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New Fixed Point Results via a Graph Structure

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Abstract: The main aim of this paper is to introduce and study some fixed point results for rational multivalued G-contraction and F-Khan-type multivalued contraction mappings on a metric space with a graph. At the end, we give an illustrative example.

Keywords: fixed point; directed graph; metric space; multivalued map

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1. Introduction

Without doubt, the two most important results in fixed point theory are the Banach contraction principle and the Tarski fixed point theorem. These results have been the subject of many generalizations and extensions, either by extending the contractive condition or changing the structure of the space. In [1], Echenique gave a proof of Tarski fixed point theorem by using graphs combining fixed point techniques and graph theory. Then, Jachymski [2] developed another structure in the fixed point theory of metric spaces by replacing the structure of orders by the structure of graphs on metric spaces. He obtained sufficient conditions for self-mappings to be a Picard operator on a metric space endowed with a graph. Fixed point theory and graph theory provide an intersection between the theories of fixed point results, which give the conditions under which (single or multivalued) mappings have solutions, and graph theory, which uses mathematical structures to illustrate the relationship between ordered pairs of elements in terms of their vertices and directed edges. Note that metric fixed point and graph theory have common application environments. In the multivalued case, the authors in [3] proved a fixed point theorem for Mizoguchi–Takahashi-type contractions on a metric space endowed with a graph. For further results in this direction, we refer to [4–11]. Recently, in [12], the authors introduced a new concept of contractions called F-Khan contractions and proved a related fixed point theorem.

In this paper, we present new fixed point results for variant multivalued mappings via a graph structure. To do this, we introduce new types of contractions called rational-type multivalued G-contractions and F-Khan-type multivalued contractions. We ensure that related mappings (involving a graph structure) have a fixed point. At the end, we consider an example to show the validity of our obtained results and to ensure that the known results in the literature are not applicable.

2. Preliminaries

Let (X, d) be a metric space. $P(X)$ denotes the family of all nonempty subsets of X; $CB(X)$ denotes the family of all nonempty, closed, and bounded subsets of X; and $K(X)$ denotes the family of all nonempty compact subsets of X. The Pompeiu–Hausdorff metric $H : CB(X) \times CB(X) \to [0, \infty)$ is defined as

$$H(A, B) = \max\{\sup_{\alpha \in A} D(\alpha, B), \sup_{\beta \in B} D(\beta, B)\}$$

where $A, B \in CB(X)$ and $D(\alpha, B) = \inf_{\beta \in B} d(\alpha, \beta)$. Using the Pompeiu–Hausdorff metric H, Nadler [13] established that every multivalued contraction mapping on a complete metric space has a fixed point. Inspired by his result, various fixed point results concerning multivalued contraction mappings appeared in the last several decades [14–22].

On the other hand, Jachymski [2] used a graph structure instead of partial orders to prove some important fixed point results. Additionally, Bojor [23] established some fixed point results in a metric space endowed with a graph. We can also find some fixed point results on a metric space with a graph structure in [4,5,7,9,24].

Now, we recall some definitions and lemmas.

Definition 1. *Ref. [2] Let X be a non-empty set and Δ denote the diagonal of the cartesian product $X \times X$. A directed graph or digraph G is characterized by a nonempty set $V(G)$ of its vertices and the set $E(G) \subset V(G) \times V(G)$ of its directed edges. A digraph is reflexive if any vertex admits a loop. For a digraph $G = (V, E)$,*

(i) *If whenever $(a, b) \in E(G) \Rightarrow (b, a) \notin E(G)$, then the digraph G is called an oriented graph.*
(ii) *A digraph G is transitive whenever $[(a, b) \in E(G)$ and $(b, c) \in E(G)] \Rightarrow (a, c) \in E(G)$, for any $a, b, c \in V(G)$.*
(iii) *A path of G is a sequence $x_0, x_1, x_2, ..., x_n, ...$ with $(x_i, x_{i+1}) \in E(G)$ for each $i \in \mathbb{N}$.*
(iv) *G is connected if there is a path between each two vertices, and it is weakly connected if $\widetilde{G}$ is connected, where $\widetilde{G}$ corresponds to the undirected graph obtained from G by ignoring the direction of edges.*
(v) *Let G^{-1} be the graph obtained from G by reversing the direction of edges. Thus,*

$$E(G^{-1}) = \{(a, b)\} \in X \times X : (b, a) \in E(G).$$

(vi) *We call (V', E') a subgraph of G if $V' \subseteq V(G)$ and $E' \subseteq E(G)$ and for any edge $(a, b) \in E'$, $a, b \in V'$.*

We refer to [25] for more details on graph theory.

Definition 2. *Ref. [4] Let (X, d) be a metric space endowed with a graph G such that $V(G) = X$ and let $T : X \to CB(X)$ be a multivalued mapping. T has the weakly graph-preserving (WGP) property, whenever for each $a \in X$ and $b \in Ta$ with $(a, b) \in E(G)$ implies $(b, c) \in E(G)$ for all $c \in Tb$.*

Definition 3. *Ref. [21] Let $F : (0, \infty) \to \mathbb{R}$ be a function verifying the following conditions:*

$(F1)$ *For all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta, F(\alpha) < F(\beta)$;*
$(F2)$ *For any positive real sequence $\{a_n\}$,*

$$\lim_{n\to\infty} a_n = 0 \text{ iff } \lim_{n\to\infty} F(a_n) = -\infty;$$

$(F3)$ *There is $k \in (0, 1)$ so that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$;*
$(F4)$ *$F(\inf A) = \inf F(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.*

We mean by $F \in \mathcal{F}$ if F verifies the conditions $(F1)$–$(F3)$, and by $F \in \mathcal{F}_*$ if the conditions $(F1)$–$(F4)$ hold. Clearly, $\mathcal{F}_* \subset \mathcal{F}$.

Lemma 1. *Ref. [4] Let (X, d) be a metric space and $T : X \to P(X)$ be an upper semi-continuous mapping such that Tr is closed for all $r \in X$. If $r_n \to r_0$, $t_n \to t_0$, and $t_n \in Tr_n$, then $t_0 \in Tr_0$.*

A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ (where $\mathbb{R}^+ = [0, \infty)$) is said to be a comparison function if it satisfies the following conditions:

(i) φ is monotonically increasing;
(ii) $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \in \mathbb{R}^+$.

Meanwhile, if φ satisfies (i) and the following condition:

(iii) $\sum_{n=0}^{\infty} \varphi^n(t)$ is convergent for each $t \geq 0$,

then φ is named to be a (c)-comparison function.

Lemma 2. *Ref. [26] Let (X, d) be a metric space. Let $A, B \subset X$, and $q > 1$. Then, for every $a \in A$ there exists $b \in B$ such that*

$$d(a, b) \leq qH(A, B).$$

3. Main Results

3.1. On Rational-Type Multivalued G-Contractions

Theorem 1. *Let G be a directed graph on a complete metric space (X, d) and $f : X \to CB(X)$ be a multivalued mapping. Suppose that f is upper semi-continuous and a WGP mapping. Suppose that:*

(i) f is a rational multivalued G-contraction of type I, that is, there is a strictly increasing (c)-comparison function φ such that

$$H(fa, fb) \leq \varphi(M(a, b)) \tag{1}$$

for all $(a, b) \in E(G)$, where

$$M(a,b) = \max\left\{ \begin{array}{l} d(a,b), \frac{D(a,fa)+D(b,fb)}{2}, \frac{D(a,fb)+D(b,fa)}{2}, \\ \frac{D(a,fa)D(b,fb)}{d(a,b)}, \frac{D(b,fb)[1+D(a,fa)]}{1+d(a,b)} \end{array} \right\};$$

(ii) $M_f = \{a \in X : (a, u) \in E(G) \text{ for } u \in fa\}$ is nonempty.

Then, f admits a fixed point.

Proof. Set $a_0 \in M_f$. There exists $a_1 \in fa_0$ such that $(a_0, a_1) \in E(G)$. So, we can use the condition (i) for a_0 and a_1. Then, we have

$$\begin{aligned}
D(a_1, fa_1) &\leq H(fa_0, fa_1) \\
&\leq \varphi(M(a_0, a_1)) \\
&= \varphi\left(\max\left\{ \begin{array}{l} d(a_0,a_1), \frac{D(a_0,fa_0)+D(a_1,fa_1)}{2}, \frac{D(a_0,fa_1)+D(a_1,fa_0)}{2}, \\ \frac{D(a_0,fa_0)D(a_1,fa_1)}{d(a_0,a_1)}, \frac{D(a_1,fa_1)[1+D(a_0,fa_0)]}{1+d(a_0,a_1)} \end{array} \right\} \right) \\
&\leq \varphi\left(\max\left\{ \begin{array}{l} d(a_0,a_1), \frac{d(a_0,a_1)+D(a_1,fa_1)}{2}, \frac{D(a_0,fa_1)+d(a_1,a_1)}{2}, \\ \frac{d(a_0,a_1)D(a_1,fa_1)}{d(a_0,a_1)}, \frac{D(a_1,fa_1)[1+d(a_0,a_1)]}{1+d(a_0,a_1)} \end{array} \right\} \right) \\
&\leq \varphi\left(\max\left\{ d(a_0,a_1), \frac{d(a_0,a_1)+D(a_1,fa_1)}{2}, D(a_1,fa_1), \right\} \right) \\
&\leq \varphi(\max\{d(a_0,a_1), D(a_1,fa_1), \}) \\
&\leq \varphi(d(a_0,a_1)).
\end{aligned}$$

Given $\sigma > 1$ an arbitrary constant. Therefore, from Lemma 2, there exists $a_2 \in fa_1$ such that

$$d(a_1, a_2) \leq \sqrt{\sigma} H(fa_0, fa_1).$$

Recall that $H(fa_0, fa_1) \leq \varphi(d(a_0, a_1))$, so one writes

$$d(a_1, a_2) \leq \sqrt{\sigma}\varphi(d(a_0, a_1)) < \sigma\varphi(d(a_0, a_1)).$$

Since φ is strictly increasing, we have

$$0 < \varphi(d(a_1, a_2)) < \varphi(\sigma\varphi(d(a_0, a_1))).$$

Set $\sigma_1 = \frac{\varphi(\sigma\varphi(d(a_0,a_1)))}{\varphi(d(a_1,a_2))}$. Then, $\sigma_1 > 1$. Since $(a_0, a_1) \in E(G)$, $a_1 \in fa_0$ and $a_2 \in fa_1$, using the WGP property, one writes $(a_1, a_2) \in E(G)$. Then,

$$\begin{aligned}
D(a_2, fa_2) &\leq H(fa_1, fa_2) \\
&\leq \varphi(M(a_1, a_2)) \\
&= \varphi\left(\max\left\{\begin{array}{l} d(a_1, a_2), \frac{D(a_1,fa_1)+D(a_2,fa_2)}{2}, \frac{D(a_1,fa_2)+D(a_2,fa_1)}{2}, \\ \frac{D(a_1,fa_1)D(a_2,fa_2)}{d(a_1,a_2)}, \frac{D(a_2,fa_2)[1+D(a_1,fa_1)]}{1+d(a_1,a_2)} \end{array}\right\}\right) \\
&\leq \varphi\left(\max\left\{\begin{array}{l} d(a_1, a_2), \frac{d(a_1,a_2)+D(a_2,fa_2)}{2}, \frac{D(a_1,fa_2)+d(a_2,a_2)}{2}, \\ \frac{d(a_1,a_2)D(a_2,fa_2)}{d(a_1,a_2)}, \frac{D(a_2,fa_2)[1+d(a_1,a_2)]}{1+d(a_1,a_2)} \end{array}\right\}\right) \\
&= \varphi\left(\max\left\{d(a_1, a_2), \frac{d(a_1, a_2) + D(a_2, fa_2)}{2}, D(a_2, fa_2), \right\}\right) \\
&= \varphi(\max\{d(a_1, a_2), D(a_2, fa_2), \}) \\
&\leq \varphi(d(a_1, a_2)) \\
&< \sqrt{\sigma_1}\varphi(d(a_1, a_2)).
\end{aligned}$$

Therefore, from Lemma 2 there exists $a_3 \in fa_2$ such that

$$d(a_2, a_3) \leq \sqrt{\sigma_1} H(fa_1, fa_2) < \sigma_1\varphi(d(a_2, a_1)) = \varphi(\sigma\varphi(d(a_0, a_1))).$$

Since φ is strictly increasing, we have

$$0 < \varphi(d(a_2, a_3)) < \varphi^2(\sigma\varphi(d(a_0, a_1))).$$

Get $\sigma_2 = \frac{\varphi^2(\sigma\varphi(d(a_0,a_1)))}{\varphi(d(a_2,a_3))}$. Then, $\sigma_2 > 1$. Continuing, we construct the sequence $\{a_n\}$ in X so that $a_{n+1} \in fa_n$, $(a_n, a_{n+1}) \in E(G)$ and

$$d(a_n, a_{n+1}) \leq \varphi^n(\sigma\varphi(d(a_0, a_1))).$$

To prove that $\{a_n\}$ is a Cauchy sequence, take $m, n \in \mathbb{N}$ with $m > n$. Consider

$$\begin{aligned}
d(a_n, a_m) &\leq \sum_{k=n}^{m-1} d(a_k, a_{k+1}) \\
&\leq \sum_{k=n}^{m-1} \varphi^k(\sigma\varphi(d(a_0, a_1))).
\end{aligned}$$

Since φ is a (c)-comparison function, the series on the right-hand side converges, and so $d(a_n, a_m) \to 0$ as $n, m \to \infty$. That is, $\{a_n\}$ is a Cauchy sequence in (X, d), which is complete, so $\{a_n\}$ is convergent to some $z \in X$, that is, $\lim\limits_{n\to\infty} a_n = z$. Since f is upper semi-continuous, using Lemma 1, we find $z \in fz$. That is, f has a fixed point in X. $\square$

Consider the following property:

$$(P)\text{-}property: \left\{ \begin{array}{c} \text{for any } \{a_n\} \text{ in } X, \text{ if } a_n \to a \text{ and } (a_n, a_{n+1}) \in E(G), \\ \text{then there is a subsequence } \{a_{n_k}\} \text{ with } (a_{n_k}, a) \in E(G). \end{array} \right.$$

Theorem 2. *Let G be a directed graph on a complete metric space (X,d) and $f : X \to CB(X)$ be a multivalued mapping such that the following conditions are verified:*

(i) *f is a rational multivalued G-contraction of type II, that is, there is a strictly increasing (c)-comparison function φ such that*

$$H(fa, fb) \le \varphi(N(a,b))$$

for all $(a,b) \in E(G)$, where

$$N(a,b) = \max\left\{ d(a,b), \frac{D(a,fa) + D(b,fb)}{2}, \frac{D(a,fb) + D(b,fa)}{2}, \frac{D(a,fa)D(b,fb)}{1 + H(fa,fb)} \right\};$$

(ii) *M_f (defined in Theorem 1) is nonempty;*
(iii) *The (P)-property is satisfied;*
(iv) *f is a WGP mapping.*

Then, f has a fixed point.

Proof. Take $a_0 \in M_f$. There is $a_1 \in fa_0$ such that $(a_0, a_1) \in E(G)$. So, we can use the condition (i) for a_0 and a_1. Then, we have

$$\begin{aligned}
D(a_1, fa_1) &\le H(fa_0, fa_1) \\
&\le \varphi(N(a_0, a_1)) \\
&= \varphi\left(\max\left\{ \begin{array}{c} d(a_0,a_1), \frac{D(a_0,fa_0)+D(a_1,fa_1)}{2}, \\ \frac{D(a_0,fa_1)+D(a_1,fa_0)}{2}, \frac{D(a_0,fa_0)D(a_1,fa_1)}{1+H(fa_0,fa_1)} \end{array} \right\} \right) \\
&\le \varphi\left(\max\left\{ \begin{array}{c} d(a_0,a_1), \frac{d(a_0,a_1)+D(a_1,fa_1)}{2}, \\ \frac{D(a_0,fa_1)+d(a_1,a_1)}{2}, \frac{d(a_0,a_1)D(a_1,fa_1)}{D(a_1,fa_1)} \end{array} \right\} \right) \\
&\le \varphi\left(\max\left\{ d(a_0,a_1), \frac{d(a_0,a_1) + D(a_1,fa_1)}{2}, D(a_1,fa_1), \right\} \right) \\
&\le \varphi(\max\{d(a_0,a_1), D(a_1,fa_1), \}) \\
&= \varphi(d(a_0,a_1)) \\
&< \sqrt{\sigma}\varphi(d(a_0,a_1))
\end{aligned}$$

where $\sigma > 1$ is a constant. Therefore, from Lemma 2, there exists $a_2 \in fa_1$ such that

$$d(a_1, a_2) \le \sqrt{\sigma} H(fa_0, fa_1) < \sigma\varphi(d(a_0, a_1)).$$

Since φ is strictly increasing, we have

$$0 < \varphi(d(a_1, a_2)) < \varphi(\sigma\varphi(d(a_0, a_1))).$$

Take $\sigma_1 = \frac{\varphi(\sigma\varphi(d(a_0,a_1)))}{\varphi(d(a_1,a_2))}$. We have $\sigma_1 > 1$. In view of $(a_0, a_1) \in E(G)$, $a_1 \in fa_0$, $a_2 \in fa_1$, and using the WGP property, one writes $(a_1, a_2) \in E(G)$. Then,

$$\begin{aligned}
D(a_2, fa_2) &\leq H(fa_1, fa_2) \\
&\leq \varphi(N(a_1, a_2)) \\
&= \varphi\left(\max\left\{\begin{array}{l} d(a_1, a_2), \frac{D(a_1,fa_1)+D(a_2,fa_2)}{2}, \\ \frac{D(a_1,fa_2)+D(a_2,fa_1)}{2}, \frac{D(a_1,fa_1)D(a_2,fa_2)}{1+H(fa_1,fa_2)} \end{array}\right\}\right) \\
&\leq \varphi\left(\max\left\{\begin{array}{l} d(a_1, a_2), \frac{d(a_1,a_2)+D(a_2,fa_2)}{2}, \\ \frac{D(a_1,fa_2)+d(a_2,a_2)}{2}, \frac{d(a_1,a_2)D(a_2,fa_2)}{D(a_2,fa_2)} \end{array}\right\}\right) \\
&= \varphi\left(\max\left\{d(a_1, a_2), \frac{d(a_1, a_2) + D(a_2, fa_2)}{2}, D(a_2, fa_2), \right\}\right) \\
&= \varphi(\max\{d(a_1, a_2), D(a_2, fa_2), \}) \\
&\leq \varphi(d(a_1, a_2)) \\
&< \sqrt{\sigma_1}\varphi(d(a_1, a_2)).
\end{aligned}$$

Therefore, from Lemma 2, there exists $a_3 \in fa_2$ such that

$$d(a_2, a_3) \leq \sqrt{\sigma_1}H(fa_1, fa_2) < \sigma_1\varphi(d(a_2, a_1)) = \varphi(\sigma\varphi(d(a_0, a_1))).$$

Since φ is strictly increasing, we have

$$0 < \varphi(d(a_2, a_3)) < \varphi^2(\sigma\varphi(d(a_0, a_1))).$$

Set $\sigma_2 = \frac{\varphi^2(\sigma\varphi(d(a_0,a_1)))}{\varphi(d(a_2,a_3))}$. Then, $\sigma_2 > 1$. Proceeding again, we construct the sequence $\{a_n\}$ in X such that $a_{n+1} \in fa_n$, $(a_n, a_{n+1}) \in E(G)$ and

$$d(a_n, a_{n+1}) \leq \varphi^n(\sigma\varphi(d(a_0, a_1))).$$

We will show that $\{a_n\}$ is a Cauchy sequence. Let $m, n \in \mathbb{N}$ with $m > n$. Consider

$$\begin{aligned}
d(a_n, a_m) &\leq \sum_{k=n}^{m-1} d(a_k, a_{k+1}) \\
&\leq \sum_{k=n}^{m-1} \varphi^k(\sigma\varphi(d(a_0, a_1))).
\end{aligned}$$

Since φ is a (c)-comparison function, the series on the right-hand side converges, and so we get $d(a_n, a_m) \to 0$ as $n, m \to \infty$. That is, $\{a_n\}$ is a Cauchy sequence in (X, d), which is complete, hence $\{a_n\}$ is convergent to some $z \in X$, that is, $\lim_{n\to\infty} a_n = z$. By the (P)-property, there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $(a_{n_k}, z) \in E(G)$ for each $k \in \mathbb{N}$. Now, assume that $D(z, fz) > 0$. Since $\lim_{n\to\infty} D(a_{n_k}, a_{n_k+1}) = 0$, $\lim_{n\to\infty} D(a_{n_k}, z) = 0$, there is $n_0 \in \mathbb{N}$ such that for $n_k > n_0$

$$D(a_{n_k}, a_{n_k+1}) < \frac{1}{3}D(z, fz) \tag{2}$$

and there is $n_1 \in \mathbb{N}$ so that for $n_k > n_1$

$$D(a_{n_k}, z) < \frac{1}{3}D(z, fz). \tag{3}$$

If we take $n_k > \max\{n_0, n_1\}$, then by (2) and (3), we have

$$\begin{array}{rcl} D(a_{n_k+1}, fz)) & \leq & H(fa_{n_k}, fz) \\ & \leq & \varphi(N(a_{n_k}, z)) \\ & \leq & \varphi\left(\max\left\{\begin{array}{l} d(a_{n_k}, z), \frac{D(a_{n_k}, fa_{n_k})+D(z,fz)}{2}, \\ \\ \frac{D(a_{n_k}, fz)+D(z, fa_{n_k})}{2}, \frac{D(a_{n_k}, fa_{n_k})D(z,fz)}{1+H(fa_{n_k}, fz)}, \end{array}\right\}\right) \\ & \leq & \varphi\left(\max\left\{\begin{array}{l} \frac{1}{3}D(z, fz), \frac{\frac{1}{3}D(z,fz)+D(z,fz)}{2}, \\ \\ \frac{D(a_{n_k}, fz)+D(z, fa_{n_k})}{2}, \frac{\frac{1}{3}D(z,fz)D(z,fz)}{D(a_{n_{k+1}}, fz)}, \end{array}\right\}\right). \end{array}$$

Taking $k \to \infty$, we have $D(z, fz) \leq \varphi(D(z, fz)) < D(z, fz)$, which is a contradiction. Thus, $D(z, fz) = 0$ and since fz is closed, we deduce $z \in fz$. That is, f admits a fixed point. $\square$

If we assume that $f : X \to K(X)$ in the previous theorems, we do not need the strictly increasing property of φ. This corresponds to the following result.

Theorem 3. *Let (X, d) be a complete metric space, G be a directed graph on X and $f : X \to K(X)$ be a multivalued mapping. Assume that f is upper semi-continuous and a WGP mapping. Suppose that:*

(i) *There is a (c)-comparison function φ such that*

$$H(fa, fb) \leq \varphi(N(a, b))$$

for all $(a, b) \in E(G)$;

(ii) *M_f is nonempty.*

Then, f has a fixed point.

Proof. Set $a_0 \in M_f$. There is $a_1 \in fa_0$ such that $(a_0, a_1) \in E(G)$. So, we can use the condition (i) for a_0 and a_1. Then, we have

$$\begin{array}{rcl} D(a_1, fa_1) & \leq & H(fa_0, fa_1) \\ & \leq & \varphi(N(a_0, a_1)) \\ & = & \varphi\left(\max\left\{\begin{array}{l} d(a_0, a_1), \frac{D(a_0, fa_0)+D(a_1, fa_1)}{2}, \\ \\ \frac{D(a_0, fa_1)+D(a_1, fa_0)}{2}, \frac{D(a_0, fa_0)D(a_1, fa_1)}{1+H(fa_0, fa_1)} \end{array}\right\}\right) \\ & \leq & \varphi(d(a_0, a_1)). \end{array}$$

Since fa_1 is compact, there exists $a_2 \in fa_1$ such that $d(a_1, a_2) = D(a_1, fa_1)$. So,

$$d(a_1, a_2)) \leq \varphi(d(a_0, a_1)).$$

Since $(a_0, a_1) \in E(G)$, $a_1 \in fa_0$, and $a_2 \in fa_1$, using the WGP property, we get $(a_1, a_2) \in E(G)$. Then,

$$\begin{array}{rcl} D(a_2, fa_2) & \leq & H(fa_1, fa_2) \\ & \leq & \varphi(N(a_1, a_2)) \\ & \leq & \varphi(d(a_1, a_2)). \end{array}$$

Since fa_2 is compact, again there exists $a_3 \in fa_2$ such that $d(a_2, a_3) = d(a_2, fa_2)$. Therefore, we have

$$d(a_2, a_3) \leq \varphi(d(a_2, a_1)).$$

Continuing this process, we construct the sequence $\{a_n\}$ in X such that $a_{n+1} \in fa_n$, $(a_n, a_{n+1}) \in E(G)$, and

$$\begin{aligned} d(a_n, a_{n+1}) &\leq \varphi(d(a_{n-1}, a_n)) \\ &\leq \varphi^2(d(a_{n-2}, a_{n-1})) \\ &\;\;\vdots \\ &\leq \varphi^n(d(a_0, a_1)). \end{aligned}$$

The rest of the proof can be finished as in Theorem 2. □

3.2. On F-Khan Contractions

Let G be a directed graph on a metric space X and T be a mapping from X to $CB(X)$. Define

$$T_G = \{(\sigma_1, \sigma_2) \in E(G) : H(T\sigma_1, T\sigma_2) > 0\},$$

$$X_T = \{\sigma_1 \in X : (\sigma_1, \sigma_2) \in E(G) \text{ for some } \sigma_2 \in T\sigma_1\},$$

and

$$L(\sigma_1, \sigma_2) = \max\left\{ \begin{array}{c} d(\sigma_1, \sigma_2), D(\sigma_1, T\sigma_1), D(\sigma_2, T\sigma_2), \\ \frac{D(\sigma_1, T\sigma_1)D(\sigma_1, T\sigma_2) + D(\sigma_2, T\sigma_2)D(\sigma_2, T\sigma_1)}{\max\{D(\sigma_1, T\sigma_2), D(\sigma_2, T\sigma_1)\}} \end{array} \right\}.$$

Definition 4. *Let (X, d) be a metric space and $T : X \to CB(X)$ be a mapping. We say that T is a multivalued F-Khan contraction if there are $F \in \mathcal{F}$ and $\Upsilon > 0$ such that for all $\max\{D(\sigma_1, T\sigma_2), D(\sigma_2, T\sigma_1)\} \neq 0$, we have*

$$\Upsilon + F(H(T\sigma_1, T\sigma_2)) \leq F(L(\sigma_1, \sigma_2)) \tag{4}$$

for $\sigma_1, \sigma_2 \in X$ with $(\sigma_1, \sigma_2) \in T_G$.

With the assumptions of upper semi-continuity and having the weak graph preservation of the mapping T, Theorems 4 and 5 hold.

Theorem 4. *Let (X, d) be a complete metric space endowed with a directed graph G and $T : X \to K(X)$ be a multivalued F-Khan contraction. If the set X_T is nonempty, then T admits a fixed point.*

Proof. If T has no fixed point, $D(\sigma, T\sigma) > 0$ for all $\sigma \in X$. Let $\sigma_0 \in X_T$. Then, $(\sigma_0, \sigma_1) \in E(G)$ for some $\sigma_1 \in T\sigma_0$. Then, we get

$$0 < D(\sigma_1, T\sigma_1) \leq H(T\sigma_0, T\sigma_1).$$

Thus, $(\sigma_0, \sigma_1) \in T_G$. Using condition (4) for σ_0 and σ_1, we have

$$\begin{aligned} F(D(\sigma_1, T\sigma_1)) &\leq F(H(T\sigma_0, T\sigma_1)) \leq F(L(\sigma_0, \sigma_1)) - \Upsilon, \\ &= F\left(\max\left\{ \begin{array}{c} d(\sigma_0, \sigma_1), D(\sigma_0, T\sigma_0), D(\sigma_1, T\sigma_1), \\ \frac{D(\sigma_0, T\sigma_0)D(\sigma_0, T\sigma_1) + D(\sigma_1, T\sigma_1)D(\sigma_1, T\sigma_0)}{\max\{D(\sigma_0, T\sigma_1), D(\sigma_1, T\sigma_0)\}} \end{array} \right\}\right) - \Upsilon \\ &\leq F(d(\sigma_0, \sigma_1)) - \Upsilon. \end{aligned} \tag{5}$$

Due to the compactness of $T\sigma_1$, there is $\sigma_2 \in T\sigma_1$ such that $d(\sigma_1, \sigma_2) = D(\sigma_1, T\sigma_1)$. So, we get

$$F(d(\sigma_1, \sigma_2)) \leq F(d(\sigma_0, \sigma_1)) - \Upsilon.$$

Since $(\sigma_0, \sigma_1) \in E(G)$, $\sigma_1 \in T\sigma_0$, and $\sigma_2 \in T\sigma_1$, by the WGP property, one writes $(\sigma_1, \sigma_2) \in E(G)$. In view of $0 < D(\sigma_2, T\sigma_2) \le H(T\sigma_1, T\sigma_2)$, we get that $(\sigma_1, \sigma_2) \in T_G$. Then,

$$F(D(\sigma_2, T\sigma_2)) \le F(H(T\sigma_1, T\sigma_2)) \le F(L(\sigma_1, \sigma_2)) - \Upsilon. \tag{6}$$

Proceeding in a similar way, we can get

$$L(\sigma_1, \sigma_2) \le \max\{d(\sigma_1, \sigma_2), D(\sigma_2, T\sigma_2)\}.$$

By (6), we obtain

$$F(D(\sigma_2, T\sigma_2)) \le F(d(\sigma_1, \sigma_2)) - \Upsilon. \tag{7}$$

Again, the compactness of $T\sigma_2$ implies that there is $\sigma_3 \in T\sigma_2$ such that $d(\sigma_2, \sigma_3) = D(\sigma_2, T\sigma_2)$. We have

$$F(d(\sigma_2, \sigma_3)) \le F(d(\sigma_2, \sigma_1)) - \Upsilon.$$

Similarly, we construct the sequence $\{\sigma_n\}$ in σ such that $\sigma_{n+1} \in T\sigma_n$, $(\sigma_n, \sigma_{n+1}) \in T_G$ and

$$F(d(\sigma_n, \sigma_{n+1})) \le F(d(\sigma_{n-1}, \sigma_n)) - \Upsilon, \quad \text{for all } n \in \mathbb{N}. \tag{8}$$

Denote $a_n = d(\sigma_n, \sigma_{n+1})$, then $a_n > 0$ and from (8), $\{a_n\}$ is a decreasing real sequence, so there is $\delta \ge 0$ such that $\lim_{n\to\infty} a_n = \delta$. We have

$$\begin{aligned} F(a_n) &\le F(a_{n-1}) - \Upsilon \\ &\le F(a_{n-2}) - 2\Upsilon \\ &\quad \vdots \\ &\le F(a_0) - n\Upsilon. \end{aligned} \tag{9}$$

The right-hand side of (9) goes to $-\infty$ when $n \to +\infty$. Hence, $\lim_{n\to\infty} F(a_n) = -\infty$. Using $(F2)$, one writes

$$\delta = \lim_{n\to\infty} a_n = 0.$$

Due to $(F3)$, there is $k \in (0,1)$ such that

$$\lim_{n\to\infty} a_n^k F(a_n) = 0.$$

By Inequality (9), we have

$$a_n^k F(a_n) - a_n^k F(a_0) \le -a_n^k n\Upsilon \le 0, \tag{10}$$

which is verified for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (10), we find that

$$\lim_{n\to\infty} n a_n^k = 0. \tag{11}$$

From (11), there is $n_1 \in \mathbb{N}$ such that $n a_n^k \le 1$ for all $n \ge n_1$. Thus,

$$a_n \le \frac{1}{n^{1/k}} \tag{12}$$

for all $n \geq n_1$. Now, we claim that the sequence $\{\sigma_n\}$ is Cauchy. For this, take $m, n \in \mathbb{N}$ with $m > n \geq n_1$. Hence,

$$\begin{aligned} d(\sigma_n, \sigma_m) &\leq d(\sigma_n, \sigma_{n+1}) + d(\sigma_{n+1}, \sigma_{n+2}) + \cdots + d(\sigma_{m-1}, \sigma_m) \\ &= a_n + a_{n+1} + \cdots + a_{m-1} \\ &= \sum_{i=n}^{m-1} a_i \\ &\leq \sum_{i=n}^{\infty} a_i \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Since $k \in (0,1)$, the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ converges, so $d(\sigma_n, \sigma_m) \to 0$ as $n, m \to \infty$, that is, $\{\sigma_n\}$ is a Cauchy sequence in the complete metric space (X, d). Then, $\{\sigma_n\}$ is convergent to some $w \in X$. Using the upper semi-continuity of T and Lemma 1, we get $w \in Tw$, which is a contradiction with our assumption. Then, T has a fixed point. □

Theorem 5. *Let (X, d) be a complete metric space endowed with a directed graph G and $T : X \to K(X)$ be a multivalued F-Khan contraction (with $F \in \mathcal{F}_*$). If X_T is nonempty, then T has a fixed point.*

Proof. If there is no fixed point of T, then $D(\sigma, T\sigma) > 0$ for each $\sigma \in X$. Let $\sigma_0 \in X_T$. Then $(\sigma_0, \sigma_1) \in E(G)$ for some $\sigma_1 \in T\sigma_0$. Then, we get

$$0 < D(\sigma_1, T\sigma_1) \leq H(T\sigma_0, T\sigma_1).$$

Thus, $(\sigma_0, \sigma_1) \in T_G$. In view of $(F4)$ and using the condition (4) for σ_0 and σ_1, we have

$$\begin{aligned} F(D(\sigma_1, T\sigma_1)) &= F(\inf\{d(\sigma_1, v) : v \in T\sigma_1\}) \\ &= \inf\{F(d(\sigma_1, v) : v \in T\sigma_1)\} \end{aligned}$$

and by (5) we have

$$\inf\{F(d(\sigma_1, v) : v \in T\sigma_1)\} < F(d(\sigma_0, \sigma_1)) - \frac{\Upsilon}{2}.$$

Thus, there is $\sigma_2 \in T\sigma_1$ so that

$$F(d(\sigma_1, \sigma_2)) \leq F(d(\sigma_0, \sigma_1)) - \frac{\Upsilon}{2}.$$

The rest of the proof follows as in the proof of Theorem 4. □

Theorem 6. *Let (X, d) be a complete metric space endowed with a directed graph G such that:*

$$\begin{array}{c} \text{for any } \{x_n\} \text{ in } X, \text{ if } x_n \to x \text{ and } (x_n, x_{n+1}) \in E(G), \\ \text{then there is a subsequence } \{x_{n_k}\} \text{ with } (x_{n_k}, x) \in E(G). \end{array} \tag{13}$$

Let $T : X \to K(X)$ be a multivalued F-Khan contraction (resp. $T : X \to CB(X)$ be a multivalued F-Khan contraction with $F \in \mathcal{F}_*$). Suppose that T is a WGP mapping and X_T is nonempty. If F is continuous, then T admits a fixed point.

Proof. Assume that T has no fixed point. In a similar way as in the proof of Theorem 4 (resp. Theorem 5), we construct the sequence $\{\sigma_n\}$ such that $\sigma_n \to w$ for some $w \in X$. By the property (13), there is a subsequence $\{\sigma_{n_k}\}$ of $\{\sigma_n\}$ such that $(\sigma_{n_k}, w) \in E(G)$ for each

$k \in \mathbb{N}$. Since $\lim_{n\to\infty} \sigma_n = w$ and $D(w, Tw) > 0$, then there is no natural number n_0 such that $D(\sigma_{n_k+1}, Tw) = 0$ for each $n_k \geq n_0$. Therefore, for all $n_k \geq n_0$

$$H(T\sigma_{n_k}, Tw) > 0,$$

thus $(\sigma_{n_k}, w) \in T_G$ for all $n_k \geq n_0$. From (4) and (F1), we have

$$\begin{aligned} F(D(\sigma_{n_k+1}, Tw)) &\leq F(H(T\sigma_{n_k}, Tw)) - \Upsilon \\ &\leq F(L(\sigma_{n_k}, w)) - \Upsilon \\ &\leq F\left(\max\left\{\begin{array}{c} d(\sigma_{n_k}, w), D(\sigma_{n_k}, T\sigma_{n_k}), D(w, Tw), \\ \frac{D(\sigma_{n_k}, T\sigma_{n_k})D(\sigma_{n_k}, Tw) + D(w, Tw)D(w, T\sigma_{n_k})}{\max\{D(\sigma_{n_k}, Tw), D(w, T\sigma_{n_k})\}} \end{array}\right\}\right) - \Upsilon \\ &\leq F\left(\max\left\{\begin{array}{c} d(\sigma_{n_k}, w), d(\sigma_{n_k}, \sigma_{n_{k+!}}), D(w, Tw), \\ \frac{d(\sigma_{n_k}, \sigma_{n_{k+1}})D(\sigma_{n_k}, Tw) + D(w, Tw)d(w, \sigma_{n_{k+1}})}{\max\{D(\sigma_{n_k}, Tw), D(w, T\sigma_{n_k})\}} \end{array}\right\}\right) - \Upsilon \end{aligned}$$

for all $n_k \geq n_0$. Letting $k \to \infty$ and due to the continuity of F, we obtain that $\Upsilon + F(D(w, Tw)) \leq F(D(w, Tw))$, which is a contradiction, so T admits a fixed point. □

Corollary 1. *Let (X, d) be a complete metric space endowed with a directed graph G and $T : X \to K(X)$ be a mapping. Suppose that there are $F \in \mathcal{F}$ and $\Upsilon > 0$ such that*

$$\Upsilon + F(H(Tv, T\sigma)) \leq F(d(v, \sigma))$$

for $v, \sigma \in X$ with $(v, \sigma) \in T_G$. If T is upper semi-continuous and a WGP mapping and the set X_T is nonempty, then T has a fixed point.

Corollary 2. *Let (X, d) be a complete metric space endowed with a directed graph G and $T : X \to CB(X)$ be a mapping. Suppose that there are $F \in \mathcal{F}_*$ and $\Upsilon > 0$ such that*

$$\Upsilon + F(H(Tv, T\sigma)) \leq F(d(v, \sigma))$$

for $v, \sigma \in X$ with $(v, \sigma) \in T_G$. Assume that T is upper semi-continuous and a WGP mapping and the set X_T is nonempty. Then, T admits a fixed point.

Example 1. *Let $X = \left\{r_\ell = \frac{\ell(\ell+1)}{2}; \ell \geq 1,\ \ell \text{ is an integer}\right\} \cup \{0\}$ and $d(\iota, \jmath) = |\iota - \jmath|$. Then (X, d) is a complete metric space. Define $T : X \to CB(X)$ by*

$$T\iota = \begin{cases} \{0\} & , \quad \iota = 0 \\ \{r_1\} & , \quad \iota = r_1 \\ \{r_1, r_2, ..., r_{\ell-1}\} & , \quad \iota = r_\ell, \ell \geq 2 \end{cases}$$

and a graph on X by $V(G) = X$ and

$$E(G) = \{(\iota, \jmath) | \iota = \jmath \text{ or } \iota = r_\ell, \jmath = r_p\ , p < \ell\}.$$

Then, T is upper semi-continuous and a WGP mapping. Now, we show that T is a multivalued F-Khan contraction with $F(\varrho) = \varrho + \ln\varrho$ and $\Upsilon = 1$. Then, for any $(\iota, \ell) \in E(G)$ with $T\iota \neq T\ell$, we consider two cases:

(i) If $\iota = r_\ell, \ell \geq 2$ and $\jmath = r_1$, we have

$$\frac{H(T\iota, T\jmath)}{M(\iota, \jmath)} e^{H(T\iota, T\jmath) - M(\iota, \jmath)} = \frac{r_{\ell-1} - 1}{r_\ell - 1} e^{r_{\ell-1} - r_\ell} < e^{-1}.$$

(ii) If $\iota = r_\ell, \jmath = r_p, \ell > p > 1$, we have

$$\frac{H(T\iota, T\jmath)}{M\iota, \jmath)} e^{H(T\iota,T\jmath)-M(\iota,\jmath)} = \frac{\ell+p-1}{\ell+p+1} e^{-\ell+p} < e^{-1}.$$

Thus, all assumptions of Theorem 4 (or Theorem 5) hold. Therefore, T has a fixed point. Moreover, if the graph on X is not considered, the contractive condition is not satisfied. In fact, taking $\iota = 0$ and $\jmath = r_1$, we have $H(T\iota, T\jmath) = 1$ and $d(\iota, \jmath) = 1$, so for all $F \in \mathcal{F}$ and $\Upsilon > 0$, we get

$$\Upsilon + F(H(T\iota, T\jmath)) > F(d(\iota, \jmath)).$$

4. Concluding Remarks

As it is known, combining some branches is an activity in different fields of science, especially in mathematics. One of them is the combination of fixed point theory and graph theory. In this paper, we present new fixed point results for multivalued mappings. To do this, we introduce new types of contractions called rational-type multivalued G-contractions and F-Khan-type multivalued contractions. We ensure that related mappings have a fixed point. At the end, we consider an example that shows the importance of graph on the contractive condition. Thus, we show here that the defined contraction does not even satisfy the condition of contraction in a metric space without graph, but it provides the condition of a contraction in a metric space with graph and has a fixed point, which shows us the importance of graph structure.

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