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Generalized and Extended Versions of Ankeny–Rivlin and Improved, Generalized, and Extended Versions of Rivlin Type Inequalities for the s^{th} Derivative of a Polynomial

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Abstract: Let $p(z)$ be a polynomial of degree n having no zeros in $|z| < 1$. In this paper, we generalize and extend a well-known result proven by Ankeny and Rivlin for the s^{th} derivative of the polynomial. Furthermore, another well-known result proven by Rivlin is also improved, generalized and extended for the s^{th} derivative of $p(z)$. Our results also give a number of interesting consequences as special cases.

Keywords: s^{th} derivative; polynomial; maximum modulus; zeros

MSC: 30C10; 30C15

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1. Introduction

Let $p(z)$ be a polynomial of degree n with real or complex coefficients. We denote $M(p, r) = \max_{|z|=r} |p(z)|$ and $m(p, r) = \min_{|z|=r} |p(z)|$.

If $p(z)$ is a polynomial of degree n , then it is well-known that:

$$M(p', 1) \leq n M(p, 1). \quad (1)$$

The result is best possible and the equality holds in (1) for $p(z) = \alpha z^n$, α being a complex number. This inequality, which is known as Bernstein's inequality, was proven by Bernstein [1], although it was also proven by Riesz [2] about 12 years before Bernstein.

Erdős considered the class of polynomials $p(z)$ of degree n having no zeros in $|z| < 1$ and conjectured that:

$$M(p', 1) \leq \frac{n}{2} M(p, 1). \quad (2)$$

The result is sharp and the equality in (2) holds for $p(z) = \alpha z^n + \beta$, where α and β are complex numbers such that $|\alpha| = |\beta|$. This inequality (2) was later verified by Lax [3] in 1944.

It is also well-known that if $p(z)$ is a polynomial of degree n , then:

$$M(p, R) \leq R^n M(p, 1), \quad R \geq 1 \quad (3)$$

and

$$M(p, r) \geq r^n M(p, 1), \quad r \leq 1. \quad (4)$$

Inequality (3) is a consequence of the Maximum Modulus Principle [4] (p. 137 Problem III 269). Equality in (3) holds if $p(z)$ is a constant multiple of z^n . Whereas, inequality (4) is due to Zarantonello and Varga [5].

Ankeny and Rivlin [6] improved inequality (3) for the class of polynomials $p(z)$ of degree n having no zeros in $|z| < 1$. In fact, they proved:

Theorem 1. *If $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then:*

$$M(p, R) \leq \frac{R^n + 1}{2} M(p, 1), \quad R \geq 1. \quad (5)$$

Equality in (5) holds for $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.

While for the same class of polynomials, Rivlin [7] improved (4) by proving the following result.

Theorem 2. *If $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then:*

$$M(p, r) \geq \left(\frac{r+1}{2} \right)^n M(p, 1), \quad r \leq 1. \quad (6)$$

Equality in (6) occurs for $p(z) = \left(\frac{\alpha + \beta z}{n} \right)^n$, $|\alpha| = |\beta|$.

Govil [8] generalized Theorem 2 by proving the following.

Theorem 3. *If $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then for $0 < r \leq R \leq 1$,*

$$M(p, r) \geq \left(\frac{1+r}{1+R} \right)^n M(p, R). \quad (7)$$

The result is best possible for the polynomial $p(z) = \left(\frac{\alpha + \beta z}{n} \right)^n$, $|\alpha| = |\beta|$.

There are many extensions of inequality (5) (see Govil [8], Qazi [9], Bidkham and Dewan [10], Govil and Qazi [11], Govil et al. [12], Mir [13]). Some of the results by the above authors considered a more general class of polynomials while some also involve certain coefficients in the inequalities.

Jain [14] proved a generalization of Theorem 1 by simultaneously considering polynomials not vanishing in $|z| < k$, $k \geq 1$ as well as the s^{th} derivative of the polynomials. In fact, he proved:

Theorem 4. *If $p(z)$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for $0 \leq s < n$,*

$$M(p^{(s)}, R) \leq \frac{1}{2} \left\{ \frac{d^s}{dR^s} (R^n + k^n) \right\} \left(\frac{2}{1+k} \right)^n M(p, 1), \quad R \geq k \quad (8)$$

and

$$M(p^{(s)}, R) \leq \left(\frac{1}{R^s + k^s} \right) \left[\left\{ \frac{d^s}{dx^s} (1 + x^n) \right\}_{x=1} \right] \times \left(\frac{R+k}{1+k} \right)^n M(p, 1), \quad 1 \leq R \leq k. \quad (9)$$

Equality holds in (8) with $k = 1$ and $s = 0$ for $p(z) = z^n + 1$ and in (9) for $s = 1$ and $p(z) = (z + k)^n$.

The paper is organized as follows. Section 2 includes preliminary results in the form of lemmas which are required to prove the main results and other claims. In Section 3, we state the main theorems of the paper and their implications on the existing results with the help of corollaries and remarks. Section 4 gives the proofs of the main theorems and in Section 5, a conclusion with some possible future prospects of the results is given.

2. Lemmas

The following lemmas are required to prove the results which are stated in the next section.

Lemma 1. If $p(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then:

$$\max_{|z|=1} |p'(z)| \leq n \frac{1 + k|\lambda|}{1 + k^2 + 2k|\lambda|} \max_{|z|=1} |p(z)|; \quad (10)$$

furthermore:

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \frac{(1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu - \lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|) + k(n-1)|\mu - \lambda^2|} \max_{|z|=1} |p(z)|, \quad (11)$$

where,

$$\lambda = \frac{ka_1}{na_0} \quad \text{and} \quad \mu = \frac{2k^2}{n(n-1)} \frac{a_2}{a_0}. \quad (12)$$

The above lemma is due to Govil et al. [15]. In their paper [15], the authors mentioned that the bound given by (11) improves over the bound given by (10) because of the fact:

$$(n-1)|\mu - \lambda^2| \leq (1-|\lambda|^2). \quad (13)$$

However, the authors did not discuss it. For the sake of completeness, we give a brief proof of it in Lemma 3 by applying the next lemma.

Lemma 2. Let $a, b, c, d > 0$ be real numbers such that $c \leq d$. If $a \leq b$ then:

$$\frac{a+c}{b+c} \leq \frac{a+d}{b+d}. \quad (14)$$

Proof of Lemma 2. Suppose that $c < d$. Then inequality (14) follows easily as:

$$a \leq b \implies a(d-c) \leq b(d-c),$$

which simplifies to:

$$(a+c)(b+d) \leq (a+d)(b+c),$$

which gives the desired conclusion of the lemma.

For the case $c = d$ equality in (14) holds trivially. $\square$

We now show that the bound given by (11) is smaller than the bound given by (10).

Lemma 3. Let $p(z) = \sum_{v=0}^n a_v z^v$ be a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then:

$$\frac{n}{1+k} \frac{(1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu - \lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|) + k(n-1)|\mu - \lambda^2|} \leq n \frac{1+k|\lambda|}{1+k^2+2k|\lambda|}, \quad (15)$$

where λ and μ are given by (12).

Proof of Lemma 3. Put $a = (1-|\lambda|)(1+k^2|\lambda|)$, $b = (1-|\lambda|)(1-k+k^2+k|\lambda|)$, $c = k(n-1)|\mu - \lambda^2|$, and $d = k(1-|\lambda|^2)$ in Lemma 2.

Now by (13), $c \leq d$ and it is easy to see that $a \leq b$. Hence, it follows from inequality (14) of Lemma 2 that:

$$\frac{(1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu - \lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|) + k(n-1)|\mu - \lambda^2|} \leq \frac{(1-|\lambda|)(1+k^2|\lambda|) + k(1-|\lambda|^2)}{(1-|\lambda|)(1-k+k^2+k|\lambda|) + k(1-|\lambda|^2)}.$$

Equivalently,

$$\begin{aligned} & \frac{n}{1+k} \frac{(1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu - \lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|) + k(n-1)|\mu - \lambda^2|} \\ & \leq \frac{n}{1+k} \frac{(1-|\lambda|)(1+k^2|\lambda|) + k(1-|\lambda|^2)}{(1-|\lambda|)(1-k+k^2+k|\lambda|) + k(1-|\lambda|^2)} \\ & = \frac{n}{1+k} \frac{1+k^2|\lambda| + k(1+|\lambda|)}{1-k+k^2+k|\lambda| + k(1+|\lambda|)} \\ & = n \frac{1+k|\lambda|}{1+k^2+2k|\lambda|}, \end{aligned}$$

which is the desired conclusion of the lemma. $\square$

Lemma 4. Let $p(z) = \sum_{v=0}^n a_v z^v$ be a polynomial of degree n having no zeros in $|z| < k$, $k > 0$, then for $0 < t \leq k$,

$$\frac{n}{t+k} \frac{(1-|\lambda|)(t^2+k^2|\lambda|) + kt(n-1)|\mu - \lambda^2|}{(1-|\lambda|)(t^2-kt+k^2+kt|\lambda|) + kt(n-1)|\mu - \lambda^2|} \leq n \frac{t+k|\lambda|}{t^2+k^2+2kt|\lambda|}, \quad (16)$$

where λ and μ are given by (12).

Proof of Lemma 4. Let $P(z) = p(tz)$, where $0 < t \leq k$. Since $p(z) \neq 0$ in $|z| < k$, then $P(z) \neq 0$ in $|z| < \frac{k}{t}$, where $\frac{k}{t} \geq 1$. Thus, applying Lemma 3 to the polynomial $P(z)$, we obtain:

$$\frac{n}{1+\frac{k}{t}} \frac{(1-|\lambda|)\left(1+\frac{k^2}{t^2}|\lambda|\right) + \frac{k}{t}(n-1)|\mu - \lambda^2|}{\left(1-\frac{k}{t}+\frac{k^2}{t^2}+\frac{k}{t}|\lambda|\right) + \frac{k}{t}(n-1)|\mu - \lambda^2|} \leq n \frac{1+\frac{k}{t}|\lambda|}{1+\frac{k^2}{t^2}+2\frac{k}{t}|\lambda|} \quad (17)$$

where,

$$\lambda = \frac{\frac{k}{t}(a_1 t)}{n a_0} = \frac{k a_1}{n a_0} \quad \text{and} \quad \mu = \frac{2\left(\frac{k}{t}\right)^2 (a_2 t^2)}{n(n-1)a_0} = \frac{2k^2}{n(n-1)} \frac{a_2}{a_0}.$$

Inequality (17), on simplification assumes (16). $\square$

Lemma 5. Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| \leq 1$. If $p(z)$ is a polynomial of degree at most n such that:

$$|p(z)| \leq |P(z)| \quad \text{on } |z| = 1,$$

then for $0 \leq s < n$,

$$|p^{(s)}(z)| \leq |P^{(s)}(z)| \quad \text{for } |z| \geq 1. \quad (18)$$

Lemma 6. If $p(z)$ is a polynomial of degree at most n , then for $0 \leq s < n$,

$$|p^{(s)}(z)| + |q^{(s)}(z)| \leq \left\{ \left| \frac{d^s}{dz^s}(1) \right| + \left| \frac{d^s}{dz^s}(z^n) \right| \right\} M(p, 1), \quad |z| \geq 1, \quad (19)$$

where,

$$q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}.$$

Lemmas 5 and 6 are due to Jain [14].

Lemma 7. If $p(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k > 0$, then for $0 < r \leq R \leq k$,

$$M(p, R) \leq \exp\left(n \int_r^R \frac{E_{t,k}}{t+k} dt\right) M(p, r), \quad (20)$$

where,

$$E_{t,k} = \frac{(1 - |\lambda|)(t^2 + k^2|\lambda|) + kt(n-1)|\mu - \lambda^2|}{(1 - |\lambda|)(t^2 - kt + k^2 + kt|\lambda|) + kt(n-1)|\mu - \lambda^2|}, \quad (21)$$

λ and μ are given by (12).

Proof of Lemma 7. Let $P(z) = p(tz)$, where $0 < t \leq k$. Since $p(z) \neq 0$ in $|z| < k$, then $P(z) \neq 0$ in $|z| < \frac{k}{t}$, where $\frac{k}{t} \geq 1$. Thus, applying inequality (11) of Lemma 1 to the polynomial $P(z)$, we obtain:

$$\begin{aligned} \max_{|z|=1} |P'(z)| &\leq \frac{n}{1 + \frac{k}{t}} \left\{ \frac{(1 - |\lambda|) \left(1 + \frac{k^2}{t^2} |\lambda|\right) + \frac{k}{t} (n-1) |\mu - \lambda^2|}{(1 - |\lambda|) \left(1 - \frac{k}{t} + \frac{k^2}{t^2} + \frac{k}{t} |\lambda|\right) + \frac{k}{t} (n-1) |\mu - \lambda^2|} \right\} \\ &\quad \times \max_{|z|=1} |P(z)|, \end{aligned} \quad (22)$$

where,

$$\lambda = \frac{\frac{k}{t}(a_1 t)}{n a_0} = \frac{k a_1}{n a_0} \quad \text{and} \quad \mu = \frac{2 \left(\frac{k}{t}\right)^2 (a_2 t^2)}{n(n-1) a_0} = \frac{2 k^2}{n(n-1)} \frac{a_2}{a_0}.$$

Inequality (22) on simplification becomes:

$$\max_{|z|=t} |p'(z)| \leq \frac{n}{t+k} \frac{(1 - |\lambda|)(t^2 + k^2|\lambda|) + kt(n-1)|\mu - \lambda^2|}{(1 - |\lambda|)(t^2 - kt + k^2 + kt|\lambda|) + kt(n-1)|\mu - \lambda^2|} \max_{|z|=t} |p(z)|,$$

which is equivalent to:

$$M(p', t) \leq \frac{n}{t+k} E_{t,k} M(p, t), \quad (23)$$

where $E_{t,k}$ is as defined in (21).

We know for $0 < r \leq R \leq k$ and $0 \leq \theta < 2\pi$:

$$\begin{aligned} \left| p(Re^{i\theta}) - p(re^{i\theta}) \right| &= \left| \int_r^R p'(te^{i\theta}) dt \right| \leq \int_r^R \left| p'(te^{i\theta}) \right| dt \\ \text{i.e., } \left| p(Re^{i\theta}) \right| &\leq \left| p(re^{i\theta}) \right| + \int_r^R \left| p'(te^{i\theta}) \right| dt, \end{aligned}$$

which implies on taking maximum over θ ,

$$M(p, R) \leq M(p, r) + \int_r^R M(p', t) dt. \quad (24)$$

Inequality (24), in conjunction with (23), gives:

$$M(p, R) \leq M(p, r) + n \int_r^R \frac{E_{t,k}}{t+k} M(p, t) dt. \quad (25)$$

Let $\phi(R)$ denote the right hand side of inequality (25). Then,

$$\phi'(R) = \frac{n}{R+k} E_{R,k} M(p, R) \quad (26)$$

$$\text{and } M(p, R) \leq \phi(R). \quad (27)$$

From (26) and (27), we further have:

$$\begin{aligned} \phi'(R) &\leq \frac{n}{R+k} E_{R,k} \phi(R) \\ \text{i.e., } \phi'(R) - \frac{n}{R+k} E_{R,k} \phi(R) &\leq 0. \end{aligned} \quad (28)$$

Multiplying (28) by:

$$\exp\left(-n \int \frac{E_{R,k}}{R+k} dR\right),$$

we have:

$$\frac{d}{dR} \left\{ \phi(R) \exp\left(-n \int \frac{E_{R,k}}{R+k} dR\right) \right\} \leq 0. \quad (29)$$

Hence, the function:

$$\psi(R) = \phi(R) \exp\left(-n \int \frac{E_{R,k}}{R+k} dR\right)$$

is a non-increasing function of R in $(0, k]$. Therefore, for $0 < r \leq R \leq k$,

$$\psi(r) \geq \psi(R). \quad (30)$$

Since $\phi(R) \geq M(p, R)$ and $\phi(r) = M(p, r)$, we have from (30):

$$M(p, r) \geq M(p, R) \exp\left(-n \int_r^R \frac{E_{t,k}}{t+k} dt\right).$$

Equivalently,

$$M(p, R) \leq \exp\left(n \int_r^R \frac{E_{t,k}}{t+k} dt\right) M(p, r)$$

which completes the proof of Lemma 7. $\square$

Lemma 8. If $p(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree $n \geq 3$ having no zeros in $|z| < k, k > 0$, then for $0 < r \leq \rho \leq k$,

$$\begin{aligned} M(p, \rho) &\leq \exp\left(n \int_r^\rho \frac{E_{t,k}}{t+k} dt\right) M(p, r) \\ &\quad - \frac{1}{k^n} \left\{ \rho^n - r^n \exp\left(n \int_r^\rho \frac{E_{t,k}}{t+k} dt\right) \right\} m(p, k), \end{aligned} \quad (31)$$

where $E_{t,k}$ is given by (21).

Proof of Lemma 8. If $p(z)$ has a zero on $|z| = k$, then $m(p, k) = 0$ and because of Lemma 7, the result holds trivially. Hence, without loss of generality, we assume that $p(z)$ has all its zeros in $|z| > k$, so that $m > 0$ and $\frac{z^n}{p(z)}$ is analytic in $|z| \leq k$. Then, by Maximum Modulus Principle [4] (p. 137, problem III 269), we have for $|z| \leq k$:

$$\left| \frac{z^n}{p(z)} \right| \leq \max_{|z|=k} \left| \frac{z^n}{p(z)} \right| \leq \frac{k^n}{\min_{|z|=k} |p(z)|} = \frac{k^n}{m}.$$

Hence, we have:

$$\frac{m|z|^n}{k^n} \leq |p(z)| \quad \text{for } |z| \leq k.$$

By Rouché's Theorem, for every α with $|\alpha| < 1$, the polynomial $p(z) + \frac{\alpha m z^n}{k^n}$ has no zeros in $|z| < k, k \geq 1$. Hence, $p(z) + \frac{\alpha m z^n}{k^n}$ has all its zeros in $|z| \geq k, k \geq 1$. Thus, applying Lemma 7 to the polynomial $p(z) + \frac{\alpha m z^n}{k^n}$, we have for $0 < r \leq \rho \leq k$ and every α with $|\alpha| < 1$:

$$\max_{|z|=\rho} \left| p(z) + \frac{\alpha m z^n}{k^n} \right| \leq \exp\left(n \int_r^\rho \frac{E_{t,k}}{t+k} dt\right) \max_{|z|=r} \left| p(z) + \frac{\alpha m z^n}{k^n} \right|.$$

This implies:

$$\max_{|z|=\rho} \left| p(z) + \frac{\alpha m z^n}{k^n} \right| \leq \exp\left(n \int_r^\rho \frac{E_{t,k}}{t+k} dt\right) \left\{ M(p, r) + \frac{|\alpha| m r^n}{k^n} \right\}. \quad (32)$$

Choosing the argument of α suitably such that:

$$\max_{|z|=\rho} \left| p(z) + \frac{\alpha m z^n}{k^n} \right| = M(p, \rho) + \frac{|\alpha| m \rho^n}{k^n},$$

then inequality (32) gives:

$$M(p, \rho) \leq \exp \left(n \int_r^\rho \frac{E_{t,k}}{t+k} dt \right) M(p, r) \frac{|\alpha|}{k^n} \left\{ \rho^n - r^n \exp \left(n \int_r^\rho \frac{E_{t,k}}{t+k} dt \right) \right\} m(p, k).$$

Taking the limit as $|\alpha| \rightarrow 1$, we get:

$$M(p, \rho) \leq \exp \left(n \int_r^\rho \frac{E_{t,k}}{t+k} dt \right) M(p, r) - \frac{1}{k^n} \left\{ \rho^n - r^n \exp \left(n \int_r^\rho \frac{E_{t,k}}{t+k} dt \right) \right\} m(p, k),$$

which completes the proof of Lemma 8. $\square$

The next lemma was proved by Aziz and Rather Corollary 1 in [16].

Lemma 9. If $p(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for $1 \leq s < n$,

$$\max_{|z|=1} |p^{(s)}(z)| \leq n(n-1) \dots (n-s+1) \times \left\{ \frac{C(n, s) |a_0| + |a_s| k^{s+1}}{C(n, s) |a_0| (1 + k^{s+1}) + |a_s| k^{s+1} (k^{s-1} + 1)} \right\} \max_{|z|=1} |p(z)| \quad (33)$$

and

$$\frac{1}{C(n, s)} \left| \frac{a_s}{a_0} \right| k^s \leq 1,$$

where $C(n, s) = \frac{n!}{(n-s)! s!}$.

Remark 1. From Lemma 9, it is easy to conclude that for $0 \leq s < n$,

$$\max_{|z|=1} |p^{(s)}(z)| \leq \left\{ \frac{d^s}{dx^s} (x^n + 1) \right\}_{x=1} \times \left\{ \frac{C(n, s) |a_0| + |a_s| k^{s+1}}{C(n, s) |a_0| (1 + k^{s+1}) + |a_s| k^{s+1} (k^{s-1} + 1)} \right\} \max_{|z|=1} |p(z)| \quad (34)$$

and

$$\frac{1}{C(n, s)} \left| \frac{a_s}{a_0} \right| k^s \leq 1. \quad (35)$$

The following result is due to Jain Remark 1 in [17].

Lemma 10. If $p(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k > 0$, then:

$$\left(\frac{R^2 + k^2 + 2kR|\delta|}{r^2 + k^2 + 2kr|\delta|} \right)^{\frac{n}{2}} \leq \left(\frac{R+k}{r+k} \right)^n \quad \text{for } 0 < r \leq R \leq k, \quad (36)$$

where, $\delta = \frac{ka_1}{na_0}$ has absolute value ≤ 1 (see Lemma 1 in [9]).

Lemma 11. Let $p(z) = \sum_{v=0}^n a_v z^v$ be a polynomial of degree n having no zeros in $|z| < k$, $k > 0$, then:

$$\exp \left(n \int_r^R \frac{E_{t,k}}{t+k} dt \right) \leq \left(\frac{R^2 + k^2 + 2kR|\lambda|}{r^2 + k^2 + 2kr|\lambda|} \right)^{\frac{n}{2}}, \quad 0 < r \leq R \leq k \quad (37)$$

and also:

$$\exp \left(n \int_r^R \frac{E_{t,k}}{t+k} dt \right) \leq \left(\frac{R+k}{r+k} \right)^n, \quad 0 < r \leq R \leq k. \quad (38)$$

In particular, we have for $r = 1$ and $R = k$,

$$\frac{1}{k^n} \exp \left(n \int_1^k \frac{E_{t,k}}{t+k} dt \right) \leq \left(\frac{2+2|\lambda|}{1+k^2+2k|\lambda|} \right)^{\frac{n}{2}}, \quad k \geq 1 \quad (39)$$

and

$$\frac{1}{k^n} \exp \left(n \int_1^k \frac{E_{t,k}}{t+k} dt \right) \leq \left(\frac{2}{1+k} \right)^n, \quad k \geq 1, \quad (40)$$

where $E_{t,k}$ and λ are given by (21) and (12) respectively.

Proof of Lemma 11. Integrating on both sides of inequality (16) of Lemma 4 with respect to t from r to R , where $0 < r \leq R \leq k$, we get:

$$n \int_r^R \frac{E_{t,k}}{t+k} dt \leq n \int_r^R \frac{t+k|\lambda|}{t^2+k^2+2kt|\lambda|} dt.$$

Equivalently,

$$\begin{aligned} \exp \left(n \int_r^R \frac{E_{t,k}}{t+k} dt \right) &\leq \exp \left(n \int_r^R \frac{t+k|\lambda|}{t^2+k^2+2kt|\lambda|} dt \right) \\ &= \left(\frac{R^2 + k^2 + 2kR|\lambda|}{r^2 + k^2 + 2kr|\lambda|} \right)^{\frac{n}{2}}, \end{aligned}$$

which is inequality (37).

Again for $0 < r \leq R \leq k$, combining (36) with (37), we get (38).

Furthermore, put $R = k$ and $r = 1$ in (37) and (38), we get (39) and (40) respectively. $\square$

Lemma 12. If $p(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for $0 \leq s < n$ and $1 \leq R \leq k$,

$$\frac{1}{R^s} F_{R,k} \exp \left(n \int_1^R \frac{E_{t,k}}{t+k} dt \right) \leq \left(\frac{1}{R^s + k^s} \right) \left(\frac{R+k}{1+k} \right)^n \quad (41)$$

where here and throughout the paper,

$$F_{R,k} = \frac{C(n,s)|a_0|R^{s+1} + |a_s|R^s k^{s+1}}{C(n,s)|a_0|(R^{s+1} + k^{s+1}) + |a_s|Rk^{s+1}(k^{s-1} + R^{s-1})}, \quad (42)$$

$C(n,s) = \frac{n!}{(n-s)!s!}$, and $E_{t,k}$ is given by (21).

Proof of Lemma 12. Taking $r = 1$ in Lemma 11, we get from (37):

$$\exp\left(n \int_1^R \frac{E_{t,k}}{t+k} dt\right) \leq \left(\frac{R+k}{1+k}\right)^n. \quad (43)$$

Now, for $1 \leq R \leq k$ and $0 \leq s < n$, consider:

$$\frac{1}{R^s + k^s} - \frac{F_{R,k}}{R^s}$$

i.e.,

$$\frac{1}{R^s + k^s} - \frac{C(n,s)|a_0|R^{s+1} + |a_s|R^s k^{s+1}}{C(n,s)|a_0|R^s(R^{s+1} + k^{s+1}) + |a_s|R^{s+1}k^{s+1}(k^{s-1} + R^{s-1})}$$

i.e.,

$$\frac{(k-R)R^s k^s \{C(n,s)|a_0| - |a_s|k^s\}}{(R^s + k^s)(C(n,s)|a_0|R^s(R^{s+1} + k^{s+1}) + |a_s|R^{s+1}k^{s+1}(k^{s-1} + R^{s-1}))},$$

which is non-negative because, by (35) of Remark 1, $C(n,s)|a_0| - |a_s|k^s \geq 0$ and the fact $R \leq k$. Thus, we have:

$$\frac{F_{R,k}}{R^s} \leq \frac{1}{R^s + k^s}. \quad (44)$$

Combining (43) and (44) we get the desired inequality (41). $\square$

3. Main Results

In this paper, by involving certain coefficients of the polynomial, we obtain a generalization and refinement of Theorem 4. Moreover, our results are found to generalize and improve or generalize some other known inequalities. More precisely, we prove:

Theorem 5. Let $p(z) = \sum_{v=0}^n a_v z^v$ be a polynomial of degree n having no zeros in $|z| < k$, $k > 0$, then for $0 \leq s < n$,

$$M(p^{(s)}, R) \leq \frac{1}{2k^n} \frac{d^s}{dR^s} (R^n + k^n) \times \exp\left(n \int_r^k \frac{E_{t,k}}{t+k} dt\right) M(p, r), \quad 0 < r \leq k \leq R \quad (45)$$

and

$$M(p^{(s)}, R) \leq \frac{1}{R^s} \left\{ \frac{d^s}{dx^s} (x^n + 1) \right\}_{x=1} \times F_{R,k} \exp\left(n \int_r^R \frac{E_{t,k}}{t+k} dt\right) M(p, r), \quad 0 < r \leq R \leq k, \quad (46)$$

where $E_{t,k}$ and $F_{R,k}$ are given by (21) and (42) respectively.

Equality occurs in (45) when $s = 0, k = 1 = r$ for $p(z) = z^n + 1$ and in (46) when $s = 1, r = R$ for $p(z) = (z + k)^n$.

Taking $r = 1$ in Theorem 5, we have the following improvement of Theorem 4 proven by Jain [14].

Corollary 1. Let $p(z) = \sum_{v=0}^n a_v z^v$ be a polynomial of degree n having no zeros in $|z| < k, k \geq 1$, then for $0 \leq s < n$,

$$M(p^{(s)}, R) \leq \frac{1}{2k^n} \frac{d^s}{dR^s} (R^n + k^n) \times \exp \left(n \int_1^k \frac{E_{t,k}}{t+k} dt \right) M(p, 1), \quad R \geq k \quad (47)$$

and

$$M(p^{(s)}, R) \leq \frac{1}{R^s} \left\{ \frac{d^s}{dx^s} (x^n + 1) \right\}_{x=1} \times F_{R,k} \exp \left(n \int_1^R \frac{E_{t,k}}{t+k} dt \right) M(p, 1), \quad 1 \leq R \leq k. \quad (48)$$

Remark 2. In the light of (40) of Lemma 11 and (41) of Lemma 12, Corollary 1 is an improvement of Theorem 4 due to Jain [14].

Remark 3. In view of (39) of Lemma 11, inequality (47), under the same hypotheses, gives an improved bound over a result due to Mir ([13], Theorem 2) for $R \geq k$ while the corresponding bound for $1 \leq R \leq k$ is supplemented by (48).

Remark 4. Taking $s = 0$ in Theorem 5, we have the following result.

Corollary 2. Let $p(z) = \sum_{v=0}^n a_v z^v$ be a polynomial of degree n having no zeros in $|z| < k, k > 0$, then:

$$M(p, R) \leq \frac{1}{2k^n} (R^n + k^n) \times \exp \left(n \int_r^k \frac{E_{t,k}}{t+k} dt \right) M(p, r), \quad 0 < r \leq k \leq R \quad (49)$$

and

$$M(p, R) \leq \exp \left(n \int_r^R \frac{E_{t,k}}{t+k} dt \right) M(p, r), \quad 0 < r \leq R \leq k. \quad (50)$$

Inequality (49) reduces to inequality (5) when we set $r = k = 1$, and therefore is a generalization of (5) due to Ankeny and Rivlin [6]. Whereas, by inequality (38) of Lemma 11, inequality (50) is an improvement of a result proven by Jain Lemma 6 in [18]. Moreover, (50) is also an improvement of another result proven by Jain Remark 1 in [17].

Remark 5. If $s = 0$ and $k = 1$ in Theorem 5, we have the next result which generalizes Theorem 1 proven by Ankeny and Rivlin [6] and improves Theorem 3 due to Govil [8].

Again putting $k = 1$ in Corollary 2, we have:

Corollary 3. If $p(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n having no zeros in $|z| < 1$, then:

$$M(p, R) \leq \frac{R^n + 1}{2} \exp \left(n \int_r^1 \frac{E_{t,1}}{t+1} dt \right) M(p, r), \quad 0 < r \leq 1 \leq R \quad (51)$$

and

$$M(p, R) \leq \exp \left(n \int_r^R \frac{E_{t,1}}{t+1} dt \right) M(p, r), \quad 0 < r \leq R \leq 1. \quad (52)$$

Remark 6. If we put $r = 1$, inequality (51) reduces to (5) due to Ankeny and Rivlin [6]. For $k = 1$, inequality (38) gives:

$$\exp \left(n \int_r^R \frac{E_{t,1}}{t+1} dt \right) \leq \left(\frac{R+1}{r+1} \right)^n,$$

which shows that the bound of (52) improves over the bound given by inequality (7) due to Govil [8].

Remark 7. If we take $s = 0$ and $r = 1$ in Theorem 5, we have the following result which improves the result proven by Aziz and Mohammad Theorem 2 in [19].

Corollary 4. If $p(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then:

$$M(p, R) \leq \frac{1}{2k^n} (R^n + k^n) \exp \left(n \int_1^k \frac{E_{t,k}}{t+k} dt \right) M(p, 1), \quad R \geq k \quad (53)$$

and

$$M(p, R) \leq \exp \left(n \int_1^R \frac{E_{t,k}}{t+k} dt \right) M(p, 1), \quad 1 \leq R \leq k. \quad (54)$$

Inequality (53) is a generalization of (5). Taking $r = 1$ in (38), we obtain:

$$\exp \left(n \int_1^R \frac{E_{t,k}}{t+k} dt \right) \leq \left(\frac{R+k}{1+k} \right)^n,$$

which verifies that inequality (54) is an improvement of the result mentioned in Remark 7.

Remark 8. Furthermore, if we take $s = 0$ and $R = k = 1$ in Theorem 5, the two inequalities reduce to a single inequality which is found to extend and improve inequality (6) due to Rivlin [7] in the form of the following result.

Corollary 5. If $p(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n having no zeros in $|z| < 1$, then:

$$M(p, r) \geq \frac{1}{\exp \left(n \int_r^1 \frac{E_{t,1}}{t+1} dt \right)} M(p, 1), \quad r \leq 1. \quad (55)$$

For $R = k = 1$ in (38), we get:

$$\exp\left(n \int_r^1 \frac{E_{t,1}}{t+1} dt\right) \leq \left(\frac{2}{r+1}\right)^n,$$

which shows that (55) is an improvement of inequality (6) proven by Rivlin [7].

If $s = 1$, Theorem 5 in particular reduces to the following result which has some implications on some known inequalities involving the derivative of $p(z)$.

Corollary 6. If $p(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k > 0$, then:

$$M(p', R) \leq \frac{1}{2k^n} n R^{n-1} \exp\left(n \int_r^k \frac{E_{t,k}}{t+k} dt\right) M(p, r), \quad 0 < r \leq k \leq R \quad (56)$$

and

$$M(p', R) \leq n \left(\frac{R + k|\lambda|}{R^2 + k^2 + 2kR|\lambda|} \right) \exp\left(n \int_r^R \frac{E_{t,k}}{t+k} dt\right) M(p, r), \quad 0 < r \leq R \leq k. \quad (57)$$

Remark 9. If we choose $r = 1$ in Corollary 6, we have the next result which is an improvement of the result due to Bidkham and Dewan Theorem 3 in [10].

Corollary 7. If $p(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then:

$$M(p', R) \leq \frac{1}{2k^n} n R^{n-1} \exp\left(n \int_1^k \frac{E_{t,k}}{t+k} dt\right) M(p, 1), \quad R \geq k \quad (58)$$

and

$$M(p', R) \leq n \left(\frac{R + k|\lambda|}{R^2 + k^2 + 2kR|\lambda|} \right) \exp\left(n \int_1^R \frac{E_{t,k}}{t+k} dt\right) M(p, 1), \quad 1 \leq R \leq k. \quad (59)$$

Putting $r = 1$ in (38) of Lemma 11, we have:

$$\exp\left(n \int_1^R \frac{E_{t,k}}{t+k} dt\right) \leq \left(\frac{R+k}{1+k}\right)^n.$$

We can show that:

$$\left(\frac{R + k|\lambda|}{R^2 + k^2 + 2kR|\lambda|}\right) \left(\frac{R+k}{1+k}\right)^n \leq \frac{(R+k)^{n-1}}{(1+k)^n},$$

which is equivalent to:

$$(R + k|\lambda|)(R + k) \leq R^2 + k^2 + 2kR|\lambda|,$$

and it simplifies to:

$$kR(1 - |\lambda|) \leq k^2(1 - |\lambda|),$$

which clearly holds as $|\lambda| \leq 1$ and $R \leq k$.

Thus, inequality (59) is an improvement of the result mentioned in Remark 9. On the other hand, inequality (58) gives the corresponding bound when $R \geq k$.

Remark 10. Again putting $s = 1$, $r = 1$, $k = 1$, $R = 1$ in Theorem 5, both the inequalities reduce to the single well-known inequality (2) conjectured by Erdős and later verified by Lax [3].

We further improve Theorem 5 by involving $m(p, k)$. In fact, we prove:

Theorem 6. Let $p(z) = \sum_{v=0}^n a_v z^v$ be a polynomial of degree $n \geq 3$ having no zeros in $|z| < k$, $k > 0$, then for $0 \leq s < n$,

$$M(p^{(s)}, R) \leq \frac{1}{2k^n} \frac{d^s}{dR^s} (R^n + k^n) \left[\exp \left(n \int_r^k \frac{E_{t,k}}{t+k} dt \right) M(p, r) - \frac{1}{k^n} \left\{ k^n - r^n \exp \left(n \int_r^k \frac{E_{t,k}}{t+k} dt \right) \right\} m(p, k) \right], \quad 0 < r \leq k \leq R \quad (60)$$

and

$$M(p^{(s)}, R) \leq \frac{1}{R^s} \left\{ \frac{d^s}{dx^s} (x^n + 1) \right\}_{x=1} F_{R,k} \left[\exp \left(n \int_r^R \frac{E_{t,k}}{t+k} dt \right) M(p, r) - \frac{1}{k^n} \left\{ R^n - r^n \exp \left(n \int_r^R \frac{E_{t,k}}{t+k} dt \right) \right\} m(p, k) \right], \quad 0 < r \leq R \leq k, \quad (61)$$

where $E_{t,k}$ and $F_{R,k}$ are respectively given by (21) and (42).

Equality occurs in (60) when $s = 0$, $k = 1 = r$ for $p(z) = z^n + 1$ and in (61) when $s = 1$, $r = R$ for $p(z) = (z + k)^n$.

The following result which further improves Corollary 1, is obtained by taking $r = 1$ in Theorem 6.

Corollary 8. Let $p(z) = \sum_{v=0}^n a_v z^v$ be a polynomial of degree $n \geq 3$ having no zeros in $|z| < k$, $k \geq 1$, then for $0 \leq s < n$,

$$M(p^{(s)}, R) \leq \frac{1}{2k^n} \frac{d^s}{dR^s} (R^n + k^n) \left[\exp \left(n \int_1^k \frac{E_{t,k}}{t+k} dt \right) M(p, 1) - \frac{1}{k^n} \left\{ k^n - \exp \left(n \int_1^k \frac{E_{t,k}}{t+k} dt \right) \right\} m(p, k) \right], \quad R \geq k \quad (62)$$

and

$$M(p^{(s)}, R) \leq \frac{1}{R^s} \left\{ \frac{d^s}{dx^s} (x^n + 1) \right\}_{x=1} F_{R,k} \left[\exp \left(n \int_1^R \frac{E_{t,k}}{t+k} dt \right) M(p, 1) - \frac{1}{k^n} \left\{ R^n - \exp \left(n \int_1^R \frac{E_{t,k}}{t+k} dt \right) \right\} m(p, k) \right], \quad 1 \leq R \leq k, \quad (63)$$

where $E_{t,k}$ and $F_{R,k}$ are respectively given by (21) and (42).

Corollary 8 has special importance that it gives Ankeny and Rivlin's analogue of the s^{th} derivative separately by two different forms of bound according as $R \geq k$ and $1 \leq R \leq k$.

Remark 11. In view of (39), inequality (62) gives a better bound than a result due to Mir Theorem 3 in [13] concerning the estimate of $M(p^{(s)}, R)$. Whereas, inequality (63) gives an improved and generalized version of the result of Bidkham and Dewan Theorem 3 in [10] concerning s^{th} derivative.

Remark 12. Putting $s = 0$ and $k = 1$, (62) reduces to (5) due to Ankeny and Rivlin [6].

Remark 13. Putting $s = 0$, $R = k = 1$ in Theorem 6, the two inequalities coincide to the following result which further extends and improves (6) due to Rivlin [7].

Corollary 9. If $p(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree $n \geq 3$ having no zeros in $|z| < 1$, then:

$$M(p, r) \geq \frac{M(p, 1)}{\exp\left(n \int_r^1 \frac{E_{t,1}}{t+1} dt\right)} + \left(\frac{1}{\exp\left(n \int_r^1 \frac{E_{t,1}}{t+1} dt\right)} - r^n \right) m(p, 1), \quad 0 < r \leq 1, \quad (64)$$

where $E_{t,1}$ is obtained by taking $k = 1$ in (21).

For $0 < r \leq 1$, we have $\left(\frac{1+r}{2}\right)^n \geq r^n$. Also from (38) for $R = k = 1$:

$$\frac{1}{\exp\left(n \int_r^1 \frac{E_{t,1}}{t+1} dt\right)} \geq \left(\frac{1+r}{2}\right)^n \geq r^n.$$

Hence,

$$\frac{1}{\exp\left(n \int_r^1 \frac{E_{t,1}}{t+1} dt\right)} - r^n \geq 0. \quad (65)$$

Thus, Corollary 9 is a more improved version of Corollary 5, which further improves Theorem 2 due to Rivlin [7].

Remark 14. Again, as in Remark 10, if we take $s = 1$, $r = 1$, $k = 1$, $R = 1$ in (60) and (61) of Theorem 6; both the inequalities also reduce to inequality (2) conjectured by Erdős and later verified by Lax [3].

4. Proofs of the Theorems

We first prove the more refined Theorem 6.

Proof of Theorem 6. First we prove inequality (60). Since $p(z) \neq 0$ in $|z| < k$, then $T(z) = p(kz) \neq 0$ in $|z| < 1$.

Let $H(z) = z^n \overline{T\left(\frac{1}{\bar{z}}\right)}$, then $H(z)$ has all its zeros in $|z| \leq 1$ and:

$$|T(z)| \leq |H(z)| \quad \text{for } |z| = 1.$$

Applying Lemma 5 to the polynomials $T(z)$ and $H(z)$, for $0 \leq s < n$ and $t \geq 1$, we have:

$$\left|T^{(s)}(te^{i\theta})\right| \leq \left|H^{(s)}(te^{i\theta})\right|, \quad 0 \leq \theta < 2\pi. \quad (66)$$

Furthermore by Lemma 6 for $0 \leq s < n$,

$$\left|T^{(s)}(te^{i\theta})\right| + \left|H^{(s)}(te^{i\theta})\right| \leq \left\{\frac{d^s}{dt^s}(1+t^n)\right\}M(T,1), \quad 0 \leq \theta < 2\pi. \quad (67)$$

Combining (66) and (67), we get:

$$\begin{aligned} \left|T^{(s)}(te^{i\theta})\right| &\leq \frac{1}{2} \left\{\frac{d^s}{dt^s}(1+t^n)\right\}M(T,1) \\ \text{i.e., } \left|p^{(s)}(kte^{i\theta})\right| &\leq \frac{1}{2k^s} \left\{\frac{d^s}{dt^s}(1+t^n)\right\}M(p,k). \end{aligned} \quad (68)$$

Putting $kt = R$ in (68), we have:

$$\left|p^{(s)}(Re^{i\theta})\right| \leq \frac{1}{2k^n} \frac{d^s}{dR^s}(R^n + k^n)M(p,k) \quad \text{for } R \geq k.$$

Thus,

$$M(p^{(s)}, R) \leq \frac{1}{2k^n} \frac{d^s}{dR^s}(R^n + k^n)M(p,k) \quad \text{for } R \geq k. \quad (69)$$

Putting $\rho = k$ in inequality (31), we have for $0 < r \leq k$:

$$\begin{aligned} M(p,k) &\leq \exp\left(n \int_r^k \frac{E_{t,k}}{t+k} dt\right) M(p,r) \\ &\quad - \frac{1}{k^n} \left\{k^n - r^n \exp\left(n \int_r^k \frac{E_{t,k}}{t+k} dt\right)\right\} m(p,k). \end{aligned} \quad (70)$$

Hence, inequality (69), in conjunction with inequality (70), we have:

$$\begin{aligned} M(p^{(s)}, R) &\leq \frac{1}{2k^n} \frac{d^s}{dR^s}(R^n + k^n) \left[\exp\left(n \int_r^k \frac{E_{t,k}}{t+k} dt\right) M(p,r) \right. \\ &\quad \left. - \frac{1}{k^n} \left\{k^n - \exp\left(n \int_r^k \frac{E_{t,k}}{t+k} dt\right)\right\} m(p,k) \right] \quad \text{for } 0 < r \leq k \leq R, \end{aligned}$$

which is inequality (60).

Next we prove inequality (61). Let $P(z) = p(Rz)$ where $0 < R \leq k$. Since $p(z) \neq 0$ in $|z| < k$, $k > 0$, then $P(z)$ does not vanish in $|z| < \frac{k}{R}$, $\frac{k}{R} \geq 1$. Thus, applying (34) of Remark 1 to the polynomial $P(z)$, we have for $0 \leq s < n$ and $0 < R \leq k$:

$$R^s M(p^{(s)}, R) \leq \left\{\frac{d^s}{dx^s}(x^n + 1)\right\}_{x=1} F_{R,k} M(p,R). \quad (71)$$

Putting $\rho = R$ in (31) of Lemma 8, we get:

$$M(p, R) \leq \exp \left(n \int_r^R \frac{E_{t,k}}{t+k} dt \right) M(p, r) - \frac{1}{k^n} \left\{ R^n - r^n \exp \left(n \int_r^R \frac{E_{t,k}}{t+k} dt \right) \right\} m(p, k). \quad (72)$$

Hence, from (71) and (72) we obtain for $0 \leq s < n$ and $0 < r \leq R \leq k$:

$$M(p^{(s)}, R) \leq \frac{1}{R^s} \left\{ \frac{d^s}{dx^s} (x^n + 1) \right\}_{x=1} F_{R,k} \left[\exp \left(n \int_r^R \frac{E_{t,k}}{t+k} dt \right) M(p, r) - \frac{1}{k^n} \left\{ R^n - r^n \exp \left(n \int_r^R \frac{E_{t,k}}{t+k} dt \right) \right\} m(p, k) \right],$$

which completes the proof of Theorem 6. $\square$

Proof of Theorem 5. The proof of Theorem 5 follows along the lines of Theorem 6, on applying (20) of Lemma 7 instead of applying (31) of Lemma 8. $\square$

5. Conclusions

The well-known result due to Ankeny and Rivlin (Theorem 1), and some related results are generalized and extended by proving inequalities for the s^{th} derivative involving certain coefficients of the polynomials. In addition, the result due to Rivlin (Theorem 2) and some related results are generalized, improved, and extended in a similar manner. It would be of interest to further extend these results to polar derivatives of a higher order and L^p norm inequalities.

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