero dissipative if* $d(H^2, v) = 0$

3. Derivation of the New Method

Consider the coefficients of a class of four-stage explicit hybrid methods defined in (1) as stated in Table 1.

Table 1. Coefficients of a class of four-stage explicit hybrid methods defined in (1).

0	0	0	0	0	0	0
1	σ_2	μ_2	a_{21}	0	0	0
c_3	σ_3	μ_3	a_{31}	a_{32}	0	0
c_4	σ_4	μ_4	a_{41}	a_{42}	a_{43}	0
	σ_5	μ_5	b_1	b_2	b_3	b_4

Using these coefficients, $P(H^2, v)$ is given by

$$P(H^2, v) = -H^2\mu_2((H^2a_{43})b_4 - b_3)(H^2a_{32}) + H^2(H^2a_{43})b_4c_3\mu_3 + H^2(H^2a_{42})b_4\mu_2 + (-b_3c_3\mu_3 - b_4c_4\mu_4 - b_2\mu_2)H^2 + \mu_5$$

Setting $b_1 = 0$, $a_{32} = 0$, $a_{42} = 0$, and $a_{43} = 0$, then solving the order conditions for fourth-order hybrid method as listed in [6]

$$\begin{aligned}
&b_1 + b_2 + b_3 + b_4 = 1 \\
&b_2 + b_3c_3 + b_4c_4 = 0 \\
&b_2 + b_3c_3^2 + b_4c_4^2 = \tfrac{1}{6} \\
&b_2a_{21} + b_3a_{31} + b_3a_{32} + b_4a_{41} + b_4a_{42} + b_4a_{43} = \tfrac{1}{12} \\
&b_2 + b_3c_3^3 + b_4c_4^3 = 0 \\
&b_2a_{21} + b_3c_3a_{31} + b_3c_3a_{32} + b_4c_4a_{41} + b_4c_4a_{42} + b_4c_4a_{43} = \tfrac{1}{12} \\
&b_3a_{32} + b_4a_{42} + b_4a_{43}c_3 = 0
\end{aligned}$$

yields

$$b_2 = \frac{6c_4^2 - 1}{6(-1 + c_4)(7c_4 + 2)}, b_3 = \frac{216c_4^3 + 108c_4^2 + 18c_4 + 1}{6(7c_4 + 2)(6c_4^2 + 2c_4 + 1)}, \ b_4 = -\frac{5}{6(-1 + c_4)(6c_4^2 + 2c_4 + 1)}$$

$$a_{31} = \frac{-2 - 5c_4 + 7c_4^2 + a_{21}(2 - 12c_4^2)}{2(1 + 6c_4)^2},$$

$$a_41 = -\frac{7}{10}c_4^2 + \frac{6}{5}c_4^2a_{21} + \frac{1}{2}c_4 + \frac{1}{5} - \frac{1}{5}a_{21},$$

$$c_3 = -\frac{c_4 + 1}{1 + 6c_4}$$

where c_4 and a_{21} are free parameters. By experiment, we choose $c_4 = -\frac{1}{2}$ to make $P(H^2, v)$ as close as possible to 1 as $v \to 0$. In order to obtain a_{21}, σ_i and μ_i, we associate each stage formula of the method with linear operator $L[y(t)]$ as follows:

$$\begin{aligned}
L_1[y(t)] &= y(t + h) - 2\sigma_2 y(t) + \mu_2 y(t - h) - h^2 a_{21} y''(t) \\
L_2[y(t)] &= y(t + c_3h) - \sigma_3(1 + c_3)y(t) + \mu_3c_3y(t - h) - h^2(a_{31}y''(t) + a_{32}y''(t + h)) \\
L_3[y(t)] &= y(t + c_4h) - \sigma_4(1 + c_4)y(t) + \mu_4c_4y(t - h) - h^2(a_{41}y''(t) + a_{42}y''(t + h) + a_{43}y''(t + c_3h)) \\
L_4[y(t)] &= y(t + h) - 2\sigma_5 y(t) + \mu_5 y(t - h) - h^2(b_1y''(t) + b_2y''(t + h) + b_3y''(t + c_3h) + b_4y''(t + c_4h))
\end{aligned}$$

Assume that $v = wh$. Setting $L_1[e^{wx}] = 0$, $L_1[\sin(wx)] = 0$ and $L_1[\cos(wx)] = 0$ results in

$$a_{21} = \frac{e^{2v} - 2e^v + 1}{e^v v^2}, \sigma_2 = \frac{e^{2v} + 2\cos(v)e^v - 2e^v + 1}{2e^v}, \ \mu_2 = 1$$

This implies $a_{31} = \frac{-(e^{2v} - 2e^v + 1)}{8e^v v^2} + \frac{9}{32}$ and $a_{41} = -\frac{9}{40} + \frac{e^{2v} - 2e^v + 1}{10e^v v^2}$. Finally, by setting

$$L_2[\sin(wx)] = 0, \ L_2[\cos(wx)] = 0, \ L_3[\sin(wx)] = 0, \ L_3[\cos(wx)] = 0, \ L_4[\sin(wx)] = 0, \text{ and } L_4[\cos(wx)] = 0$$

we have

$$\sigma_3 = \frac{1}{40\sin(v)e^v}(9e^v v^2 \sin(v) + 32\cos(v/4)\sin(v)e^v + 32\sin(v/4)\cos(v)e^v - 4\sin(v)e^{2v} + 8\sin(v)e^v - 4\sin(v))$$

$$\sigma_4 = \frac{1}{20\sin(v)e^v}(-9e^v v^2 \sin(v) + 40\cos(v/2)\sin(v)e^v - 40\sin(v/2)\cos(v)e^v + 4\sin(v)e^{2v} - 8\sin(v)e^v + 4\sin(v))$$

$$\sigma_5 = \tfrac{1}{27\sin(v)}(\cos(v)\sin(v)v^2 + 8\cos(v)\sin(v/4)v^2 - 5\cos(v)\sin(v/2)v^2 + 8\sin(v)\cos(v/4)v^2 + 5\sin(v)\cos(v/2)v^2 + 27\cos(v)\sin(v))$$

$$\mu_3 = \frac{4\sin(v/4)}{\sin(v)},\ \mu_4 = \frac{2\sin(v/2)}{\sin(v)},$$

$$\mu_5 = \frac{1}{27\sin(v)}(v^2\sin(v) + 16v^2\sin(v/4) - 10v^2\sin(v/2) + 27\sin(v))$$

The resulting method is denoted by MEHM. This method has the following quantities:

$$P(H^2, v) = \frac{(-2H^2 + 2v^2 + 54)\cos^3(v/4) + (6H^2 - 6v^2 - 27)\cos(v/4) - 4H^2 + 4v^2}{54\cos^3(v/4) - 27\cos(v/4)}$$

$$S(H^2, v) = \tfrac{1}{108(2\cos^3(v/4) - \cos(v/4))}(128\cos^7(v/4)v^2 + 3456\cos^7(v/4) - 192\cos^5(v/4)v^2 + 256\cos^4(v/4)v^2 \\ -5184cos^5(v/4) + 80\cos^3(v/4)v^2 - 192\cos^2(v/4)v^2 + 2160\cos^3(v/4) + 12\cos(v/4)v^2 + 16v^2 - 216\cos(v/4) + \\ (-128\cos^7(v/4) + 192\cos^5(v/4) - 18\cos^3(v/4)v^2 - 256\cos^4(v/4) - 80\cos^3(v/4) + 9\cos(v/4)v^2 + \\ 192\cos^2(v/4) - 12\cos(v/4) - 16)H^2 + (18\cos^3(v/4) - 9\cos(v/4))H^4)$$

It is also noted that $\lim_{v\to 0} P(H^2, v) = 1$ and $\lim_{v\to 0} S(H^2, v) = -H^2 + \frac{1}{12}H^4 + 2$, with $\frac{S(H^2,v)}{2}$ being the rational approximation for the cosine as $v \to 0$. The method is considered to be zero dissipative whenever $v \to 0$. Solving $-H^2 + \frac{1}{12}H^4 + 2 < 2$ for $H > 0$, we obtain $H < 2\sqrt{3}$. It is also observed that the local truncation error is $O(h^6)$ as $v \to 0$. The region of absolute stability of this method depicted using Maple 2020 software is shown below in Figure 1.

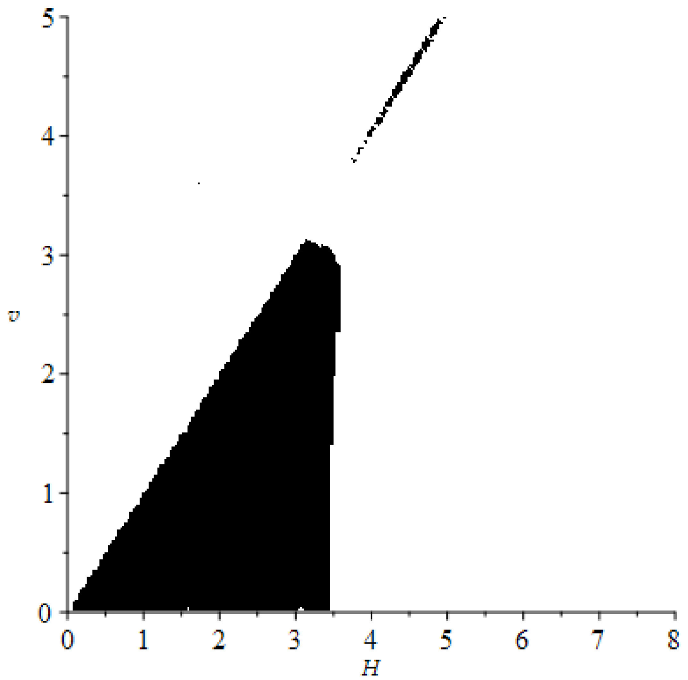

Figure 1. Region of absolute stability of the proposed method.

4. Results

The new and existing codes are abbreviated as follows.

MEHM: The modified explicit hybrid method with four stages derived in this paper.

EHM5IIPA: The phase-fitted and amplification-fitted explicit hybrid method with four stages derived in [8]. This method was derived based on the fifth-order hybrid method of the form (2).

Several problems are used to provide numerical comparisons in a constant step-size setting. Maximum global errors produced by each method are tabulated in Tables 2–5. All numerical computations have been done in Maple 2020 software with 20 precision digits.

Problem 1 (Prothero–Robinson problem)

Source: D'Ambrosio et al. [9]

$$y''(t) = -(y(t) - e^{-\mu t})) + \mu^2 e^{(-\mu t)}, y(0) = 1, y'(0) = -\mu, 0 \le t \le 10$$

Exact solution: $y(t) = e^{-\mu t}$. We use $v = h$ in computing the numerical solutions for $\mu = 1$, with MEHM and EHM5IIPA codes.

Table 2. Maximum global error in solving Problem 1.

Step-Size	MEHM	EHM5IIPA
0.4	8.12463×10^{-6}	3.03912×10^{-5}
0.2	4.72859×10^{-7}	1.19831×10^{-6}
0.1	2.80407×10^{-8}	4.23368×10^{-8}
0.05	1.69979×10^{-9}	1.41116×10^{-9}
0.025	1.04445×10^{-10}	4.55621×10^{-11}

Problem 2 (Duffing equation)

Source: Yusufoğlu [10]

$$y''(t) + 3y(t) - 2y^3(t) = \cos(t)\sin(2t), y(0) = 0, y'(0) = 1, 0 \le t \le 20$$

Exact solution:$y(t) = \sin(t)$. For MEHM and EHM5IIPA codes, $v = h$ was used.

Table 3. Maximum global error in solving Problem 2.

Step-Size	MEHM	EHM5IIPA
0.4	2.48225×10^{-14}	1.02736
0.2	5.51845×10^{-13}	2.71483×10^{-1}
0.1	2.95522×10^{-13}	8.20955×10^{-2}
0.05	3.76672×10^{-12}	2.97346×10^{-3}
0.025	4.66915×10^{-12}	9.77418×10^{-5}

Problem 3 (The well-known two-body problem)

Source: Franco [11]

$$y_1'' = -\frac{y_1}{(y_1^2 + y_2^2)^{(3/2)}}, y_1(0) = 1 - e, y_1'(0) = 0, y_2'' = -\frac{y_2}{(y_1^2 + y_2^2)^{(3/2)}}, y_2(0) = 0, y_2'(0) = \sqrt{\frac{1+e}{1-e}}, 0 \le t \le 20$$

Exact solution:$y_1(t) = \cos(R) - e, y_2(t) = \sqrt{1-e^2}\sin(R)$, where R satisfies the Kepler's equation $t = R - e\sin(R)$ and e is the eccentricity of the orbit. In this numerical experiment, we consider the case $e = 0.03$. For MEHM and EHM5IIPA codes, $v = h$.

Table 4. Maximum global error in solving Problem 3.

Step-Size	MEHM	EHM5IIPA
0.4	1.42361×10^{-2}	2.29762×10^{-1}
0.2	9.29187×10^{-4}	1.98607×10^{-3}
0.1	6.00156×10^{-5}	1.82083×10^{-4}
0.05	3.81442×10^{-6}	6.89947×10^{-6}
0.025	2.40430×10^{-7}	2.27004×10^{-7}

Problem 4 (Kramarz's system)

Source: D'Ambrosio et al. [12]

$$y''(t) = \begin{pmatrix} \mu - 2 & 2\mu - 2 \\ 1 - \mu & 1 - 2\mu \end{pmatrix} y(t), y(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, y'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $\mu = 2500$ and $0 \leq t \leq 5$.

Exact solution: $y_1(t) = 2\cos(t), y_2(t) = -\cos(t)$. For both codes, $v = h$ is used.

Table 5. Maximum global error in solving Problem 4.

Step-Size	MEHM	EHM5IIPA
0.05	1.16031×10^{-16}	1.74602×10^{-16}
0.025	1.72165×10^{-16}	1.67037×10^{-15}
0.0125	5.41637×10^{-15}	5.97021×10^{-16}
0.00625	7.41002×10^{-15}	1.49427×10^{-14}
0.003125	2.45548×10^{-14}	5.03433×10^{-14}

5. Discussion and Conclusions

In this paper, a modified explicit hybrid method with four stages was proposed. The derivation of the method is based on the modified formula of hybrid method given in (1) while taking into consideration $\lim_{v \to 0} P(H^2, v)$. For this method, it was our intention to achieve $\lim_{v \to 0} P(H^2, v) = 1$ in such a way that $\frac{S(H^2,v)}{2}$ is the rational approximation for the cosine, as studied by Coleman [13]. Moreover, the first stage of the modified formula is imposed to exactly integrate e^{wx}, while the remaining stages are imposed to exactly integrate sin(wx) and cos(wx) where $w \in \mathbf{C}$. The maximum global errors of the new method were tabulated and compared with that of the phase-fitted and amplification fi-ted hybrid method in [8]. From the numerical results, the new method was observed to achieve high accuracy in solving the Duffing equation and Kramarz's system. Furthermore, the new method performs with better accuracy for bigger step-sizes than that of the existing method for solving both the Prothero and Robinson and the two-body problems. Hence, this study offers evidence that, by taking into account $\lim_{v \to 0} P(H^2, v)$, the resulting modified explicit hybrid method is capable of solving second-order ordinary differential equations $y''(t) = f(t, y)$.

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