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Robust Solutions for Uncertain Continuous-Time Linear Programming Problems with Time-Dependent Matrices

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Abstract: The uncertainty for the continuous-time linear programming problem with time-dependent matrices is considered in this paper. In this case, the robust counterpart of the continuous-time linear programming problem is introduced. In order to solve the robust counterpart, it will be transformed into the conventional form of the continuous-time linear programming problem with time-dependent matrices. The discretization problem is formulated for the sake of numerically calculating the ϵ-optimal solutions, and a computational procedure is also designed to achieve this purpose.

Keywords: approximate solutions; continuous-time linear programming problems; ϵ-optimal solutions; robust counterpart; weak duality

MSC: 90C05; 90C46; 90C90

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1. Introduction

The theory of the continuous-time linear programming problem originated from the "bottleneck problem" proposed by Bellman [1]. A continuous-time linear programming problem with constant matrices was formulated and studied rigorously by Tyndall [2,3]. In this paper, we study the time-dependent matrices that is more complicated than the constant matrices. Although Levinson [4] also studied the problem of time-dependent matrices, the numerical methodology was not proposed to effectively calculate the optimal solution. Wu [5] tried to design a computational procedure to calculate the approximate optimal solutions. When the data in the continuous-time linear programming problem with time-dependent matrices are uncertain, a robust counterpart should be established and solved, which is the purpose of this paper.

The separated continuous-time linear programming problem is a subclass of the continuous-time linear programming problem, and it can be used to model the job-shop scheduling problems. Owing to its special structure, many researchers such as Anderson et al. [6–8], Fleischer and Sethuraman [9], and Pullan [10–14] have paid much attention to investigating its optimal solutions without providing the numerical technique. On the other hand, many interesting theoretical results have also been established by Meidan and Perold [15], Papageorgiou [16], and Schechter [17]. From the computational viewpoint, Weiss [18] designed a simplex-like algorithm that can be used to solve the separated continuous-time linear programming problem. In Wu [5,19], different computational procedures have been proposed to solve the continuous-time linear programming problem in which the functions are assumed to be piecewise continuous rather than continuous on the time interval $[0, T]$. Especially, Wu [5] solved the more complicated problem that involves time-dependent matrices. In this paper, we consider the uncertain continuous-time linear programming problem with time-dependent matrices, which is more complicated than the problem studied in Wu [5].

There are many types of continuous-time optimization problems that have been theoretically studied without considering the numerical issue. For example, the nonlinear types of continuous-time optimization problems have been studied by Farr and Hanson [20,21], Grinold [22,23], Hanson and Mond [24], Reiland [25,26], Reiland and Hanson [27], and Singh [28]. On the other hand, Rojas-Medar et al. [29], and Singh and Farr [30] studied the nonsmooth continuous-time optimization problems. Additionally, Nobakhtian and Pouryayevali [31,32] studied the nonsmooth continuous-time multiobjective programming problems. Especially, the continuous-time fractional programming problems have been theoretically investigated by Zalmai [33–36]. From the numerical viewpoint, Wen and Wu [37–39] have developed many different numerical techniques to solve the continuous-time linear fractional programming problems. As a matter of fact, numerically solving the continuous-time optimization problems is a difficult task. Even for a median size of continuous-time optimization problems, a great deal of computer resources may be needed.

Solving optimization problems that involve uncertain data has attracted many researchers. The pioneering work for solving the stochastic optimization problem was initiated by Dantzig [40], in which the uncertain data were driven by the observed probabilities. The main difficulty is how to fit the uncertain data using some known exact probability distribution function. Alternatively, the so-called robust optimization might pave another avenue to model the optimization problems with uncertain data. The basic idea of robust optimization assumes that each uncertain data should fall into a predetermined set. In other words, the uncertainty can be circumscribed beforehand. For example, the real-valued data can be assumed to fall into a bounded closed interval in $\mathbb{R}$ for convenience. Ben-Tal and Nemirovski [41,42], and El Ghaoui [43,44] proposed to solve the so-called robust optimization problems by assuming the optimization problems with uncertain data falling into uncertainty sets. The interested readers may refer to the articles contributed by Averbakh and Zhao [45], Ben-Tal et al. [46], Bertsimas et al. [47–49], Chen et al. [50], Erdoğan and Iyengar [51], Zhang [52], and the references therein. Wu [53] proposed a computational procedure to solve the robust continuous-time linear programming problem with constant matrices. In this paper, we solve the robust continuous-time linear programming problem with time-dependent matrices by designing a practical algorithm. We emphasize that the problems with time-dependent matrices are more complicated than the problems with constant matrices. In Section 2, we formulate a robust continuous-time linear programming problem with time-dependent matrices and transform it into a conventional form of the continuous-time linear programming problem with time-dependent matrices under some algebraic calculation. In Section 3, in order to numerically solve the desired problems, a discretization problem is introduced. In Section 4, we derive the analytic formula of error bound. We also introduce the concept of an ϵ-optimal solution to obtain the approximate solution. In Section 5, the convergent property of approximate solutions is studied. In Section 6, we design a computational procedure and provide a numerical example to demonstrate the usefulness of this practical algorithm.

2. Robust Continuous-Time Linear Programming Problems

We consider the following continuous-time linear programming problem with time-dependent matrices:

$$
\begin{aligned}
\text{(CLP)} \qquad \max \quad & \sum_{j=1}^{q} \int_0^T a_j(t) \cdot z_j(t) dt \\
\text{subject to} \quad & \sum_{j=1}^{q} B_{ij}(t) \cdot z_j(t) \leq c_i(t) + \sum_{j=1}^{q} \int_0^t K_{ij}(t,s) \cdot z_j(s) ds \\
& \quad \text{for all } t \in [0,T] \text{ and } i = 1, \cdots, p; \\
& z_j \in L^2[0,T] \text{ and } z_j(t) \geq 0 \text{ for all } j = 1, \cdots, q \text{ and } t \in [0,T],
\end{aligned}
$$

where B_{ij} and K_{ij} are the nonnegative real-valued functions defined on $[0,T]$ and $[0,T] \times [0,T]$, respectively, for $i = 1, \cdots, p$ and $j = 1, \cdots, q$. When the real-valued functions c_i are assumed to be nonnegative on $[0,T]$ for $i = 1, \cdots, p$, it is obvious that the primal problem (CLP) is feasible with a trivial feasible solution $z_j(t) = 0$ for all $j = 1, \cdots, q$.

The functions a_j and c_i can be assumed to be certain or uncertain data for $i = 1, \cdots, p$ and $j = 1, \cdots, q$. When the functions a_j and c_i are assumed to be uncertain, they are considered to be the pointwise-uncertain. In other words, given each $t \in [0, T]$, the uncertain data $a_j(t)$ and $c_i(t)$ will be assumed to fall into the uncertainty sets $\mathcal{V}_{a_j}(t)$ and $\mathcal{V}_{c_i}(t)$, respectively, which are predetermined by the decision-makers. When the functions a_j and c_i are assumed to be certain, the function values $a_j(t)$ and $c_i(t)$ are assumed to be certain for each $t \in [0, T]$. When the function values $a_j(t)$ and $c_i(t)$ are assumed to be certain, we can also consider the uncertainty sets $\mathcal{V}_{a_j}(t) = \{a_j(t)\}$ and $\mathcal{V}_{c_i}(t) = \{a_j(t)\}$ to be the singleton sets. We denote by $I^{(\mathbf{a})}$ and $I^{(\mathbf{c})}$ the sets of indices in which the functions a_j and c_i are assumed to be uncertain, respectively. In other words, if $j \in I^{(\mathbf{a})}$, the function a_j is uncertain and, if $i \in I^{(\mathbf{c})}$, the function c_i is uncertain.

We also assume that some of the functions $B_{ij}(t)$ and $K_{ij}(t, s)$ are pointwise-uncertain by similarly considering the uncertainty sets $\mathcal{U}_{B_{ij}}(t)$ and $\mathcal{U}_{K_{ij}}(t, s)$, respectively. Given any fixed $i \in \{1, 2, \cdots, p\}$, we also denote by $I_i^{(B)}$ and $I_i^{(K)}$ the set of indices in which the functions B_{ij} and K_{ij} are assumed to be uncertain. Therefore, $I_i^{(B)}$ and $I_i^{(K)}$ are subsets of $\{1, 2, \cdots, q\}$.

The robust counterpart of the original continuous-time linear programming problem (CLP) is assumed to take each data in the corresponding uncertainty set, and it is formulated as follows:

$$
\begin{aligned}
\text{(RCLP)} \qquad \max \quad & \sum_{j=1}^{q} \int_0^T a_j(t) \cdot z_j(t) dt \\
\text{subject to} \quad & \sum_{j=1}^{q} B_{ij}(t) \cdot z_j(t) \leq c_i(t) + \sum_{j=1}^{q} \int_0^t K_{ij}(t, s) \cdot z_j(s) ds \\
& \qquad \text{for all } t \in [0, T] \text{ and } i = 1, \cdots, p; \\
& z_j \in L^2[0, T] \text{ and } z_j(t) \geq 0 \text{ for } j = 1, \cdots, q \text{ and } t \in [0, T]; \\
& \text{for all } a_j(t) \in \mathcal{V}_{a_j}(t) \text{ for all } t \in [0, T] \text{ and } j = 1, \cdots, q; \\
& \text{for all } c_i(t) \in \mathcal{V}_{c_i}(t) \text{ for all } t \in [0, T] \text{ and } i = 1, \cdots, p; \\
& \text{for all } B_{ij}(t) \in \mathcal{U}_{B_{ij}}(t) \text{ for all } t \in [0, T], i = 1, \cdots, p \text{ and } j = 1, \cdots, q; \\
& \text{for all } K_{ij}(t, s) \in \mathcal{U}_{K_{ij}}(t, s) \\
& \qquad \text{for all } (t, s) \in [0, T] \times [0, T], i = 1, \cdots, p \text{ and } j = 1, \cdots, q.
\end{aligned}
$$

The robust counterpart (RCLP) as shown above is a semi-infinite problem, which has infinitely many numbers of constraints. Therefore, it is really hard to solve. Usually, the uncertainty sets are taken to be bounded closed intervals in $\mathbb{R}$. When the uncertainty sets $\mathcal{U}_{B_{ij}}(t), \mathcal{U}_{K_{ij}}(t, s)$, $\mathcal{V}_{a_j}(t)$, and $\mathcal{V}_{c_i}(t)$ are taken to be bounded closed intervals in $\mathbb{R}$, the semi-infinite problem (RCLP) can be transformed into a conventional continuous-time linear programming problem with time-dependent matrices. Now, the uncertainty sets are described below.

- For $B_{ij}(t)$ with $j \in I_i^{(B)}$ and $K_{ij}(t, s)$ with $j \in I_i^{(K)}$, the uncertain data $B_{ij}(t)$ and $K_{ij}(t, s)$ are assumed to fall into the bounded closed intervals given by

$$
\mathcal{U}_{B_{ij}}(t) = \left[B_{ij}^{(0)}(t) - \widehat{B}_{ij}(t), B_{ij}^{(0)}(t) + \widehat{B}_{ij}(t) \right]
$$

and

$$
\mathcal{U}_{K_{ij}}(t, s) = \left[K_{ij}^{(0)}(t, s) - \widehat{K}_{ij}(t, s), K_{ij}^{(0)}(t, s) + \widehat{K}_{ij}(t, s) \right],
$$

respectively, where $B_{ij}^{(0)}(t) \geq 0$ and $K_{ij}^{(0)}(t,s) \geq 0$ denote the known nominal data of $B_{ij}(t)$ and $K_{ij}(t,s)$, respectively, and $\widehat{B}_{ij}(t) \geq 0$ and $\widehat{K}_{ij}(t,s) \geq 0$ denote the uncertainties such that

$$B_{ij}^{(0)}(t) - \widehat{B}_{ij}(t) \geq 0 \text{ and } K_{ij}^{(0)}(t,s) - \widehat{K}_{ij}(t,s) \geq 0.$$

For $j \notin I_i^{(B)}$, we also use the notation $B_{ij}^{(0)}(t)$ to denote the certain data with uncertainty $\widehat{B}_{ij}(t) = 0$, and use the notation $K_{ij}^{(0)}(t,s)$ to denote the certain data with uncertainty $\widehat{K}_{ij}(t,s) = 0$ for $j \notin I_i^{(K)}$.

- For a_j with $j \in I^{(\mathbf{a})}$ and c_i with $i \in I^{(\mathbf{c})}$, the uncertain data a_j and c_i are assumed to fall into the bounded closed intervals given by

$$\mathcal{V}_{a_j}(t) = \left[a_j^{(0)}(t) - \widehat{a}_j(t), a_j^{(0)}(t) + \widehat{a}_j(t)\right] \text{ and } \mathcal{V}_{c_i}(t) = \left[c_i^{(0)}(t) - \widehat{c}_i(t), c_i^{(0)}(t) + \widehat{c}_i(t)\right],$$

where $a_j^{(0)}(t)$ and $c_i^{(0)}(t)$ denote the known nominal data of $a_j(t)$ and $c_i(t)$, respectively, and $\widehat{a}_j(t) \geq 0$ and $\widehat{c}_i(t) \geq 0$ denote the uncertainties of $a_j(t)$ and $c_i(t)$, respectively. For $j \notin I^{(\mathbf{a})}$, we also use the notation $a_j^{(0)}(t)$ to denote the certain data with uncertainties $\widehat{a}_j(t) = 0$ and use the notation $c_i^{(0)}(t)$ to denote the certain data with uncertainties $\widehat{c}_j(t) = 0$ for $i \notin I^{(\mathbf{c})}$.

Under the above settings, the robust counterpart (RCLP) is written as follows:

$$\begin{aligned}
\text{(RCLP)} \quad \max \quad & \sum_{j \notin I^{(\mathbf{a})}} \int_0^T a_j^{(0)}(t) \cdot z_j(t)dt + \sum_{j \in I^{(\mathbf{a})}} \int_0^T a_j(t) \cdot z_j(t)dt \\
\text{subject to} \quad & \sum_{\{j: j \notin I_i^{(B)}\}} B_{ij}^{(0)}(t) \cdot z_j(t) + \sum_{\{j: j \in I_i^{(B)}\}} B_{ij}(t) \cdot z_j(t) \\
& \quad \leq c_i^{(0)}(t) + \sum_{\{j: j \notin I_i^{(K)}\}} \int_0^t K_{ij}^{(0)}(t,s) \cdot z_j(s)ds + \sum_{\{j: j \in I_i^{(K)}\}} \int_0^t K_{ij}(t,s) \cdot z_j(s)ds \\
& \qquad \text{for } i \notin I^{(\mathbf{c})} \text{ and for all } t \in [0,T]; \\
& \sum_{\{j: j \notin I_i^{(B)}\}} B_{ij}^{(0)}(t) \cdot z_j(t) + \sum_{\{j: j \in I_i^{(B)}\}} B_{ij}(t) \cdot z_j(t) \\
& \quad \leq c_i(t) + \sum_{\{j: j \notin I_i^{(K)}\}} \int_0^t K_{ij}^{(0)}(t,s) \cdot z_j(s)ds + \sum_{\{j: j \in I_i^{(K)}\}} \int_0^t K_{ij}(t,s) \cdot z_j(s)ds \\
& \qquad \text{for } i \in I^{(\mathbf{c})} \text{ and for all } t \in [0,T]; \\
& z_j \in L^2[0,T] \text{ and } z_j(t) \geq 0 \text{ for } j = 1, \cdots, q \text{ and } t \in [0,T]; \\
& \text{for all } B_{ij}(t) \in \mathcal{U}_{B_{ij}}(t) \text{ with } j \in I_i^{(B)} \text{ for all } t \in [0,T]; \\
& \text{for all } K_{ij}(t,s) \in \mathcal{U}_{K_{ij}}(t,s) \text{ with } j \in I_i^{(K)} \text{ for all } (t,s) \in [0,T] \times [0,T]; \\
& \text{for all } a_j(t) \in \mathcal{V}_{a_j}(t) \text{ with } j \in I^{(\mathbf{a})} \text{ for all } t \in [0,T]; \\
& \text{for all } c_i(t) \in \mathcal{V}_{c_i}(t) \text{ with } i \in I^{(\mathbf{c})} \text{ for all } t \in [0,T].
\end{aligned}$$

We convert the above semi-infinite problem (RCLP) into a conventional continuous-time linear programming problem with time-dependent matrices. We first rewrite the problem (RCLP) as the following equivalent form:

$$
\begin{aligned}
\text{(RCLP1)} \quad & \max \quad \phi \\
& \text{subject to} \quad \phi \leq \sum_{j \notin I^{(\mathbf{a})}} \int_0^T a_j^{(0)}(t) \cdot z_j(t)dt + \sum_{j \in I^{(\mathbf{a})}} \int_0^T a_j(t) \cdot z_j(t)dt \\
& \sum_{\{j:j \notin I_i^{(B)}\}} B_{ij}^{(0)}(t) \cdot z_j(t) + \sum_{\{j:j \in I_i^{(B)}\}} B_{ij}(t) \cdot z_j(t) \\
& \quad \leq c_i^{(0)}(t) + \sum_{\{j:j \notin I_i^{(K)}\}} \int_0^t K_{ij}^{(0)}(t,s) \cdot z_j(s)ds + \sum_{\{j:j \in I_i^{(K)}\}} \int_0^t K_{ij}(t,s) \cdot z_j(s)ds \\
& \qquad \text{for } i \notin I^{(\mathbf{c})} \text{ and for all } t \in [0,T]; \\
& \sum_{\{j:j \notin I_i^{(B)}\}} B_{ij}^{(0)}(t) \cdot z_j(t) + \sum_{\{j:j \in I_i^{(B)}\}} B_{ij}(t) \cdot z_j(t) \\
& \quad \leq c_i(t) + \sum_{\{j:j \notin I_i^{(K)}\}} \int_0^t K_{ij}^{(0)}(t,s) \cdot z_j(s)ds + \sum_{\{j:j \in I_i^{(K)}\}} \int_0^t K_{ij}(t,s) \cdot z_j(s)ds \\
& \qquad \text{for } i \in I^{(\mathbf{c})} \text{ and for all } t \in [0,T]; \\
& \phi \in \mathbb{R}; \\
& z_j \in L^2[0,T] \text{ and } z_j(t) \geq 0 \text{ for } j = 1, \cdots, q \text{ and } t \in [0,T]; \\
& \text{for all } B_{ij}(t) \in \mathcal{U}_{B_{ij}}(t) \text{ with } j \in I_i^{(B)} \text{ for all } t \in [0,T]; \\
& \text{for all } K_{ij}(t,s) \in \mathcal{U}_{K_{ij}}(t,s) \text{ with } j \in I_i^{(K)} \text{ for all } (t,s) \in [0,T] \times [0,T]; \\
& \text{for all } a_j(t) \in \mathcal{V}_{a_j}(t) \text{ with } j \in I^{(\mathbf{a})} \text{ for all } t \in [0,T]; \\
& \text{for all } c_i(t) \in \mathcal{V}_{c_i}(t) \text{ with } i \in I^{(\mathbf{c})} \text{ for all } t \in [0,T].
\end{aligned}
$$

Given any fixed $i \in \{1, \cdots, p\}$, for $j \in I_i^{(B)}$, since $z_j(t) \geq 0$ and $B_{ij}(t) \leq B_{ij}^{(0)}(t) + \widehat{B}_{ij}(t)$, we have

$$
\sum_{\{j:j \notin I_i^{(B)}\}} B_{ij}^{(0)}(t) \cdot z_j(t) + \sum_{\{j:j \in I_i^{(B)}\}} B_{ij}(t) \cdot z_j(t) \leq \sum_{j=1}^q B_{ij}^{(0)}(t) \cdot z_j(t) + \sum_{\{j:j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot z_j(t). \quad (1)
$$

Similarly, for $j \in I_i^{(K)}$, since $z_j(s) \geq 0$ and $K_{ij}^{(0)}(t,s) - \widehat{K}_{ij}(t,s) \leq K_{ij}(t,s)$, we also have

$$
\begin{aligned}
& \sum_{\{j:j \notin I_i^{(K)}\}} \int_0^t K_{ij}^{(0)}(t,s) \cdot z_j(s)ds + \sum_{\{j:j \in I_i^{(K)}\}} \int_0^t K_{ij}(t,s) \cdot z_j(s)ds \\
& \quad \geq \sum_{j=1}^q \int_0^t K_{ij}^{(0)}(t,s) \cdot z_j(s)ds - \sum_{\{j:j \in I_i^{(K)}\}} \int_0^t \widehat{K}_{ij}(t,s) \cdot z_j(s)ds. \quad (2)
\end{aligned}
$$

Using the inequalities (1) and (2), we consider the following cases.

- For $i \in I^{(\mathbf{c})}$, since $c_i^{(0)}(t) - \widehat{c}_i(t) \leq c_i(t)$ for all $t \in [0, T]$, we obtain

$$\begin{aligned}
&\sum_{\{j:j\notin I_i^{(B)}\}} B_{ij}^{(0)}(t) \cdot z_j(t) + \sum_{\{j:j\in I_i^{(B)}\}} B_{ij}(t) \cdot z_j(t) \\
&\quad - \sum_{\{j:j\notin I_i^{(K)}\}} \int_0^t K_{ij}^{(0)}(t,s) \cdot z_j(s)ds - \sum_{\{j:j\in I_i^{(K)}\}} \int_0^t K_{ij}(t,s) \cdot z_j(s)ds - c_i(t) \\
&\leq \left(\sum_{j=1}^q B_{ij}^{(0)}(t) \cdot z_j(t) + \sum_{\{j:j\in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot z_j(t) \right) \\
&\quad - \left(\sum_{j=1}^q \int_0^t K_{ij}^{(0)}(t,s) \cdot z_j(s)ds - \sum_{\{j:j\in I_i^{(K)}\}} \int_0^t \widehat{K}_{ij}(t,s) \cdot z_j(s)ds \right) - \left(c_i^{(0)}(t) - \widehat{c}_i(t) \right)
\end{aligned}$$

for all $B_{ij}(t) \in \mathcal{U}_{B_{ij}}(t)$, $K_{ij}(t,s) \in \mathcal{U}_{K_{ij}}(t,s)$, and $c_i(t) \in \mathcal{V}_{c_i}(t)$, which implies

$$\begin{aligned}
&\max_{B_{ij},K_{ij},c_i} \left\{ \sum_{\{j:j\notin I_i^{(B)}\}} B_{ij}^{(0)}(t) \cdot z_j(t) + \sum_{\{j:j\in I_i^{(B)}\}} B_{ij}(t) \cdot z_j(t) \right. \\
&\qquad \left. - \sum_{\{j:j\notin I_i^{(K)}\}} \int_0^t K_{ij}^{(0)}(t,s) \cdot z_j(s)ds - \sum_{\{j:j\in I_i^{(K)}\}} \int_0^t K_{ij}(t,s) \cdot z_j(s)ds - c_i(t) \right\} \\
&= \left(\sum_{j=1}^q B_{ij}^{(0)}(t) \cdot z_j(t) + \sum_{\{j:j\in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot z_j(t) \right) \\
&\quad - \left(\sum_{j=1}^q \int_0^t K_{ij}^{(0)}(t,s) \cdot z_j(s)ds - \sum_{\{j:j\in I_i^{(K)}\}} \int_0^t \widehat{K}_{ij}(t,s) \cdot z_j(s)ds \right) - \left(c_i^{(0)}(t) - \widehat{c}_i(t) \right),
\end{aligned} \tag{3}$$

where the equality can be attained.

- For $i \notin I^{(\mathbf{c})}$, we obtain

$$\begin{aligned}
&\sum_{\{j:j\notin I_i^{(B)}\}} B_{ij}^{(0)}(t) \cdot z_j(t) + \sum_{\{j:j\in I_i^{(B)}\}} B_{ij}(t) \cdot z_j(t) \\
&\quad - \sum_{\{j:j\notin I_i^{(K)}\}} \int_0^t K_{ij}^{(0)}(t,s) \cdot z_j(s)ds - \sum_{\{j:j\in I_i^{(K)}\}} \int_0^t K_{ij}(t,s) \cdot z_j(s)ds - c_i^{(0)}(t) \\
&\leq \left(\sum_{j=1}^q B_{ij}^{(0)}(t) \cdot z_j(t) + \sum_{\{j:j\in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot z_j(t) \right) \\
&\quad - \left(\sum_{j=1}^q \int_0^t K_{ij}^{(0)}(t,s) \cdot z_j(s)ds - \sum_{\{j:j\in I_i^{(K)}\}} \int_0^t \widehat{K}_{ij}(t,s) \cdot z_j(s)ds \right) - c_i^{(0)}(t),
\end{aligned}$$

which implies

$$\begin{aligned}
&\max_{B_{ij},K_{ij}}\left\{\sum_{\{j:j\notin I_i^{(B)}\}} B_{ij}^{(0)}(t)\cdot z_j(t)+\sum_{\{j:j\in I_i^{(B)}\}} B_{ij}(t)\cdot z_j(t)\right.\\
&\qquad\left.-\sum_{\{j:j\notin I_i^{(K)}\}}\int_0^t K_{ij}^{(0)}(t,s)\cdot z_j(s)ds-\sum_{\{j:j\in I_i^{(K)}\}}\int_0^t K_{ij}(t,s)\cdot z_j(s)ds-c_i^{(0)}(t)\right\}\\
&=\left(\sum_{j=1}^{q} B_{ij}^{(0)}(t)\cdot z_j(t)+\sum_{\{j:j\in I_i^{(B)}\}}\widehat{B}_{ij}(t)\cdot z_j(t)\right)\\
&\quad-\left(\sum_{j=1}^{q}\int_0^t K_{ij}^{(0)}(t,s)\cdot z_j(s)ds-\sum_{\{j:j\in I_i^{(K)}\}}\int_0^t \widehat{K}_{ij}(t,s)\cdot z_j(s)ds\right)-c_i^{(0)}(t),
\end{aligned}\tag{4}$$

where the equality can be attained. Since $a_j^{(0)}(t)-\widehat{a}_j(t)\le a_j(t)$ for $j\in I^{(\mathbf{a})}$ and $t\in[0,T]$, we have

$$\begin{aligned}
&\sum_{j=1}^{q}\int_0^T a_j^{(0)}(t)\cdot z_j(t)dt-\sum_{j\in I^{(\mathbf{a})}}\int_0^T \widehat{a}_j(t)\cdot z_j(t)dt\\
&\quad\le\sum_{j\notin I^{(\mathbf{a})}}\int_0^T a_j^{(0)}(t)\cdot z_j(t)dt+\sum_{j\in I^{(\mathbf{a})}}\int_0^T a_j(t)\cdot z_j(t)dt,
\end{aligned}$$

which implies

$$\begin{aligned}
&\min_{a_j}\left\{\sum_{j\notin I^{(\mathbf{a})}}\int_0^T a_j^{(0)}(t)\cdot z_j(t)dt+\sum_{j\in I^{(\mathbf{a})}}\int_0^T a_j(t)\cdot z_j(t)dt\right\}\\
&\quad=\sum_{j=1}^{q}\int_0^T a_j^{(0)}(t)\cdot z_j(t)dt-\sum_{j\in I^{(\mathbf{a})}}\int_0^T \widehat{a}_j(t)\cdot z_j(t)dt,
\end{aligned}\tag{5}$$

where the equality can also be attained. From the Equalities (3)–(5), it follows that $(\phi,\mathbf{z}(t))=(\phi,z_1(t),\cdots,z_n(t))$ is a feasible solution of problem (RCLP1) if and only if it satisfies the following inequalities:

$$\begin{aligned}
&\sum_{j=1}^{q} B_{ij}^{(0)}(t)\cdot z_j(t)+\sum_{\{j:j\in I_i^{(B)}\}}\widehat{B}_{ij}(t)\cdot z_j(t)\\
&\quad\le\sum_{j=1}^{q}\int_0^t K_{ij}^{(0)}(t,s)\cdot z_j(s)ds-\sum_{\{j:j\in I_i^{(K)}\}}\int_0^t \widehat{K}_{ij}(t,s)\cdot z_j(s)ds+c_i^{(0)}(t)-\widehat{c}_i(t)\text{ for } i\in I^{(\mathbf{c})};\\
&\sum_{j=1}^{q} B_{ij}^{(0)}(t)\cdot z_j(t)+\sum_{\{j:j\in I_i^{(B)}\}}\widehat{B}_{ij}(t)\cdot z_j(t)\\
&\quad\le\sum_{j=1}^{q}\int_0^t K_{ij}^{(0)}(t,s)\cdot z_j(s)ds-\sum_{\{j:j\in I_i^{(K)}\}}\int_0^t \widehat{K}_{ij}(t,s)\cdot z_j(s)ds+c_i^{(0)}(t)\text{ for } i\notin I^{(\mathbf{c})};\\
&\phi\le\sum_{j=1}^{q}\int_0^T a_j^{(0)}(t)\cdot z_j(t)dt-\sum_{j\in I^{(\mathbf{a})}}\int_0^T \widehat{a}_j(t)\cdot z_j(t)dt.
\end{aligned}$$

This shows that problem (RCLP1) is equivalent to the following problem:

$$
\begin{aligned}
\text{(RCLP2)} \qquad \max \quad & \phi \\
\text{subject to} \quad & \phi \leq \sum_{j=1}^{q} \int_0^T a_j^{(0)}(t) \cdot z_j(t)dt - \sum_{j \in I^{(\mathbf{a})}} \int_0^T \widehat{a}_j(t) \cdot z_j(t)dt; \\
& \sum_{j=1}^{q} B_{ij}^{(0)}(t) \cdot z_j(t) + \sum_{\{j:j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot z_j(t) \leq \left(c_i^{(0)}(t) - \widehat{c}_i(t)\right) \\
& \quad + \left(\sum_{j=1}^{q} \int_0^t K_{ij}^{(0)}(t,s) \cdot z_j(s)ds - \sum_{\{j:j \in I_i^{(K)}\}} \int_0^t \widehat{K}_{ij}(t,s) \cdot z_j(s)ds \right) \\
& \quad \text{for all } t \in [0,T] \text{ and } i \in I^{(\mathbf{c})}; \\
& \sum_{j=1}^{q} B_{ij}^{(0)}(t) \cdot z_j(t) + \sum_{\{j:j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot z_j(t) \leq c_i^{(0)}(t) \\
& \quad + \left(\sum_{j=1}^{q} \int_0^t K_{ij}^{(0)}(t,s) \cdot z_j(s)ds - \sum_{\{j:j \in I_i^{(K)}\}} \int_0^t \widehat{K}_{ij}(t,s) \cdot z_j(s)ds \right) \\
& \quad \text{for all } t \in [0,T] \text{ and } i \notin I^{(\mathbf{c})}; \\
& \phi \in \mathbb{R}; \\
& z_j \in L^2[0,T] \text{ and } z_j(t) \geq 0 \text{ for } j = 1, \cdots, q \text{ and } t \in [0,T]
\end{aligned}
$$

which can also be rewritten as the following continuous-time linear programming problem:

$$
\begin{aligned}
\text{(RCLP3)} \qquad \max \quad & \sum_{j=1}^{q} \int_0^T a_j^{(0)}(t) \cdot z_j(t)dt - \sum_{j \in I^{(\mathbf{a})}} \int_0^T \widehat{a}_j(t) \cdot z_j(t)dt \\
\text{subject to} \quad & \sum_{j=1}^{q} B_{ij}^{(0)}(t) \cdot z_j(t) + \sum_{\{j:j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot z_j(t) \leq \left(c_i^{(0)}(t) - \widehat{c}_i(t)\right) \\
& \quad + \left(\sum_{j=1}^{q} \int_0^t K_{ij}^{(0)}(t,s) \cdot z_j(s)ds - \sum_{\{j:j \in I_i^{(K)}\}} \int_0^t \widehat{K}_{ij}(t,s) \cdot z_j(s)ds \right) \\
& \quad \text{for all } t \in [0,T] \text{ and } i \in I^{(\mathbf{c})}; \\
& \sum_{j=1}^{q} B_{ij}^{(0)}(t) \cdot z_j(t) + \sum_{\{j:j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot z_j(t) \leq c_i^{(0)}(t) \\
& \quad + \left(\sum_{j=1}^{q} \int_0^t K_{ij}^{(0)}(t,s) \cdot z_j(s)ds - \sum_{\{j:j \in I_i^{(K)}\}} \int_0^t \widehat{K}_{ij}(t,s) \cdot z_j(s)ds \right) \\
& \quad \text{for all } t \in [0,T] \text{ and } i \notin I^{(\mathbf{c})}; \\
& z_j \in L^2[0,T] \text{ and } z_j(t) \geq 0 \text{ for } j = 1, \cdots, q \text{ and } t \in [0,T].
\end{aligned}
$$

The duality theory in continuous-time linear programming problem states that the dual problem of (RCLP3) can be formulated as follows:

$$
\begin{aligned}
\text{(DRCLP3)} \qquad \max \quad & \sum_{i=1}^{p} \int_0^T c_i^{(0)}(t) \cdot w_i(t)dt - \sum_{i \in I^{(\mathbf{c})}} \int_0^T \widehat{c}_i(t) \cdot w_i(t)dt \\
\text{subject to} \quad & \sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot w_i(t) + \sum_{\{i:j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot w_i(t) \geq \left(a_j^{(0)}(t) - \widehat{a}_j(t) \right) \\
& \quad + \left(\sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot w_i(s)ds - \sum_{\{i:j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot w_i(s)ds \right) \\
& \quad \text{for all } t \in [0,T] \text{ and } j \in I^{(\mathbf{a})}; \\
& \sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot w_i(t) + \sum_{\{i:j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot w_i(t) \geq a_j^{(0)}(t) \\
& \quad + \left(\sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot w_i(s)ds - \sum_{\{i:j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot w_i(s)ds \right) \\
& \quad \text{for all } t \in [0,T] \text{ and } j \notin I^{(\mathbf{a})}; \\
& w_i \in L^2[0,T] \text{ and } w_i(t) \geq 0 \text{ for } i = 1, \cdots, p \text{ and } t \in [0,T].
\end{aligned}
$$

3. Discretization

In order to efficiently develop the numerical method, the following conditions are assumed to be satisfied.

- For $i = 1, \cdots, p$ and $j = 1, \cdots, q$, the functions $B_{ij}^{(0)}$ and $K_{ij}^{(0)}$ are assumed to be nonnegative on $[0,T]$ and $[0,T] \times [0,T]$, respectively. The functions $B_{ij}^{(0)} - \widehat{B}_{ij}$ for $j \in I_i^{(B)}$ and $K_{ij}^{(0)} - \widehat{K}_{ij}$ for $j \in I_i^{(K)}$ are also assumed to be nonnegative on $[0,T]$ and $[0,T] \times [0,T]$, respectively.
- For $i = 1, \cdots, p$ and $j = 1, \cdots, q$, the functions $a_j^{(0)}$ and $c_i^{(0)}$ are assumed to be piecewise continuous on $[0,T]$. For $j \in I^{(\mathbf{a})}$ and $i \in I^{(\mathbf{c})}$, the functions $\widehat{a}_j$ and $\widehat{c}_i$ are assumed to be piecewise continuous on $[0,T]$, which also suggest that the functions $a_j^{(0)} - \widehat{a}_j$ and $c_i^{(0)} - \widehat{c}_i$ are piecewise continuous on $[0,T]$ for $j \in I^{(\mathbf{a})}$ and $i \in I^{(\mathbf{c})}$.
- For each $j = 1, \cdots, q$ and $t \in [0,T]$, the following inequality is satisfied:

$$\sum_{i=1}^{p} B_{ij}^{(0)}(t) + \sum_{\{i:j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) > 0. \tag{6}$$

- For each $i = 1, \cdots, p$ and for

$$\bar{B}_i \equiv \min_{j \in I_i^{(B)}} \inf_{t \in [0,T]} \left\{ B_{ij}^{(0)}(t) + \widehat{B}_{ij}(t) : B_{ij}^{(0)}(t) + \widehat{B}_{ij}(t) > 0 \right\}$$

and

$$\widetilde{B}_i \equiv \min_{j \notin I_i^{(B)}} \inf_{t \in [0,T]} \left\{ B_{ij}^{(0)}(t) : B_{ij}^{(0)}(t) > 0 \right\},$$

the following inequality is satisfied:

$$\min_{i=1,\cdots,p} \min\left\{ \bar{B}_i, \widetilde{B}_i \right\} = \sigma > 0.$$

In other words,

$$\sigma \leq \begin{cases} B_{ij}^{(0)}(t) + \widehat{B}_{ij}(t) & \text{if } B_{ij}^{(0)}(t) + \widehat{B}_{ij}(t) \neq 0 \text{ for } j \in I_i^{(B)} \\ B_{ij}^{(0)}(t) & \text{if } B_{ij}^{(0)}(t) \neq 0 \text{ for } j \notin I_i^{(B)}. \end{cases} \tag{7}$$

Let $\mathfrak{A}_j$, $\mathfrak{S}_i$, $\mathfrak{B}_{ij}$, and $\mathfrak{K}_{ij}$ denote the set of discontinuities of $a_j(t)$, $c_i(t)$, $B_{ij}(t)$, and $K_{ij}(t,s)$, respectively. Then, $\mathfrak{A}_j$, $\mathfrak{S}_i$, and $\mathfrak{B}_{ij}$ are finite subsets of $[0,T]$ and $\mathfrak{K}_{ij}$ is a finite subset of $[0,T] \times [0,T]$. We also write

$$\mathfrak{K}_{ij} = \mathfrak{K}_{ij}^{(1)} \times \mathfrak{K}_{ij}^{(2)},$$

where $\mathfrak{K}_{ij}^{(1)}$ and $\mathfrak{K}_{ij}^{(2)}$ are a finite subset of $[0,T]$. In order to determine the partition of the time interval $[0,T]$, we consider the following set:

$$\mathcal{D} = \left(\bigcup_{j=1}^{q} \mathfrak{A}_j\right) \bigcup \left(\bigcup_{i=1}^{p} \mathfrak{S}_i\right) \bigcup \left(\bigcup_{i=1}^{p}\bigcup_{j=1}^{q} \mathfrak{B}_{ij}\right) \bigcup \left(\bigcup_{i=1}^{p}\bigcup_{j=1}^{q} \mathfrak{K}_{ij}^{(1)}\right) \bigcup \left(\bigcup_{i=1}^{p}\bigcup_{j=1}^{q} \mathfrak{K}_{ij}^{(2)}\right) \bigcup \{0,T\}.$$

Then, $\mathcal{D}$ is a finite subset of $[0,T]$ written by

$$\mathcal{D} = \{d_0, d_1, d_2, \cdots, d_r\},$$

where, for convenience, we set $d_0 = 0$ and $d_r = T$. Let $\mathcal{P}_n$ be a partition of $[0,T]$ satisfying $\mathcal{D} \subseteq \mathcal{P}_n$. In other words, each closed $[d_v, d_{v+1}]$ is also divided into many closed subintervals.

Let

$$\mathcal{P}_n = \left\{e_0^{(n)}, e_1^{(n)}, \cdots, e_n^{(n)}\right\},$$

where $e_0^{(n)} = 0$ and $e_n^{(n)} = T$. The n closed subintervals are denoted by

$$\bar{E}_l^{(n)} = \left[e_{l-1}^{(n)}, e_l^{(n)}\right] \text{ for } l = 1, \cdots, n.$$

For convenience, we also write

$$E_l^{(n)} = \left(e_{l-1}^{(n)}, e_l^{(n)}\right) \text{ and } F_l^{(n)} = \left[e_{l-1}^{(n)}, e_l^{(n)}\right).$$

We denote by $\mathfrak{d}_l^{(n)}$ the length of closed interval $\bar{E}_l^{(n)}$. Let

$$\parallel \mathcal{P}_n \parallel = \max_{l=1,\cdots,n} \mathfrak{d}_l^{(n)}$$

and assume

$$\parallel \mathcal{P}_n \parallel \to 0 \text{ as } n \to \infty.$$

From a computational viewpoint, we also assume that there exists $n_*, n^* \in \mathbb{N}$ satisfying

$$n_* \cdot r \leq n \leq n^* \cdot r \text{ and } \parallel \mathcal{P}_n \parallel \leq \frac{T}{n^*}. \tag{8}$$

When $n_* \to \infty$, we have $n \to \infty$. In the sequel, $n \to \infty$ implicitly means that $n_* \to \infty$.

Under the above construction for the partition $\mathcal{P}_n$, it is clear that the functions $a_j^{(0)} - \widehat{a}_j$ for $j \in I^{(\mathbf{a})}$, $a_j^{(0)}$ for $j \notin I^{(\mathbf{a})}$, $c_i^{(0)} - \widehat{c}_i$ for $i \in I^{(\mathbf{c})}$, and $c_i^{(0)}$ for $i \notin I^{(\mathbf{c})}$ are continuous on each open interval $E_l^{(n)}$ for $l = 1, \cdots, n$. Now, we define

$$a_{lj}^{(n)} = \begin{cases} \inf\limits_{t \in \bar{E}_l^{(n)}} \left[a_j^{(0)}(t) - \widehat{a}_j(t) \right] & \text{for } j \in I^{(\mathbf{a})} \\ \inf\limits_{t \in \bar{E}_l^{(n)}} a_j^{(0)}(t) & \text{for } j \notin I^{(\mathbf{a})} \end{cases}$$

and

$$c_{li}^{(n)} = \begin{cases} \inf\limits_{t \in \bar{E}_l^{(n)}} \left[c_i^{(0)}(t) - \widehat{c}_i(t) \right] & \text{for } i \in I^{(\mathbf{c})} \\ \inf\limits_{t \in \bar{E}_l^{(n)}} c_i^{(0)}(t) & \text{for } i \notin I^{(\mathbf{c})} \end{cases}$$

and the vectors

$$\mathbf{a}_l^{(n)} = \left(a_{l1}^{(n)}, a_{l2}^{(n)}, \cdots, a_{lq}^{(n)} \right)^\top \in \mathbb{R}^q \text{ and } \mathbf{c}_l^{(n)} = \left(c_{l1}^{(n)}, c_{l2}^{(n)}, \cdots, c_{lp}^{(n)} \right)^\top \in \mathbb{R}^p.$$

Then,

$$a_{lj}^{(n)} \leq \left\{ \begin{array}{ll} a_j^{(0)}(t) - \widehat{a}_j(t) & \text{for } j \in I^{(\mathbf{a})} \\ a_j^{(0)}(t) & \text{for } j \notin I^{(\mathbf{a})} \end{array} \right\} \text{ and } c_{li}^{(n)} \leq \left\{ \begin{array}{ll} c_i^{(0)}(t) - \widehat{c}_i(t) & \text{for } i \in I^{(\mathbf{c})} \\ c_i^{(0)}(t) & \text{for } i \notin I^{(\mathbf{c})} \end{array} \right\} \tag{9}$$

for all $t \in \bar{E}_l^{(n)}$ and $l = 1, \cdots, n$.

For the time-dependent matrices $B(t)$ and $K(t,s)$, the (i,j)th entries of constant matrices $B_l^{(n)}$ and $K_{lk}^{(n)}$ are defined and denoted by

$$B_{lij}^{(n)} = \begin{cases} \sup\limits_{t \in \bar{E}_l^{(n)}} \left[B_{ij}^{(0)}(t) + \widehat{B}_{ij}(t) \right] & \text{for } j \in I_i^{(B)} \\ \sup\limits_{t \in \bar{E}_l^{(n)}} B_{ij}^{(0)}(t) & \text{for } j \notin I_i^{(B)} \end{cases} \tag{10}$$

and

$$K_{lkij}^{(n)} = \begin{cases} \inf\limits_{(t,s) \in \bar{E}_l^{(n)} \times \bar{E}_k^{(n)}} \left[K_{ij}^{(0)}(t,s) - \widehat{K}_{ij}(t,s) \right] & \text{for } j \in I_i^{(K)} \\ \inf\limits_{(t,s) \in \bar{E}_l^{(n)} \times \bar{E}_k^{(n)}} K_{ij}^{(0)}(t,s) & \text{for } j \notin I_i^{(K)}. \end{cases} \tag{11}$$

Then,

$$B_{lij}^{(n)} \geq \begin{cases} B_{ij}^{(0)}(t) + \widehat{B}_{ij}(t) & \text{for } j \in I_i^{(B)} \\ B_{ij}^{(0)}(t) & \text{for } j \notin I_i^{(B)} \end{cases} \tag{12}$$

for all $t \in \bar{E}_l^{(n)}$ and $l = 1, \cdots, n$, and

$$K_{lkij}^{(n)} \geq \begin{cases} K_{ij}^{(0)}(t,s) - \widehat{K}_{ij}(t,s) & \text{for } j \in I_i^{(K)} \\ K_{ij}^{(0)}(t,s) & \text{for } j \notin I_i^{(K)} \end{cases} \tag{13}$$

for all $(t,s) \in \bar{E}_l^{(n)} \times \bar{E}_k^{(n)}$ and $l, k = 1, \cdots, n$.

Remark 1. *From (7), it follows that, if $B_{lij}^{(n)} \neq 0$, then $B_{lij}^{(n)} \geq \sigma > 0$ for all $i = 1, \cdots, p$, $j = 1, \cdots, q$ and $l = 1, \cdots, n$. Given any fixed $t \in \bar{E}_l^{(n)}$, from (6), for any $j = 1, \cdots, q$, there exists $i_j \in \{1, \cdots, p\}$ such that*

$$\begin{cases} B_{i_j j}^{(0)}(t) + \widehat{B}_{i_j j}(t) > 0 & \text{if } j \in I_{i_j}^{(B)} \\ B_{i_j j}^{(0)}(t) > 0 & \text{if } j \notin I_{i_j}^{(B)} \end{cases}$$

which suggests that $B^{(n)}_{li_jj} \neq 0$, i.e., $B^{(n)}_{li_jj} \geq \sigma > 0$. In other words, for each j and l, there exists $i_{lj} \in \{1, 2, \cdots, p\}$ such that $B^{(n)}_{li_{lj}j} \geq \sigma > 0$.

Given any $n \in \mathbb{N}$ and $l = 1, \cdots, n$, we formulate the following linear programming problem:

$$
\begin{aligned}
(\mathrm{P}_n) \qquad \max \quad & \sum_{l=1}^{n} \sum_{j=1}^{q} \mathfrak{d}_l^{(n)} \cdot a_{lj}^{(n)} \cdot z_{lj} \\
\text{subject to} \quad & \sum_{j=1}^{q} B_{1ij}^{(n)} \cdot z_{1j} \leq c_{1i}^{(n)} \text{ for } i = 1, \cdots, p; \\
& \sum_{j=1}^{q} B_{lij}^{(n)} \cdot z_{lj} \leq c_{li}^{(n)} + \sum_{k=1}^{l-1} \sum_{j=1}^{q} \mathfrak{d}_k^{(n)} \cdot K_{lkij}^{(n)} \cdot z_{kj} \\
& \quad \text{for } l = 2, \cdots, n \text{ and } i = 1, \cdots, p; \\
& z_{lj} \geq 0 \text{ for } l = 1, \cdots, n \text{ and } j = 1, \cdots, q.
\end{aligned}
$$

According to the duality theory of linear programming, the dual problem of (P_n) is given by

$$
\begin{aligned}
(\widehat{\mathrm{D}}_n) \qquad \min \quad & \sum_{l=1}^{n} \sum_{i=1}^{p} c_{li}^{(n)} \cdot \widehat{w}_{li} \\
\text{subject to} \quad & \sum_{i=1}^{p} B_{lij}^{(n)} \cdot \widehat{w}_{li} \geq \mathfrak{d}_l^{(n)} a_{lj}^{(n)} + \mathfrak{d}_l^{(n)} \cdot \sum_{k=l+1}^{n} \sum_{i=1}^{p} K_{klij}^{(n)} \cdot \widehat{w}_{ki} \\
& \quad \text{for } l = 1, \cdots, n-1 \text{ and } j = 1, \cdots, q; \\
& \sum_{i=1}^{p} B_{lij}^{(n)} \cdot \widehat{w}_{ni} \geq \mathfrak{d}_n^{(n)} a_{nj}^{(n)} \text{ for } j = 1, \cdots, q; \\
& \widehat{w}_{li} \geq 0 \text{ for } l = 1, \cdots, n \text{ and } i = 1, \cdots, p.
\end{aligned}
$$

Now, let

$$
w_{li} = \frac{\widehat{w}_{li}}{\mathfrak{d}_l^{(n)}}.
$$

By dividing $\mathfrak{d}_l^{(n)}$ on both sides of the constraints, the dual problem $(\widehat{\mathrm{D}}_n)$ can be equivalently written by

$$
\begin{aligned}
(\mathrm{D}_n) \qquad \min \quad & \sum_{l=1}^{n} \sum_{i=1}^{p} \mathfrak{d}_l^{(n)} \cdot c_{li}^{(n)} \cdot w_{li} \\
\text{subject to} \quad & \sum_{i=1}^{p} B_{lij}^{(n)} \cdot w_{li} \geq a_{lj}^{(n)} + \sum_{k=l+1}^{n} \sum_{i=1}^{p} \mathfrak{d}_k^{(n)} \cdot K_{klij}^{(n)} \cdot w_{ki} \\
& \quad \text{for } l = 1, \cdots, n-1 \text{ and } j = 1, \cdots, q; \\
& \sum_{i=1}^{p} B_{nij}^{(n)} \cdot w_{ni} \geq a_{nj}^{(n)} \text{ for } j = 1, \cdots, q; \\
& w_{li} \geq 0 \text{ for } l = 1, \cdots, n \text{ and } i = 1, \cdots, p.
\end{aligned}
$$

Remark 2. *We have the following observations.*

- *If $\mathbf{c}_l^{(n)} \geq \mathbf{0}$ for all $l = 1, \cdots, n$, then the problem (P_n) is feasible, since $(\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n) = \mathbf{0}$ is a feasible solution of (P_n). If the vector-valued function $\mathbf{c}$ is nonnegative, then $\mathbf{c}_l^{(n)} \geq \mathbf{0}$ for all $l = 1, \cdots, n$, which suggests that the primal problem (P_n) is feasible.*

- *The dual problem (D_n) is always feasible for each $n \in \mathbb{N}$, which can be realized from part (i) of Proposition 1 given below.*

Recall that $\mathfrak{d}_l^{(n)}$ denotes the length of closed interval $\bar{E}_l^{(n)}$. We also define

$$\mathfrak{s}_l^{(n)} = \max_{k=l,\cdots,n} \mathfrak{d}_k^{(n)} \tag{14}$$

Then, we have

$$\mathfrak{s}_l^{(n)} = \max\left\{\mathfrak{d}_l^{(n)}, \mathfrak{d}_{l+1}^{(n)}, \cdots, \mathfrak{d}_n^{(n)}\right\} = \max\left\{\mathfrak{d}_l^{(n)}, \mathfrak{s}_{l+1}^{(n)}\right\}$$

which suggests that

$$\mathfrak{s}_l^{(n)} \geq \mathfrak{d}_l^{(n)} \text{ and } \parallel \mathcal{P}_n \parallel \geq \mathfrak{s}_l^{(n)} \geq \mathfrak{s}_{l+1}^{(n)} \text{ for } l = 1, \cdots, n-1. \tag{15}$$

For convenience, we adopt the following notations:

$$\bar{\tau}_l^{(n)} = \max_{j=1,\cdots,q} a_{lj}^{(n)} \text{ and } \tau_l^{(n)} = \max_{k=l,\cdots,n} \bar{\tau}_k^{(n)} \tag{16}$$

$$\bar{\sigma}_l^{(n)} = \min_{i=1,\cdots,p} \min_{j=1,\cdots,q} \left\{B_{lij}^{(n)} : B_{lij}^{(n)} > 0\right\} \text{ and } \sigma_l^{(n)} = \min_{k=l,\cdots,n} \bar{\sigma}_k^{(n)} \tag{17}$$

$$\bar{\nu}_l^{(n)} = \max_{k=1,\cdots,n} \max_{j=1,\cdots,q} \left\{\sum_{i=1}^{p} K_{klij}^{(n)}\right\} \text{ and } \nu_l^{(n)} = \max_{k=l,\cdots,n} \bar{\nu}_k^{(n)} \tag{18}$$

$$\bar{\phi}_l^{(n)} = \max_{k=1,\cdots,n} \max_{i=1,\cdots,p} \left\{\sum_{j=1}^{q} K_{klij}^{(n)}\right\} \text{ and } \phi_l^{(n)} = \max_{k=l,\cdots,n} \bar{\phi}_k^{(n)}$$

$$\tau = \max_{j=1,\cdots,q} \tau_j, \text{ where } \tau_j = \begin{cases} \sup_{t\in[0,T]} \left[a_j^{(0)}(t) - \widehat{a}_j(t)\right] & \text{for } j \in I^{(\mathbf{a})} \\ \sup_{t\in[0,T]} a_j^{(0)}(t) & \text{for } j \notin I^{(\mathbf{a})} \end{cases}$$

$$\zeta = \max_{i=1,\cdots,p} \zeta_i, \text{ where } \zeta_i = \begin{cases} \sup_{t\in[0,T]} \left[c_i^{(0)}(t) - \widehat{c}_i(t)\right] & \text{for } i \in I^{(\mathbf{c})} \\ \sup_{t\in[0,T]} c_i^{(0)}(t) & \text{for } i \notin I^{(\mathbf{c})} \end{cases}$$

$$\begin{aligned} \nu &= \max_{j=1,\cdots,q} \sup_{(t,s)\in[0,T]\times[0,T]} \left\{\sum_{\{i:j\in I_i^{(K)}\}} \left[K_{ij}^{(0)}(t,s) - \widehat{K}_{ij}(t,s)\right] + \sum_{\{i:j\notin I_i^{(K)}\}} K_{ij}^{(0)}(t,s)\right\} \\ &= \max_{j=1,\cdots,q} \sup_{(t,s)\in[0,T]\times[0,T]} \left\{\sum_{i=1}^{p} K_{ij}^{(0)}(t,s) - \sum_{\{i:j\in I_i^{(K)}\}} \widehat{K}_{ij}(t,s)\right\} \end{aligned} \tag{19}$$

$$\begin{aligned} \phi &= \max_{i=1,\cdots,p} \sup_{(t,s)\in[0,T]\times[0,T]} \left\{\sum_{\{i:j\in I_i^{(K)}\}} \left[K_{ij}^{(0)}(t,s) - \widehat{K}_{ij}(t,s)\right] + \sum_{\{i:j\notin I_i^{(K)}\}} K_{ij}^{(0)}(t,s)\right\} \\ &= \max_{i=1,\cdots,p} \sup_{(t,s)\in[0,T]\times[0,T]} \left\{\sum_{j=1}^{q} K_{ij}^{(0)}(t,s) - \sum_{\{j:j\in I_i^{(K)}\}} \widehat{K}_{ij}(t,s)\right\}. \end{aligned} \tag{20}$$

For each $l = 1, \cdots, n$, by Remark 1, since $B^{(n)}_{lij} \geq \sigma > 0$ for $B^{(n)}_{lij} \neq 0$ and there exists i_{lj} such that $B^{(n)}_{li_{lj}j} \geq \sigma$, it follows that $\sigma^{(n)}_l \geq \sigma$. We also have the following inequalities:

$$\sigma^{(n)}_l \leq \sigma^{(n)}_{l+1}, \quad \tau^{(n)}_l \geq \tau^{(n)}_{l+1} \text{ and } \nu^{(n)}_l \geq \nu^{(n)}_{l+1} \tag{21}$$

and

$$\bar{\tau}^{(n)}_l \leq \tau^{(n)}_l \leq \tau, \quad \bar{\nu}^{(n)}_l \leq \nu^{(n)}_l \leq \nu \text{ and } \bar{\sigma}^{(n)}_l \geq \sigma^{(n)}_l \geq \sigma > 0 \tag{22}$$

for any $n \in \mathbb{N}$.

Proposition 1. *The following statements hold true.*

(i) *Let*

$$\mathfrak{w}^{(n)}_l = \frac{\tau^{(n)}_l}{\sigma^{(n)}_l} \cdot \left(1 + \mathfrak{s}^{(n)}_l \cdot \frac{\nu^{(n)}_l}{\sigma^{(n)}_l}\right)^{n-l} \geq 0. \tag{23}$$

We write $\breve{w}^{(n)}_{li} = \mathfrak{w}^{(n)}_l$ *for* $i = 1, \cdots, p$ *and* $l = 1, \cdots, n$ *and consider the following vector:*

$$\breve{\mathbf{w}}^{(n)} = \left(\breve{\mathbf{w}}^{(n)}_1, \breve{\mathbf{w}}^{(n)}_2, \cdots, \breve{\mathbf{w}}^{(n)}_n\right)^\top \text{ with } \breve{\mathbf{w}}^{(n)}_l = \left(\breve{w}^{(n)}_{l1}, \breve{w}^{(n)}_{l2}, \cdots, \breve{w}^{(n)}_{lp}\right)^\top.$$

Then, $\breve{\mathbf{w}}^{(n)}$ *is a feasible solution of problem* (D_n) *satisfying*

$$\breve{w}^{(n)}_{li} \leq \frac{\tau}{\sigma} \cdot \exp\left(r \cdot T \cdot \frac{\nu}{\sigma}\right) \tag{24}$$

for all $n \in \mathbb{N}$, $i = 1, \cdots, p$ *and* $l = 1, \cdots, n$.

(ii) *Given any feasible solution* $\mathbf{w}^{(n)}$ *of problem* (D_n), *we define*

$$\bar{w}^{(n)}_{li} = \min\left\{w^{(n)}_{li}, \mathfrak{w}^{(n)}_l\right\}$$

for $i = 1, \cdots, p$ *and* $l = 1, \cdots, n$ *and consider the following vector:*

$$\bar{\mathbf{w}}^{(n)} = \left(\bar{\mathbf{w}}^{(n)}_1, \bar{\mathbf{w}}^{(n)}_2, \cdots, \bar{\mathbf{w}}^{(n)}_n\right)^\top \text{ with } \bar{\mathbf{w}}^{(n)}_l = \left(\bar{w}^{(n)}_{l1}, \bar{w}^{(n)}_{l2}, \cdots, \bar{w}^{(n)}_{lp}\right)^\top.$$

Then, $\bar{\mathbf{w}}^{(n)}$ *is a feasible solution of problem* (D_n) *satisfying the following inequalities:*

$$\bar{w}^{(n)}_{li} \leq \mathfrak{w}^{(n)}_l \leq \frac{\tau}{\sigma} \cdot \exp\left(r \cdot T \cdot \frac{\nu}{\sigma}\right)$$

for all $n \in \mathbb{N}$, $i = 1, \cdots, p$ *and* $l = 1, \cdots, n$. *Suppose that each* $\mathbf{c}^{(n)}_l$ *is nonnegative and that* $\mathbf{w}^{(n)}$ *is an optimal solution of problem* (D_n). *Then,* $\bar{\mathbf{w}}^{(n)}$ *is also an optimal solution of problem* (D_n).

Proof. For the proof, refer to Wu [5]. □

Proposition 2. *Suppose that the primal problem* (P_n) *is feasible with a feasible solution* $\mathbf{z}^{(n)} = (\mathbf{z}^{(n)}_1, \mathbf{z}^{(n)}_2, \cdots, \mathbf{z}^{(n)}_n)$, *where* $\mathbf{z}^{(n)}_l = (z^{(n)}_{l1}, z^{(n)}_{l2}, \cdots, z^{(n)}_{lq})^\top$ *for* $l = 1, \cdots, n$. *Then,*

$$0 \leq z^{(n)}_{lj} \leq \frac{\zeta}{\sigma} \cdot \left(1 + \parallel \mathcal{P}_n \parallel \cdot \frac{\phi}{\sigma}\right)^{l-1} \leq \frac{\zeta}{\sigma} \cdot \exp\left(r \cdot T \cdot \frac{\phi}{\sigma}\right) \tag{25}$$

for all $j = 1, \cdots, q$, $l = 1, \cdots, n$ *and* $n \in \mathbb{N}$.

Proof. For the proof, refer to Wu [5]. □

Let $\bar{\mathbf{z}}^{(n)} = (\bar{\mathbf{z}}_1^{(n)}, \bar{\mathbf{z}}_2^{(n)}, \cdots, \bar{\mathbf{z}}_n^{(n)})$ with $\bar{\mathbf{z}}_l^{(n)} = (\bar{z}_{l1}^{(n)}, \bar{z}_{l2}^{(n)}, \cdots, \bar{z}_{lq}^{(n)})^\top$ be an optimal solution of problem (P_n). We construct a vector-valued step function $\widehat{\mathbf{z}}^{(n)} : [0, T] \to \mathbb{R}^q$ as follows:

$$\widehat{\mathbf{z}}^{(n)}(t) = \begin{cases} \bar{\mathbf{z}}_l^{(n)} & \text{if } t \in F_l^{(n)} = \left[e_{l-1}^{(n)}, e_l^{(n)}\right) \text{ for } l = 1, \cdots, n \\ \bar{\mathbf{z}}_n^{(n)} & \text{if } t = T. \end{cases} \tag{26}$$

The following result will be useful for further discussions.

Proposition 3. *Suppose that $\bar{\mathbf{z}}^{(n)} = (\bar{\mathbf{z}}_1^{(n)}, \bar{\mathbf{z}}_2^{(n)}, \cdots, \bar{\mathbf{z}}_n^{(n)})$ is a feasible solution of primal problem* (P_n)*, where $\bar{\mathbf{z}}_l^{(n)} = (\bar{z}_{l1}^{(n)}, \bar{z}_{l2}^{(n)}, \cdots, \bar{z}_{lq}^{(n)})^\top$ for $l = 1, \cdots, n$. Then, the vector-valued step function $\widehat{\mathbf{z}}^{(n)}$ defined in* (26) *is a feasible solution of problem* (RCLP3).

Proof. The feasibility of $\bar{\mathbf{z}}^{(n)}$ suggests that

$$B_1^{(n)} \bar{\mathbf{z}}_1^{(n)} \le \mathbf{c}_1^{(n)} \text{ and } B_l^{(n)} \bar{\mathbf{z}}_l^{(n)} \le \mathbf{c}_l^{(n)} + \sum_{k=1}^{l-1} \mathfrak{d}_k^{(n)} K_{lk}^{(n)} \bar{\mathbf{z}}_k^{(n)} \text{ for } l = 2, \cdots, n.$$

Therefore, we obtain

$$\begin{aligned} &\sum_{j=1}^{q} B_{ij}^{(0)}(t) \cdot \bar{z}_{1j} + \sum_{\{j: j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot \bar{z}_{1j} \\ &\quad = \sum_{\{j: j \notin I_i^{(B)}\}} B_{ij}^{(0)}(t) \cdot \bar{z}_{1j} + \sum_{\{j: j \in I_i^{(B)}\}} \left[B_{ij}^{(0)}(t) + \widehat{B}_{ij}(t)\right] \cdot \bar{z}_{1j} \le \sum_{j=1}^{q} B_{1ij}^{(n)} \cdot \bar{z}_{1j} \le c_{1i}^{(n)} \end{aligned} \tag{27}$$

for $i = 1, \cdots, p$ and

$$\begin{aligned} &\sum_{j=1}^{q} B_{ij}^{(0)}(t) \cdot \bar{z}_{lj} + \sum_{\{j: j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot \bar{z}_{lj} \\ &\qquad - \sum_{k=1}^{l-1} \sum_{j=1}^{q} \mathfrak{d}_k^{(n)} \cdot K_{ij}^{(0)}(t,s) \cdot \bar{z}_{kj} + \sum_{k=1}^{l-1} \sum_{\{j: j \in I_i^{(K)}\}} \mathfrak{d}_k^{(n)} \cdot \widehat{K}_{ij}(t,s) \cdot \bar{z}_{kj} \\ &\quad = \sum_{\{j: j \notin I_i^{(B)}\}} B_{ij}^{(0)}(t) \cdot \bar{z}_{lj} + \sum_{\{j: j \in I_i^{(B)}\}} \left[B_{ij}^{(0)}(t) + \widehat{B}_{ij}(t)\right] \cdot \bar{z}_{lj} \\ &\qquad - \sum_{k=1}^{l-1} \mathfrak{d}_k^{(n)} \cdot \left\{ \sum_{\{j: j \notin I_i^{(K)}\}} K_{ij}^{(0)}(t,s) + \sum_{\{j: j \in I_i^{(K)}\}} \left[K_{ij}^{(0)}(t,s) - \widehat{K}_{ij}(t,s)\right] \right\} \cdot \bar{z}_{kj} \\ &\quad \le \sum_{j=1}^{q} B_{lij}^{(n)} \cdot \bar{z}_{lj} - \sum_{k=1}^{l-1} \mathfrak{d}_k^{(n)} \cdot K_{lkij}^{(n)} \cdot \bar{z}_{kj} \le c_{li}^{(n)} \end{aligned} \tag{28}$$

for $l = 2, \cdots, n$ and $i = 1, \cdots, p$. Two cases will be considered below.

- Suppose that $t \in F_l^{(n)}$ for $l = 2, \cdots, n$. Since $e_{l-1}^{(n)}$ is the left-end point of closed interval $\bar{E}_l^{(n)}$, we have

$$\begin{aligned}
&\sum_{j=1}^{q} B_{ij}^{(0)}(t) \cdot \widehat{z}_j^{(n)}(t) + \sum_{\{j:j\in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot \widehat{z}_j^{(n)}(t) \\
&\quad - \sum_{j=1}^{q} \int_0^t K_{ij}^{(0)}(t,s) \cdot \widehat{z}_j^{(n)}(s)ds + \sum_{\{j:j\in I_i^{(K)}\}} \int_0^t \widehat{K}_{ij}(t,s) \cdot \widehat{z}_j^{(n)}(s)ds \\
&= \sum_{j=1}^{q} B_{ij}^{(0)}(t) \cdot \widehat{z}_j^{(n)}(t) + \sum_{\{j:j\in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot \widehat{z}_j^{(n)}(t) \\
&\quad - \sum_{j=1}^{q} \sum_{k=1}^{l-1} \int_{\bar{E}_k^{(n)}} K_{ij}^{(0)}(t,s) \cdot \widehat{z}_j^{(n)}(s)ds - \sum_{j=1}^{q} \int_{e_{l-1}^{(n)}}^t K_{ij}^{(0)}(t,s) \cdot \widehat{z}_j^{(n)}(s)ds \\
&\quad + \sum_{\{j:j\in I_i^{(K)}\}} \sum_{k=1}^{l-1} \int_{\bar{E}_k^{(n)}} \widehat{K}_{ij}(t,s) \cdot \widehat{z}_j^{(n)}(s)ds + \sum_{\{j:j\in I_i^{(K)}\}} \int_{e_{l-1}^{(n)}}^t \widehat{K}_{ij}(t,s) \cdot \widehat{z}_j^{(n)}(s)ds \\
&\leq \sum_{j=1}^{q} B_{ij}^{(0)}(t) \cdot \widehat{z}_j^{(n)}(t) + \sum_{\{j:j\in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot \widehat{z}_j^{(n)}(t) \\
&\quad - \sum_{j=1}^{q} \sum_{k=1}^{l-1} \int_{\bar{E}_k^{(n)}} K_{ij}^{(0)}(t,s) \cdot \widehat{z}_j^{(n)}(s)ds + \sum_{\{j:j\in I_i^{(K)}\}} \sum_{k=1}^{l-1} \int_{\bar{E}_k^{(n)}} \widehat{K}_{ij}(t,s) \cdot \widehat{z}_j^{(n)}(s)ds \\
&\quad \text{(since } K_{ij}^{(0)}(t,s) \geq 0 \text{ for all } i \text{ and } j\text{, and } K_{ij}^{(0)}(t,s) - \widehat{K}_{ij}(t,s) \geq 0 \text{ for } j \in I_i^{(K)}) \\
&= \sum_{j=1}^{q} B_{ij}^{(0)}(t) \cdot \bar{z}_{lj} + \sum_{\{j:j\in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot \bar{z}_{lj} \\
&\quad - \sum_{k=1}^{l-1} \sum_{j=1}^{q} \mathfrak{d}_k^{(n)} \cdot K_{ij}^{(0)}(t,s) \cdot \bar{z}_{kj} + \sum_{k=1}^{l-1} \sum_{\{j:j\in I_i^{(K)}\}} \mathfrak{d}_k^{(n)} \cdot \widehat{K}_{ij}(t,s) \cdot \bar{z}_{kj} \\
&\leq c_{li}^{(n)} \text{ (by (28))} \\
&\leq \left\{ \begin{array}{ll} c_i^{(0)}(t) - \widehat{c}_i(t) & \text{for } i \in I^{(\mathbf{c})} \\ c_i^{(0)}(t) & \text{for } i \notin I^{(\mathbf{c})} \end{array} \right\} \text{ (by (9))}
\end{aligned}$$

For $l = 1$, using (27), we can similarly obtain the desired inequality.

- Suppose that $t = T$. Then, we have

$$\begin{aligned}
&\sum_{j=1}^{q} B_{ij}^{(0)}(T)\cdot\widehat{z}_j^{(n)}(T) + \sum_{\{j:j\in I_i^{(B)}\}} \widehat{B}_{ij}(T)\cdot\widehat{z}_j^{(n)}(T) \\
&\qquad - \sum_{j=1}^{q}\int_0^T K_{ij}^{(0)}(T,s)\cdot\widehat{z}_j^{(n)}(s)ds + \sum_{\{j:j\in I_i^{(K)}\}}\int_0^T \widehat{K}_{ij}(T,s)\cdot\widehat{z}_j^{(n)}(s)ds \\
&\quad = \sum_{j=1}^{q} B_{ij}^{(0)}(T)\cdot\widehat{z}_j^{(n)}(T) + \sum_{\{j:j\in I_i^{(B)}\}} \widehat{B}_{ij}(T)\cdot\widehat{z}_j^{(n)}(T) \\
&\qquad - \sum_{j=1}^{q}\sum_{k=1}^{n}\int_{\bar{E}_k^{(n)}} K_{ij}^{(0)}(T,s)\cdot\widehat{z}_j^{(n)}(s)ds + \sum_{\{j:j\in I_i^{(K)}\}}\sum_{k=1}^{n}\int_{\bar{E}_k^{(n)}} \widehat{K}_{ij}(T,s)\cdot\widehat{z}_j^{(n)}(s)ds \\
&\quad = \sum_{j=1}^{q} B_{ij}^{(0)}(T)\cdot\bar{z}_{nj} + \sum_{\{j:j\in I_i^{(B)}\}} \widehat{B}_{ij}(T)\cdot\bar{z}_{nj} \\
&\qquad - \sum_{j=1}^{q}\sum_{k=1}^{n}\mathfrak{d}_k^{(n)}\cdot K_{ij}^{(0)}(T,s)\cdot\bar{z}_{kj} + \sum_{\{j:j\in I_i^{(K)}\}}\sum_{k=1}^{n}\mathfrak{d}_k^{(n)}\cdot\widehat{K}_{ij}(T,s)\cdot\bar{z}_{kj} \\
&\quad \leq \sum_{j=1}^{q} B_{ij}^{(0)}(T)\cdot\bar{z}_{nj} + \sum_{\{j:j\in I_i^{(B)}\}} \widehat{B}_{ij}(T)\cdot\bar{z}_{nj} \\
&\qquad - \sum_{j=1}^{q}\sum_{k=1}^{n-1}\mathfrak{d}_k^{(n)}\cdot K_{ij}^{(0)}(T,s)\cdot\bar{z}_{kj} + \sum_{\{j:j\in I_i^{(K)}\}}\sum_{k=1}^{n-1}\mathfrak{d}_k^{(n)}\cdot\widehat{K}_{ij}(T,s)\cdot\bar{z}_{kj} \\
&\qquad \text{(since } K_{ij}^{(0)}(T,s)\geq 0 \text{ for all } i \text{ and } j\text{, and } K_{ij}^{(0)}(T,s)-\widehat{K}_{ij}(T,s)\geq 0 \text{ for } j\in I_i^{(K)}\text{)} \\
&\quad \leq c_{ni}^{(n)} \text{ (from (28) by taking } l = n\text{)} \\
&\quad \leq \left\{\begin{array}{ll} c_i^{(0)}(T)-\widehat{c}_i(T) & \text{for } i\in I^{(\mathbf{c})} \\ c_i^{(0)}(T) & \text{for } i\notin I^{(\mathbf{c})} \end{array}\right\} \text{ (by (9))}
\end{aligned}$$

Therefore, we conclude that $(\widehat{z}_1^{(n)},\cdots,\widehat{z}_q^{(n)})$ is a feasible solution of problem (RCLP3), and the proof is complete. □

4. Analytic Formula of the Error Bound

Given any optimization problem (P), we write $V(\mathrm{P})$ to denote the optimal objective value of problem (P). For example, the optimal objective value of problem (RCLP3) is denoted by $V(\mathrm{RCLP3})$.

Suppose that $\bar{\mathbf{z}}^{(n)}$ is an optimal solution of problem (P_n). Then, using (9), we have

$$\int_0^T (\mathbf{a}(t))^{\top}\widehat{\mathbf{z}}^{(n)}(t)dt \geq \sum_{l=1}^{n}\int_{\bar{E}_l^{(n)}}\left(\mathbf{a}_l^{(n)}\right)^{\top}\bar{\mathbf{z}}_l^{(n)}dt = \sum_{l=1}^{n}\mathfrak{d}_l^{(n)}\cdot\left(\mathbf{a}_l^{(n)}\right)^{\top}\bar{\mathbf{z}}_l^{(n)} = V(\mathrm{P}_n). \tag{29}$$

Therefore, we have

$$\begin{aligned}
V(\mathrm{RCLP3}) &\geq \int_0^T (\mathbf{a}(t))^{\top}\widehat{\mathbf{z}}^{(n)}(t)dt \text{ (by Proposition 3)} \\
&\geq V(\mathrm{P}_n) \text{ (by (29))}.
\end{aligned}$$

According to the weak duality theorem for the primal-dual pair of problems (DRCLP3) and (RCLP3), we obtain

$$V(\mathrm{DRCLP3}) \geq V(\mathrm{RCLP3}) \geq V(\mathrm{P}_n) = V(\mathrm{D}_n). \tag{30}$$

In the sequel, we want to show that

$$\lim_{n\to\infty} V(\mathrm{D}_n) = V(\mathrm{DRCLP3}). \tag{31}$$

Let $\mathbf{w}^{(n)} = (\mathbf{w}_1^{(n)}, \mathbf{w}_2^{(n)}, \cdots, \mathbf{w}_n^{(n)})$ with $\mathbf{w}_l^{(n)} = (w_{l1}^{(n)}, w_{l2}^{(n)}, \cdots, w_{lp}^{(n)})^\top$ be an optimal solution of problem (D_n). We define

$$\bar{w}_{li}^{(n)} = \min\left\{w_{li}^{(n)}, \mathfrak{w}_l^{(n)}\right\},$$

where $\mathfrak{w}_l^{(n)}$ is defined in (23), for $i = 1, \cdots, p$ and $l = 1, \cdots, n$, and consider the following vector:

$$\bar{\mathbf{w}}^{(n)} = \left(\bar{\mathbf{w}}_1^{(n)}, \bar{\mathbf{w}}_2^{(n)}, \cdots, \bar{\mathbf{w}}_n^{(n)}\right)^\top \text{ with } \bar{\mathbf{w}}_l^{(n)} = \left(\bar{w}_{l1}^{(n)}, \bar{w}_{l2}^{(n)}, \cdots, \bar{w}_{lp}^{(n)}\right)^\top$$

Then, part (ii) of Proposition 1 states that $\bar{\mathbf{w}}^{(n)}$ is an optimal solution of problem (D_n) satisfying the following inequalities:

$$\bar{w}_{li}^{(n)} \leq \mathfrak{w}_l^{(n)} \leq \frac{\tau}{\sigma} \cdot \exp\left(r \cdot T \cdot \frac{\nu}{\sigma}\right) \tag{32}$$

for all $n \in \mathbb{N}$, $i = 1, \cdots, p$ and $l = 1, \cdots, n$.

For each $l = 1, \cdots, n$ and $j = 1, \cdots, q$, we define a real-valued function $\bar{h}_{lj}^{(n)}$ on the half-open interval $F_l^{(n)} = [e_{l-1}^{(n)}, e_l^{(n)})$ given by

$$\begin{aligned}
\bar{h}_{lj}^{(n)}(t) &= \left(e_l^{(n)} - t\right) \cdot \sum_{i=1}^{p} K_{llij}^{(n)} \cdot \bar{w}_{li}^{(n)} \\
&+ \sum_{i=1}^{p} B_{lij}^{(n)} \cdot \bar{w}_{li}^{(n)} - \left(\sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot \bar{w}_{li}^{(n)} + \sum_{\{i: j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot \bar{w}_{li}^{(n)}\right) \\
&+ \int_t^{e_l^{(n)}} \left[\left(\sum_{i=1}^{p} K_{ij}^{(0)}(s,t) \cdot \bar{w}_{li}^{(n)} - \sum_{\{i: j \in I_i^{(K)}\}} \widehat{K}_{ij}(s,t) \cdot \bar{w}_{li}^{(n)}\right) - \sum_{i=1}^{p} K_{llij}^{(n)} \cdot \bar{w}_{li}^{(n)}\right] ds \\
&+ \sum_{k=l+1}^{n} \int_{\bar{E}_k^{(n)}} \left[\left(\sum_{i=1}^{p} K_{ij}^{(0)}(s,t) \cdot \bar{w}_{ki}^{(n)} - \sum_{\{i: j \in I_i^{(K)}\}} \widehat{K}_{ij}(s,t) \cdot \bar{w}_{ki}^{(n)}\right) - \sum_{i=1}^{p} K_{klij}^{(n)} \cdot \bar{w}_{ki}^{(n)}\right] ds.
\end{aligned}$$

For each $j = 1, \cdots, q$, we also define a real number

$$r_j^{(n)} = \sum_{i=1}^{p} B_{nij}^{(n)} \cdot \bar{w}_{ni}^{(n)} - \left(\sum_{i=1}^{p} B_{ij}^{(0)}(T) \cdot \bar{w}_{ni}^{(n)} + \sum_{\{i: j \in I_i^{(B)}\}} \widehat{B}_{ij}(T) \cdot \bar{w}_{ni}^{(n)}\right). \tag{33}$$

For $l = 1, \cdots, n$, let

$$\bar{\pi}_l^{(n)} = \max\left\{\max_{j \notin I^{(\mathbf{a})}} \sup_{t \in E_l^{(n)}} \left[\bar{h}_{lj}^{(n)}(t) + a_j^{(0)}(t) - a_{lj}^{(n)}\right], \max_{j \in I^{(\mathbf{a})}} \sup_{t \in E_l^{(n)}} \left[\bar{h}_{lj}^{(n)}(t) + a_j^{(0)}(t) - \widehat{a}_j(t) - a_{lj}^{(n)}\right]\right\} \tag{34}$$

and

$$\pi_l^{(n)} = \max_{k=l,\cdots,n} \bar{\pi}_k^{(n)}.$$

Then, we have

$$\pi_l^{(n)} = \max\left\{\bar{\pi}_l^{(n)}, \bar{\pi}_{l+1}^{(n)}, \cdots, \bar{\pi}_n^{(n)}\right\} = \max\left\{\bar{\pi}_l^{(n)}, \pi_{l+1}^{(n)}\right\} \tag{35}$$

which suggests that

$$\pi_l^{(n)} \geq \pi_{l+1}^{(n)} \tag{36}$$

and, for any $t \in E_l^{(n)}$,

$$\pi_l^{(n)} \geq \bar{\pi}_l^{(n)} \geq \left\{ \begin{array}{ll} \bar{h}_{lj}^{(n)}(t) + a_j^{(0)}(t) - a_{lj}^{(n)} & \text{for } j \notin I^{(\mathbf{a})} \\ \bar{h}_{lj}^{(n)}(t) + a_j^{(0)}(t) - \widehat{a}_j - a_{lj}^{(n)} & \text{for } j \in I^{(\mathbf{a})} \end{array} \right\} \tag{37}$$

for $l = 1, \cdots, n-1$. We want to prove

$$\lim_{n\to\infty} \bar{\pi}_l^{(n)} = 0 = \lim_{n\to\infty} \pi_l^{(n)}.$$

We need some useful lemmas.

Lemma 1. *For $i = 1, \cdots, p$; $j = 1, \cdots, q$; and $l = 1, \cdots, n$, we have*

$$\left\{ \begin{array}{ll} \sup\limits_{t\in E_l^{(n)}} \left[a_j^{(0)}(t) - a_{lj}^{(n)}\right] \to 0 & \textit{for } j \notin I^{(\mathbf{a})} \\ \sup\limits_{t\in E_l^{(n)}} \left[a_j^{(0)}(t) - \widehat{a}_j(t) - a_{lj}^{(n)}\right] \to 0 & \textit{for } j \in I^{(\mathbf{a})} \end{array} \right\} \textit{as } n \to \infty$$

and

$$\left\{ \begin{array}{ll} \sup\limits_{t\in E_l^{(n)}} \left[B_{lij}^{(n)} - B_{ij}^{(0)}(t)\right] \to 0 & \textit{for } j \notin I_i^{(B)} \\ \sup\limits_{t\in E_l^{(n)}} \left[B_{lij}^{(n)} - B_{ij}^{(0)}(t) - \widehat{B}_{ij}(t)\right] \to 0 & \textit{for } j \in I_i^{(B)} \end{array} \right\} \textit{as } n \to \infty.$$

Proof. According to the construction of partition $\mathcal{P}_n$, it is clear that $a_j^{(0)}$ for $j \notin I^{(\mathbf{a})}$ and $a_j^{(0)} - \widehat{a}_j$ for $j \in I^{(\mathbf{a})}$ are continuous on the open interval $E_l^{(n)} = (e_{l-1}^{(n)}, e_l^{(n)})$. Let

$$a_j = \left\{ \begin{array}{ll} a_j^{(0)} & \text{for } j \notin I^{(\mathbf{a})} \\ a_j^{(0)} - \widehat{a}_j & \text{for } j \in I^{(\mathbf{a})}. \end{array} \right.$$

Then, we also see that the function a_j is continuous on the open interval $E_l^{(n)} = (e_{l-1}^{(n)}, e_l^{(n)})$ and

$$a_{lj}^{(n)} = \inf_{t\in \bar{E}_l^{(n)}} a_j.$$

Given a decreasing sequence $\{\delta_m\}_{m=1}^{\infty}$ satisfying $\delta_m > 0$ for all m and $\delta_m \to 0$ as $n \to$, where δ_1 is defined by

$$\delta_1 = \frac{1}{2} \cdot \left(e_l^{(n)} - e_{l-1}^{(n)}\right),$$

we can define the closed interval

$$E_{lm}^{(n)} = \left[e_{l-1}^{(n)} + \delta_m, e_l^{(n)} - \delta_m\right],$$

which implies

$$E_l^{(n)} = \bigcup_{m=1}^{\infty} E_{lm}^{(n)} \text{ and } E_{lm_1}^{(n)} \subseteq E_{lm_2}^{(n)} \text{ for } m_2 > m_1. \tag{38}$$

Since $E_{lm}^{(n)} \subset E_l^{(n)}$, we see that a_j is uniformly continuous on each closed interval $E_{lm}^{(n)}$. Therefore, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies

$$\left|a_j(t_1) - a_j(t_2)\right| < \frac{\epsilon}{2} \text{ for any } t_1, t_2 \in E_{lm}^{(n)}. \tag{39}$$

Since the length of $E_l^{(n)}$ is less than or equal to $\| \mathcal{P}_n \| \leq T/n^*$ with $n^* \to \infty$ by (8), we can consider a sufficiently large $n_0 \in \mathbb{N}$ satisfying $T/n_0 < \delta$. In this case, each length of $E_l^{(n)}$ for $l = 1, \cdots, n$ is less than δ for $n \geq n_0$, which also suggests that, if $n \geq n_0$, then (39) is satisfied for any $t_1, t_2 \in E_{lm}^{(n)}$. We consider the following cases.

- Suppose that the infimum $a_{lj}^{(n)}$ is attained at $t^{(n*)} \in E_l^{(n)}$. Using (38), there exists m^* satisfying $t^{(n*)} \in E_{lm^*}^{(n)}$. Given any $t \in E_l^{(n)}$, we see that $t \in E_{lm_0}^{(n)}$ for some m_0. Let $m = \max\{m_0, m^*\}$. Using (38), it follows that $t, t^{(n*)} \in E_{lm}^{(n)}$. Therefore, we obtain

$$\left|a_j(t) - a_{lj}^{(n)}\right| = \left|a_j(t) - a_j\left(t^{(n*)}\right)\right| < \frac{\epsilon}{2}$$

  since the length of $E_{lm}^{(n)}$ is less than δ, where ϵ is independent of t because of the uniform continuity.
- Suppose that the infimum $a_{lj}^{(n)}$ is not attained at any point in $E_l^{(n)}$. The continuity of a_j on the open interval $E_l^{(n)}$ suggests that that the infimum $a_{lj}^{(n)}$ is either the right-hand limit or left-hand limit given by

$$a_{lj}^{(n)} = \lim_{t \to e_{l-1}^{(n)}+} a_j(t) \text{ or } a_{lj}^{(n)} = \lim_{t \to e_l^{(n)}-} a_j(t).$$

  Therefore, for sufficiently large n, i.e., the open interval $E_l^{(n)}$ is sufficiently small such that its length is less than δ, we can obtain

$$\left|a_j(t) - a_{lj}^{(n)}\right| < \frac{\epsilon}{2}$$

  for all $t \in E_l^{(n)}$.

From the above two cases, since $a_j(t) \geq a_{lj}^{(n)}$ for all $t \in E_l^{(n)}$, we conclude that

$$0 \leq \sup_{t \in E_l^{(n)}} \left[a_j(t) - a_{lj}^{(n)}\right] \leq \frac{\epsilon}{2} < \epsilon \text{ for } l = 1, \cdots, n.$$

This completes the proof. □

Lemma 2. *For $i = 1, \cdots, p$; $j = 1, \cdots, q$; and $l, k = 1, \cdots, n$, we have*

$$\left.\begin{cases} \sup\limits_{(s,t) \in E_k^{(n)} \times E_l^{(n)}} \left[K_{ij}^{(0)}(s,t) - K_{klij}^{(n)}\right] \to 0 & \text{for } j \notin I_i^{(K)} \\ \sup\limits_{(s,t) \in E_k^{(n)} \times E_l^{(n)}} \left[K_{ij}^{(0)}(s,t) - \widehat{K}_{ij}(s,t) - K_{klij}^{(n)}\right] \to 0 & \text{for } j \in I_i^{(K)} \end{cases}\right\} \text{as } n \to \infty$$

and

$$\left.\begin{cases} \sup\limits_{t\in E_l^{(n)}}\left[\int_{\bar{E}_k^{(n)}}\left(K_{ij}^{(0)}(s,t)-K_{klij}^{(n)}\right)\cdot\bar{w}_{ki}^{(n)}ds\right]\to 0 & \textit{for } j\notin I_i^{(K)} \\ \sup\limits_{t\in E_l^{(n)}}\left[\int_{\bar{E}_k^{(n)}}\left(K_{ij}^{(0)}(s,t)-\widehat{K}_{ij}(s,t)-K_{klij}^{(n)}\right)\cdot\bar{w}_{ki}^{(n)}ds\right]\to 0 & \textit{for } j\in I_i^{(K)} \end{cases}\right\}\ \textit{as } n\to\infty.$$

Proof. Let

$$K_{ij}=\begin{cases} K_{ij}^{(0)} & \text{for } j\notin I_i^{(K)} \\ K_{ij}^{(0)}-\widehat{K}_{ij} & \text{for } j\in I_i^{(K)}. \end{cases}$$

Then,

$$K_{klij}^{(n)}=\inf_{(s,t)\in\bar{E}_k^{(n)}\times\bar{E}_l^{(n)}}K_{ij}(s,t).$$

According to the construction of partition $\mathcal{P}_n$, we also see that K_{ij} is continuous on the open rectangle

$$E_k^{(n)}\times E_l^{(n)}=\left(e_{k-1}^{(n)},e_k^{(n)}\right)\times\left(e_{l-1}^{(n)},e_l^{(n)}\right).$$

Let $\{\delta_m\}_{m=1}^{\infty}$ be a decreasing sequence and be convergent to zero such that $\delta_m>0$ for all m, where δ_1 is defined by

$$\delta_1=\frac{1}{2}\cdot\min\left\{\left(e_k^{(n)}-e_{k-1}^{(n)}\right),\left(e_l^{(n)}-e_{l-1}^{(n)}\right)\right\}.$$

Therefore, we can define the compact rectangle

$$E_{km}^{(n)}\times E_{lm}^{(n)}=\left[e_{k-1}^{(n)}+\delta_m,e_k^{(n)}-\delta_m\right]\times\left[e_{l-1}^{(n)}+\delta_m,e_l^{(n)}-\delta_m\right].$$

The following inclusion

$$\bigcup_{m=1}^{\infty}E_{km}^{(n)}\times E_{lm}^{(n)}\subseteq E_k^{(n)}\times E_l^{(n)}$$

is obvious. For $(s,t)\in E_k^{(n)}\times E_l^{(n)}$, there exist m_1 and m_2 such that $s\in E_{km_1}^{(n)}$ and $t\in E_{lm_2}^{(n)}$, respectively. Let $m=\max\{m_1,m_2\}$. Then, we have $E_{km_1}^{(n)}\subseteq E_{km}^{(n)}$ and $E_{lm_2}^{(n)}\subseteq E_{lm}^{(n)}$. Therefore, we obtain

$$(s,t)\in E_{km_1}^{(n)}\times E_{lm_2}^{(n)}\subseteq E_{km}^{(n)}\times E_{lm}^{(n)},$$

which proves

$$E_k^{(n)}\times E_l^{(n)}=\bigcup_{m=1}^{\infty}E_{km}^{(n)}\times E_{lm}^{(n)}. \tag{40}$$

We also see that

$$E_{km_1}^{(n)}\times E_{lm_1}^{(n)}\subseteq E_{km_2}^{(n)}\times E_{lm_2}^{(n)}\text{ for } m_2>m_1. \tag{41}$$

Since $E_{km}^{(n)}\times E_{lm}^{(n)}\subset E_k^{(n)}\times E_l^{(n)}$, it follows that K_{ij} is continuous on each compact rectangle $E_{km}^{(n)}\times E_{lm}^{(n)}$, which also means that K_{ij} is uniformly continuous on each compact rectangle $E_{km}^{(n)}\times E_{lm}^{(n)}$. Therefore, given any $\epsilon>0$, there exists $\delta>0$ such that

$$|t_1-t_2|<\delta\text{ and }|s_1-s_2|<\delta$$

implies

$$\left|K_{ij}(s_1,t_1) - K_{ij}(s_2,t_2)\right| < \frac{\epsilon}{2} \tag{42}$$

for $(s_1,t_1),(s_2,t_2) \in E_{km}^{(n)} \times E_{lm}^{(n)}$. Since the length of $E_k^{(n)}$ is less than or equal to $\parallel \mathcal{P}_n \parallel \leq T/n^*$ with $n^* \to \infty$ using (8), we can consider a sufficiently large $n_0 \in \mathbb{N}$ satisfying $T/n_0 < \delta$. In this case, each length of $E_k^{(n)}$ for $k = 1,\cdots,n$ is less than δ, which means that, if $n \geq n_0$, then (42) is satisfied for any $(s_1,t_1),(s_2,t_2) \in E_{km}^{(n)} \times E_{lm}^{(n)}$. We consider the following cases.

- Suppose that the infimum $K_{klij}^{(n)}$ is attained at $(s^{(n*)}, t^{(n*)}) \in E_k^{(n)} \times E_l^{(n)}$. Using (40), there exists m^* satisfying $(s^{(n*)}, t^{(n*)}) \in E_{km^*}^{(n)} \times E_{lm^*}^{(n)}$. Given any $(s,t) \in E_k^{(n)} \times E_l^{(n)}$, we see that $(s,t) \in E_{km_0}^{(n)} \times E_{lm_0}^{(n)}$ for some m_0. Let $m = \max\{m^*, m_0\}$. Using (41), it follows that $(s,t),(s^{(n*)}, t^{(n*)}) \in E_{km}^{(n)} \times E_{lm}^{(n)}$. Then, we have

$$\left|K_{ij}(s,t) - K_{klij}^{(n)}\right| = \left|K_{ij}(s,t) - K_{ij}\left(s^{(n*)}, t^{(n*)}\right)\right| < \frac{\epsilon}{2},$$

  since the lengths of $E_{km}^{(n)}$ and $E_{lm}^{(n)}$ are less than δ, where ϵ is independent from (s,t) in $E_k^{(n)} \times E_l^{(n)}$ because of the uniform continuity.
- Suppose that the infimum $K_{klij}^{(n)}$ is not attained at any point in $E_k^{(n)} \times E_l^{(n)}$. Let

$$\mathcal{K}_{ij} = \left\{K_{ij}(s,t) : (s,t) \in E_k^{(n)} \times E_l^{(n)}\right\}.$$

  Since K_{ij} is continuous on the open rectangle $E_k^{(n)} \times E_l^{(n)}$, it follows that the infimum $K_{klij}^{(n)}$ is in the boundary of the closure of $\mathcal{K}_{ij}$ and is the limit of the function K_{ij} on $E_k^{(n)} \times E_l^{(n)}$. Therefore, for sufficiently large n, i.e., the open rectangle $E_k^{(n)} \times E_l^{(n)}$ is sufficiently small such that the lengths of $E_k^{(n)}$ and $E_l^{(n)}$ are less than δ, we have

$$\left|K_{ij}(s,t) - K_{klij}^{(n)}\right| < \frac{\epsilon}{2}$$

  for all $(s,t) \in E_k^{(n)} \times E_l^{(n)}$.

  From the above two cases, we conclude that

$$0 \leq \sup_{(s,t)\in E_k^{(n)} \times E_l^{(n)}} \left[K_{ij}(s,t) - K_{klij}^{(n)}\right] \leq \frac{\epsilon}{2} < \epsilon$$

and

$$\begin{aligned} 0 &\leq \sup_{t\in E_l^{(n)}} \left[\int_{\bar{E}_k^{(n)}} \left(K_{ij}(s,t) - K_{klij}^{(n)}\right) \cdot \varpi_{ki}^{(n)} ds\right] \\ &\leq \frac{\epsilon}{2} \cdot \eth_k^{(n)} \cdot \varpi_{ki}^{(n)} < \epsilon \cdot T \cdot \frac{\tau}{\sigma} \cdot \exp\left(r \cdot T \cdot \frac{\nu}{\sigma}\right) \text{ (using (32))}, \end{aligned}$$

which implies

$$\sup_{t\in E_l^{(n)}} \left[\int_{\bar{E}_k^{(n)}} \left(K_{ij}(s,t) - K_{klij}^{(n)}\right) \cdot \varpi_{ki}^{(n)} ds\right] \to 0 \text{ as } n \to \infty.$$

This completes the proof. □

Lemma 3. *For each $l = 1, \cdots, n$, we have*

$$\lim_{n\to\infty} \bar{\pi}_l^{(n)} = 0 = \lim_{n\to\infty} \pi_l^{(n)}.$$

Proof. It suffices to prove

$$\left.\begin{cases} \sup\limits_{t\in E_l^{(n)}} \left[\bar{h}_{lj}^{(n)}(t) + a_j^{(0)}(t) - a_{lj}^{(n)}\right] \to 0 & \text{for } j \notin I^{(\mathbf{a})} \\ \sup\limits_{t\in E_l^{(n)}} \left[\bar{h}_{lj}^{(n)}(t) + a_j^{(0)}(t) - \widehat{a}_j(t) - a_{lj}^{(n)}\right] \to 0 & \text{for } j \in I^{(\mathbf{a})} \end{cases}\right\} \text{as } n \to \infty.$$

From (8), since

$$\mathfrak{d}_l^{(n)} \leq \parallel \mathcal{P}_n \parallel \leq \frac{r \cdot T}{n} \to 0 \text{ as } n \to \infty$$

and $\varpi_{li}^{(n)}$ is bounded according to (32), it follows that

$$\left(e_l^{(n)} - t\right) \cdot K_{llij}^{(n)} \cdot \varpi_{li}^{(n)} \leq \mathfrak{d}_l^{(n)} \cdot K_{llij}^{(n)} \cdot \varpi_{li}^{(n)} \to 0 \text{ as } n \to \infty.$$

Using Lemmas 1 and 2, we have

$$\begin{aligned} \sup_{t\in E_l^{(n)}} \bar{h}_{lj}^{(n)}(t) \leq{}& \mathfrak{d}_l^{(n)} \cdot \sum_{i=1}^{p} K_{llij}^{(n)} \cdot \varpi_{li}^{(n)} + \sum_{\{i:j\notin I_i^{(B)}\}} \varpi_{li}^{(n)} \cdot \sup_{t\in E_l^{(n)}} \left[B_{lij}^{(n)} - B_{ij}^{(0)}(t)\right] \\ &+ \sum_{\{i:j\in I_i^{(B)}\}} \varpi_{li}^{(n)} \cdot \sup_{t\in E_l^{(n)}} \left[B_{lij}^{(n)} - B_{ij}^{(0)}(t) - \widehat{B}_{ij}(t)\right] \\ &+ \sum_{k=l}^{n} \sum_{\{i:j\notin I_i^{(K)}\}} \sup_{t\in E_l^{(n)}} \left[\int_{\bar{E}_k^{(n)}} \left(K_{ij}^{(0)}(s,t) - K_{klij}^{(n)}\right) \cdot \varpi_{ki}^{(n)} ds\right] \\ &+ \sum_{k=l}^{n} \sum_{\{i:j\in I_i^{(K)}\}} \sup_{t\in E_l^{(n)}} \left[\int_{\bar{E}_k^{(n)}} \left(K_{ij}^{(0)}(s,t) - \widehat{K}_{ij}(s,t) - K_{klij}^{(n)}\right) \cdot \varpi_{ki}^{(n)} ds\right] \to 0 \text{ as } n \to \infty. \end{aligned}$$

Now, for $j \notin I^{(\mathbf{a})}$, we have

$$0 \leq \sup_{t\in E_l^{(n)}} \left[\bar{h}_{lj}^{(n)}(t) + a_j^{(0)}(t) - a_{lj}^{(n)}\right] \leq \sup_{t\in E_l^{(n)}} \bar{h}_{lj}^{(n)}(t) + \sup_{t\in E_l^{(n)}} \left[a_j^{(0)}(t) - a_{lj}^{(n)}\right]$$

and, for $j \in I^{(\mathbf{a})}$, we have

$$0 \leq \sup_{t\in E_l^{(n)}} \left[\bar{h}_{lj}^{(n)}(t) + a_j^{(0)}(t) - \widehat{a}_j(t) - a_{lj}^{(n)}\right] \leq \sup_{t\in E_l^{(n)}} \bar{h}_{lj}^{(n)}(t) + \sup_{t\in E_l^{(n)}} \left[a_j^{(0)}(t) - \widehat{a}_j(t) - a_{lj}^{(n)}\right].$$

Using Lemma 1, we complete the proof. □

We define the following notations:

$$\bar{\mathfrak{k}}_l^{(n)} = \max_{j=1,\cdots,q} \left\{ \sup_{(s,t)\in[e_{l-1}^{(n)},T]\times\bar{E}_l^{(n)}} \left[\sum_{i=1}^{p} K_{ij}^{(0)}(s,t) - \sum_{\{i:j\in I_i^{(K)}\}} \widehat{K}_{ij}^{(0)}(s,t)\right]\right\} \tag{43}$$

and

$$\bar{\mathfrak{b}}_l^{(n)} = \min_{j=1,\cdots,q} \left\{ \inf_{t\in\bar{E}_l^{(n)}} \left[\sum_{i=1}^{p} B_{ij}^{(0)}(t) + \sum_{\{i:j\in I_i^{(B)}\}} \widehat{B}_{ij}^{(0)}(t)\right]\right\}. \tag{44}$$

From (7), (6) and (19), we see that

$$\bar{\mathfrak{b}}_l^{(n)} \geq \min_{j=1,\cdots,q} \left\{ \inf_{t\in[0,T]} \left[\sum_{i=1}^{p} B_{ij}^{(0)}(t) + \sum_{\{i:j\in I_i^{(B)}\}} \widehat{B}_{ij}^{(0)}(t) \right] \right\} \geq \sigma > 0 \text{ and } \bar{\mathfrak{k}}_l^{(n)} \leq \nu.$$

Let

$$\mathfrak{k}_l^{(n)} = \max_{k=l,\cdots,n} \bar{\mathfrak{k}}_k^{(n)} \leq \nu \text{ and } \mathfrak{b}_l^{(n)} = \min_{k=l,\cdots,n} \bar{\mathfrak{b}}_k^{(n)} \geq \sigma. \tag{45}$$

Then, we see that $0 < \mathfrak{b}_l^{(n)}$ and

$$\mathfrak{k}_l^{(n)} \geq \mathfrak{k}_{l+1}^{(n)} \text{ and } \mathfrak{b}_l^{(n)} \leq \mathfrak{b}_{l+1}^{(n)}. \tag{46}$$

Now, we define the real-valued functions $\mathfrak{u}^{(n)}$ and $\mathfrak{v}^{(n)}$ on $[0,T]$ by

$$\mathfrak{u}^{(n)}(t) = \begin{cases} \mathfrak{k}_l^{(n)} & \text{if } t \in F_l^{(n)} \text{ for } l = 1,\cdots,n \\ \mathfrak{k}_n^{(n)} & \text{if } t = T \end{cases}$$

and

$$\mathfrak{v}^{(n)}(t) = \begin{cases} \mathfrak{b}_l^{(n)} & \text{if } t \in F_l^{(n)} \text{ for } l = 1,\cdots,n \\ \mathfrak{b}_n^{(n)} & \text{if } t = T. \end{cases}$$

Then, we have

$$\mathfrak{u}^{(n)}(t) \leq \nu \text{ and } \mathfrak{v}^{(n)}(t) \geq \sigma \text{ for all } t \in [0,T] \tag{47}$$

From (32) and Lemmas 1 and 2, we see that the sequence $\{\bar{h}_{lj}^{(n)}\}_{n=1}^{\infty}$ is uniformly bounded. In other words, the sequence $\{\pi_l^{(n)}\}_{n=1}^{\infty}$ is uniformly bounded. Therefore, there exists a constant $\mathfrak{x}$ satisfying $\pi_l^{(n)} \leq \mathfrak{x}$ for all $n \in \mathbb{N}$ and $l = 1,\cdots,n$. Now, we define a real-valued function $\mathfrak{p}^{(n)}$ on $[0,T]$ by

$$\mathfrak{p}^{(n)}(t) = \begin{cases} \mathfrak{x}, \text{ if } t = e_{l-1}^{(n)} \text{ for } l = 1,\cdots,n \\ \pi_l^{(n)}, \text{ if } t \in E_l^{(n)} \text{ for } l = 1,\cdots,n \\ \max\left\{ \max_{j\notin I^{(\mathbf{a})}} \left\{ r_j^{(n)} + a_j^{(0)}(T) - a_{nj}^{(n)} \right\}, \right. \\ \qquad \left. \max_{j\in I^{(\mathbf{a})}} \left\{ r_j^{(n)} + a_j^{(0)}(T) - \widehat{a}_j(T) - a_{nj}^{(n)} \right\} \right\}, \text{ if } t = e_n^{(n)} = T, \end{cases}$$

where $r_j^{(n)}$ is the jth component of $\mathbf{r}^{(n)}$ in (33). Then, we have

$$\mathfrak{p}^{(n)}(t) \leq \mathfrak{x} \text{ for all } n \in \mathbb{N} \text{ and } t \in [0,T). \tag{48}$$

Let $\mathbf{1}_p = (1,1,\cdots,1)^\top \in \mathbb{R}^p$ denote a p-dimensional vector such that each component of $\mathbf{1}_p$ is 1, and let the real-valued function $\mathfrak{f}^{(n)} : [0,T] \to \mathbb{R}_+$ be defined by

$$\mathfrak{f}^{(n)}(t) = \frac{\mathfrak{p}^{(n)}(t)}{\mathfrak{v}^{(n)}(t)} \cdot \exp\left[\frac{\mathfrak{u}^{(n)}(t) \cdot (T-t)}{\mathfrak{v}^{(n)}(t)} \right] \tag{49}$$

We need a useful lemma given below

Lemma 4. *We have*

$$\mathfrak{f}^{(n)}(T)\cdot\left(\sum_{i=1}^{p}B_{ij}^{(0)}(T)+\sum_{\{i:j\in I_i^{(B)}\}}\widehat{B}_{ij}(T)\right)\geq\left\{\begin{array}{ll} r_j^{(n)}+a_j(T)-a_{nj}^{(n)} & \text{for } j\notin I^{(\mathbf{a})}\\ r_j^{(n)}+a_j(T)-\widehat{a}_j(T)-a_{nj}^{(n)} & \text{for } j\in I^{(\mathbf{a})}.\end{array}\right\}$$

For $t\in F_l^{(n)}$ and $l=1,\cdots,n$, if $j\notin I^{(\mathbf{a})}$, then

$$\begin{aligned}&\mathfrak{f}^{(n)}(t)\cdot\left(\sum_{i=1}^{p}B_{ij}^{(0)}(t)+\sum_{\{i:j\in I_i^{(B)}\}}\widehat{B}_{ij}(t)\right)\\ &\quad\geq\bar{h}_{lj}^{(n)}(t)+a_j^{(0)}(t)-a_{lj}^{(n)}+\int_t^T\mathfrak{f}^{(n)}(s)\cdot\left(\sum_{i=1}^{p}K_{ij}^{(0)}(s,t)-\sum_{\{i:j\in I_i^{(K)}\}}\widehat{K}_{ij}(s,t)\right)ds\end{aligned}$$

and, if $j\in I^{(\mathbf{a})}$, then

$$\begin{aligned}&\mathfrak{f}^{(n)}(t)\cdot\left(\sum_{i=1}^{p}B_{ij}^{(0)}(t)+\sum_{\{i:j\in I_i^{(B)}\}}\widehat{B}_{ij}(t)\right)\\ &\quad\geq\bar{h}_{lj}^{(n)}(t)+a_j^{(0)}(t)-\widehat{a}_j(t)-a_{lj}^{(n)}+\int_t^T\mathfrak{f}^{(n)}(s)\cdot\left(\sum_{i=1}^{p}K_{ij}^{(0)}(s,t)-\sum_{\{i:j\in I_i^{(K)}\}}\widehat{K}_{ij}(s,t)\right)ds.\end{aligned}$$

Moreover, the sequence of real-valued functions $\{\mathfrak{f}^{(n)}\}_{n=1}^{\infty}$ is uniformly bounded.

Proof. For $t\in F_l^{(n)}$, from (49), we have

$$\begin{aligned}\int_t^T\mathfrak{f}^{(n)}(s)ds&=\int_t^T\frac{\mathfrak{p}^{(n)}(s)}{\mathfrak{v}^{(n)}(s)}\cdot\exp\left[\frac{\mathfrak{u}^{(n)}(s)\cdot(T-s)}{\mathfrak{v}^{(n)}(s)}\right]ds\\ &=\int_t^{e_l^{(n)}}\frac{\pi_l^{(n)}}{\mathfrak{b}_l^{(n)}}\cdot\exp\left[\frac{\mathfrak{k}_l^{(n)}\cdot(T-s)}{\mathfrak{b}_l^{(n)}}\right]ds+\sum_{k=l+1}^{n}\int_{\bar{E}_k^{(n)}}\frac{\pi_k^{(n)}}{\mathfrak{b}_k^{(n)}}\cdot\exp\left[\frac{\mathfrak{k}_k^{(n)}\cdot(T-s)}{\mathfrak{b}_k^{(n)}}\right]ds\\ &\leq\int_t^{e_l^{(n)}}\frac{\pi_l^{(n)}}{\mathfrak{b}_l^{(n)}}\cdot\exp\left[\frac{\mathfrak{k}_l^{(n)}\cdot(T-s)}{\mathfrak{b}_l^{(n)}}\right]ds+\sum_{k=l+1}^{n}\int_{\bar{E}_k^{(n)}}\frac{\pi_l^{(n)}}{\mathfrak{b}_l^{(n)}}\cdot\exp\left[\frac{\mathfrak{k}_l^{(n)}\cdot(T-s)}{\mathfrak{b}_l^{(n)}}\right]ds\\ &\qquad\text{(by (36) and (46))}\\ &=\int_t^T\frac{\pi_l^{(n)}}{\mathfrak{b}_l^{(n)}}\cdot\exp\left[\frac{\mathfrak{k}_l^{(n)}\cdot(T-s)}{\mathfrak{b}_l^{(n)}}\right]ds=\frac{\pi_l^{(n)}}{\mathfrak{k}_l^{(n)}}\cdot\left(\exp\left[\frac{\mathfrak{k}_l^{(n)}\cdot(T-t)}{\mathfrak{b}_l^{(n)}}\right]-1\right)\end{aligned}\tag{50}$$

Since

$$\mathfrak{b}_l^{(n)}\cdot\mathfrak{f}^{(n)}(t)=\begin{cases}\mathfrak{r}\cdot\exp\left[\dfrac{\mathfrak{k}_l^{(n)}\cdot(T-t)}{\mathfrak{b}_l^{(n)}}\right] & \text{if } t=e_{l-1}^{(n)} \text{ for } l=1,\cdots,n\\ \pi_l^{(n)}\cdot\exp\left[\dfrac{\mathfrak{k}_l^{(n)}\cdot(T-t)}{\mathfrak{b}_l^{(n)}}\right] & \text{if } t\in E_l^{(n)} \text{ for } l=1,\cdots,n,\end{cases}$$

using (50), it follows that, for $t \in F_l^{(n)}$,

$$\mathfrak{b}_l^{(n)} \cdot \mathfrak{f}^{(n)}(t) \geq \begin{cases} \mathfrak{r} \cdot \left(1 + \frac{\mathfrak{k}_l^{(n)}}{\pi_l^{(n)}} \cdot \int_t^T \mathfrak{f}^{(n)}(s)ds\right) & \text{if } t = e_{l-1}^{(n)} \text{ for } l = 1, \cdots, n \\ \pi_l^{(n)} + \mathfrak{k}_l^{(n)} \cdot \int_t^T \mathfrak{f}^{(n)}(s)ds & \text{if } t \in E_l^{(n)} \text{ for } l = 1, \cdots, n. \end{cases}$$
$$\geq \pi_l^{(n)} + \mathfrak{k}_l^{(n)} \cdot \int_t^T \mathfrak{f}^{(n)}(s)ds \text{ (since } \pi_l^{(n)} \leq \mathfrak{r} \text{ for all } l = 1, \cdots, n). \tag{51}$$

For $t = e_n^{(n)} = T$, we also have

$$\mathfrak{b}_n^{(n)} \cdot \mathfrak{f}^{(n)}(T) = \max\left\{\max_{j \notin I^{(\mathbf{a})}} \left\{r_j^{(n)} + a_j^{(0)}(T) - a_{nj}^{(n)}\right\}, \max_{j \in I^{(\mathbf{a})}} \left\{r_j^{(n)} + a_j^{(0)}(T) - \widehat{a}_j(T) - a_{nj}^{(n)}\right\}\right\}. \tag{52}$$

For each $j = 1, \cdots, q$ and $l = 1, \cdots, n$, we consider the following cases.

- For $t = e_n^{(n)} = T$, from (44) and (52), we have

$$\mathfrak{f}^{(n)}(T) \cdot \left(\sum_{i=1}^{p} B_{ij}^{(0)}(T) + \sum_{\{i: j \in I_i^{(B)}\}} \widehat{B}_{ij}(T)\right) \geq \bar{\mathfrak{b}}_n^{(n)} \cdot \mathfrak{f}^{(n)}(T) = \mathfrak{b}_n^{(n)} \cdot \mathfrak{f}^{(n)}(T)$$
$$\geq \left\{ \begin{array}{ll} r_j^{(n)} + a_j^{(0)}(T) - a_{nj}^{(n)} & \text{for } j \notin I^{(\mathbf{a})} \\ r_j^{(n)} + a_j^{(0)}(T) - \widehat{a}_j(T) - a_{nj}^{(n)} & \text{for } j \in I^{(\mathbf{a})}. \end{array} \right\}.$$

- For $t \in F_l^{(n)}$, by (44), (51), and (37), we have

$$\mathfrak{f}^{(n)}(t) \cdot \left(\sum_{i=1}^{p} B_{ij}^{(0)}(t) + \sum_{\{i: j \in I_i^{(B)}\}} \widehat{B}_{ij}(t)\right) \geq \bar{\mathfrak{b}}_l^{(n)} \cdot \mathfrak{f}^{(n)}(t) \geq \mathfrak{b}_l^{(n)} \cdot \mathfrak{f}^{(n)}(t)$$
$$\geq \begin{cases} \bar{h}_{lj}^{(n)}(t) + a_j^{(0)}(t) - a_{lj}^{(n)} + \mathfrak{k}_l^{(n)} \cdot \int_t^T \mathfrak{f}^{(n)}(s)ds & \text{for } j \notin I^{(\mathbf{a})} \\ \bar{h}_{lj}^{(n)}(t) + a_j^{(0)}(t) - \widehat{a}_j(t) - a_{lj}^{(n)} + \mathfrak{k}_l^{(n)} \cdot \int_t^T \mathfrak{f}^{(n)}(s)ds & \text{for } j \in I^{(\mathbf{a})}. \end{cases}$$

Since
$$\mathfrak{k}_l^{(n)} \geq \bar{\mathfrak{k}}_l^{(n)} \geq K_{ij}^{(0)}(s,t) - \sum_{\{i: j \in I_i^{(K)}\}} \widehat{K}_{ij}(s,t)$$
for $(s,t) \in [e_{l-1}^{(n)}, T] \times \bar{E}_l^{(n)}$, we obtain the desired inequalities.

Finally, from (47) and (48), it is obvious that the sequence of real-valued functions $\{\mathfrak{f}^{(n)}\}_{n=1}^{\infty}$ is uniformly bounded. This completes the proof. $\square$

We define a vector-valued function $\widehat{\mathbf{w}}^{(n)}(t) : [0,T] \to \mathbb{R}^p$ by

$$\widehat{\mathbf{w}}^{(n)}(t) = \begin{cases} \bar{\mathbf{w}}_l^{(n)} + \mathfrak{f}^{(n)}(t)\mathbf{1}_p & \text{if } t \in F_l^{(n)} \text{ for } l = 1, \cdots, n \\ \bar{\mathbf{w}}_n^{(n)} + \mathfrak{f}^{(n)}(T)\mathbf{1}_p & \text{if } t = T. \end{cases} \tag{53}$$

Remark 3. *Since the sequence of real-valued functions $\{\mathfrak{f}^{(n)}\}_{n=1}^{\infty}$ is uniformly bounded by Lemma 4, from (32), we also see that the family of vector-valued functions $\{\widehat{\mathbf{w}}^{(n)}\}_{n \in \mathbb{N}}$ is also uniformly bounded.*

Proposition 4. *For any $n \in \mathbb{N}$, $\widehat{\mathbf{w}}^{(n)}$ is a feasible solution of problem* (DRCLP3).

Proof. For $l = 1, \cdots, n$, we define a real-valued function $b_j^{(n)}$ on $F_l^{(n)}$ by

$$b_j^{(n)}(t) = \left(\sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot \bar{w}_{li}^{(n)} + \sum_{\{i:j\in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot \bar{w}_{li}^{(n)} \right) - \sum_{i=1}^{p} B_{lij}^{(n)} \cdot \bar{w}_{li}^{(n)}$$
$$- \int_t^{e_l^{(n)}} \left[\left(\sum_{i=1}^{p} K_{ij}^{(0)}(s,t) \cdot \bar{w}_{li}^{(n)} - \sum_{\{i:j\in I_i^{(K)}\}} \widehat{K}_{ij}(s,t) \cdot \bar{w}_{li}^{(n)} \right) - \sum_{i=1}^{p} K_{llij}^{(n)} \cdot \bar{w}_{li}^{(n)} \right] ds$$
$$- \sum_{k=l+1}^{n} \int_{\bar{E}_k^{(n)}} \left[\left(\sum_{i=1}^{p} K_{ij}^{(0)}(s,t) \cdot \bar{w}_{ki}^{(n)} - \sum_{\{i:j\in I_i^{(K)}\}} \widehat{K}_{ij}(s,t) \cdot \bar{w}_{ki}^{(n)} \right) - \sum_{i=1}^{p} K_{klij}^{(n)} \cdot \bar{w}_{ki}^{(n)} \right]$$
$$+ \mathfrak{f}^{(n)}(t) \cdot \left(\sum_{i=1}^{p} B_{ij}^{(0)}(t) + \sum_{\{i:j\in I_i^{(B)}\}} \widehat{B}_{ij}(t) \right) - \int_t^T \mathfrak{f}^{(n)}(s) \cdot \left(\sum_{i=1}^{p} K_{ij}^{(0)}(s,t) - \sum_{\{i:j\in I_i^{(K)}\}} \widehat{K}_{ij}(s,t) \right) ds,$$

which implies

$$-b_j^{(n)}(t) + \mathfrak{f}^{(n)}(t) \cdot \left(\sum_{i=1}^{p} B_{ij}^{(0)}(t) + \sum_{\{i:j\in I_i^{(B)}\}} \widehat{B}_{ij}(t) \right) - \int_t^T \mathfrak{f}^{(n)}(s) \cdot \left(\sum_{i=1}^{p} K_{ij}^{(0)}(s,t) - \sum_{\{i:j\in I_i^{(K)}\}} \widehat{K}_{ij}(s,t) \right) ds$$
$$= \sum_{i=1}^{p} B_{lij}^{(n)} \cdot \bar{w}_{li}^{(n)} - \left(\sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot \bar{w}_{li}^{(n)} + \sum_{\{i:j\in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot \bar{w}_{li}^{(n)} \right)$$
$$+ \int_t^{e_l^{(n)}} \left[\left(\sum_{i=1}^{p} K_{ij}^{(0)}(s,t) \cdot \bar{w}_{li}^{(n)} - \sum_{\{i:j\in I_i^{(K)}\}} \widehat{K}_{ij}(s,t) \cdot \bar{w}_{li}^{(n)} \right) - \sum_{i=1}^{p} K_{llij}^{(n)} \cdot \bar{w}_{li}^{(n)} \right] ds$$
$$+ \sum_{k=l+1}^{n} \int_{\bar{E}_k^{(n)}} \left[\left(\sum_{i=1}^{p} K_{ij}^{(0)}(s,t) \cdot \bar{w}_{ki}^{(n)} - \sum_{\{i:j\in I_i^{(K)}\}} \widehat{K}_{ij}(s,t) \cdot \bar{w}_{ki}^{(n)} \right) - \sum_{i=1}^{p} K_{klij}^{(n)} \cdot \bar{w}_{ki}^{(n)} \right].$$

Therefore, by adding the term $(e_l^{(n)} - t) \cdot \sum_{i=1}^{p} K_{llij}^{(n)} \cdot \bar{w}_{li}^{(n)}$ on both sides, we obtain

$$\left(e_l^{(n)} - t\right) \cdot \sum_{i=1}^{p} K_{llij}^{(n)} \cdot \bar{w}_{li}^{(n)} - b_j^{(n)}(t) + \mathfrak{f}^{(n)}(t) \cdot \left(\sum_{i=1}^{p} B_{ij}^{(0)}(t) + \sum_{\{i:j\in I_i^{(B)}\}} \widehat{B}_{ij}(t) \right)$$
$$- \int_t^T \mathfrak{f}^{(n)}(s) \cdot \left(\sum_{i=1}^{p} K_{ij}^{(0)}(s,t) - \sum_{\{i:j\in I_i^{(K)}\}} \widehat{K}_{ij}(s,t) \right) ds = \bar{h}_{lj}^{(n)}(t),$$

which implies

$$b_j^{(n)}(t) - \left(e_l^{(n)} - t\right) \cdot \sum_{i=1}^{p} K_{llij}^{(n)} \cdot \bar{w}_{li}^{(n)}$$
$$= -\bar{h}_{lj}^{(n)}(t) + \mathfrak{f}^{(n)}(t) \cdot \left(\sum_{i=1}^{p} B_{ij}^{(0)}(t) + \sum_{\{i:j\in I_i^{(B)}\}} \widehat{B}_{ij}(t) \right)$$
$$- \int_t^T \mathfrak{f}^{(n)}(s) \cdot \left(\sum_{i=1}^{p} K_{ij}^{(0)}(s,t) - \sum_{\{i:j\in I_i^{(K)}\}} \widehat{K}_{ij}(s,t) \right) ds$$
$$\geq \left\{ \begin{array}{ll} a_j^{(0)}(t) - a_{lj}^{(n)} & \text{for } j \notin I^{(\mathfrak{a})} \\ a_j^{(0)}(t) - \widehat{a}_j(t) - a_{lj}^{(n)} & \text{for } j \in I^{(\mathfrak{a})} \end{array} \right\} \text{(by Lemma 4)} \tag{54}$$

Now, from (53), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{p} B_{ij}^{(0)}(t)\cdot \widehat{w}_i^{(n)}(t)+\sum_{\{i:j\in I_i^{(B)}\}} \widehat{B}_{ij}(t)\cdot \widehat{w}_i^{(n)}(t)\\
&\quad -\int_t^T\left(\sum_{i=1}^{p} K_{ij}^{(0)}(s,t)\cdot \widehat{w}_i^{(n)}(s)-\sum_{\{i:j\in I_i^{(K)}\}} \widehat{K}_{ij}(s,t)\cdot \widehat{w}_i^{(n)}(s)\right)ds\\
&\quad =\left(\sum_{i=1}^{p} B_{lij}^{(n)}\cdot \bar{w}_{li}^{(n)}-\int_t^{e_l^{(n)}}\sum_{i=1}^{p} K_{llij}^{(n)}\cdot \bar{w}_{li}^{(n)}ds-\sum_{k=l+1}^{n}\int_{\bar{E}_k^{(n)}}\sum_{i=1}^{p} K_{klij}^{(n)}\cdot \bar{w}_{ki}^{(n)}ds\right)+b_j^{(n)}(t)\\
&\quad =\left(\sum_{i=1}^{p} B_{lij}^{(n)}\cdot \bar{w}_{li}^{(n)}-\sum_{k=l+1}^{n}\sum_{i=1}^{p}\mathfrak{d}_k^{(n)}\cdot K_{klij}^{(n)}\cdot \bar{w}_{ki}^{(n)}\right)+b_j^{(n)}(t)-\left(e_l^{(n)}-t\right)\cdot\sum_{i=1}^{p} K_{llij}^{(n)}\cdot \bar{w}_{li}^{(n)}\\
&\quad \geq a_{lj}^{(n)}+b_j^{(n)}(t)-\left(e_l^{(n)}-t\right)\cdot\sum_{i=1}^{p} K_{llij}^{(n)}\cdot \bar{w}_{li}^{(n)}\quad \text{(by the feasibility of } \bar{\mathbf{w}}^{(n)} \text{ for problem } (\mathrm{D}_n))\\
&\quad \geq \left\{\begin{array}{ll} a_j^{(0)}(t) & \text{for } j\notin I^{(\mathbf{a})}\\ a_j^{(0)}(t)-\widehat{a}_j(t) & \text{for } j\in I^{(\mathbf{a})}\end{array}\right\}\ \text{(by (54))}
\end{aligned}
$$

Suppose that $t=T$. We define

$$
\begin{aligned}
\widehat{b}_j^{(n)}=&\left(\sum_{i=1}^{p} B_{ij}^{(0)}(T)\cdot \bar{w}_{ni}^{(n)}+\sum_{\{i:j\in I_i^{(B)}\}} \widehat{B}_{ij}(T)\cdot \bar{w}_{ni}^{(n)}\right)-\sum_{i=1}^{p} B_{nij}^{(n)}\cdot \bar{w}_{ni}^{(n)}\\
&+\mathfrak{f}^{(n)}(T)\cdot\left(\sum_{i=1}^{p} B_{ij}^{(0)}(T)+\sum_{\{i:j\in I_i^{(B)}\}} \widehat{B}_{ij}(T)\right).
\end{aligned}
$$

Then,

$$
\begin{aligned}
&-\widehat{b}_j^{(n)}+\mathfrak{f}^{(n)}(T)\cdot\left(\sum_{i=1}^{p} B_{ij}^{(0)}(T)+\sum_{\{i:j\in I_i^{(B)}\}} \widehat{B}_{ij}(T)\right)\\
&\quad =\sum_{i=1}^{p} B_{nij}^{(n)}\cdot \bar{w}_{ni}^{(n)}-\left(\sum_{i=1}^{p} B_{ij}^{(0)}(T)\cdot \bar{w}_{ni}^{(n)}+\sum_{\{i:j\in I_i^{(B)}\}} \widehat{B}_{ij}(T)\cdot \bar{w}_{ni}^{(n)}\right)=r_j^{(n)},
\end{aligned}
$$

which implies

$$
\begin{aligned}
\widehat{b}_j^{(n)}&=-r_j^{(n)}+\mathfrak{f}^{(n)}(T)\cdot\left(\sum_{i=1}^{p} B_{ij}^{(0)}(T)+\sum_{\{i:j\in I_i^{(B)}\}} \widehat{B}_{ij}(T)\right)\\
&\geq \left\{\begin{array}{ll} a_j(T)-a_{nj}^{(n)} & \text{for } j\notin I^{(\mathbf{a})}\\ a_j(T)-\widehat{a}_j(T)-a_{nj}^{(n)} & \text{for } j\in I^{(\mathbf{a})}.\end{array}\right\}\ \text{(by Lemma 4)}
\end{aligned}
\tag{55}
$$

Now, from (53), we obtain

$$\begin{aligned}
&\sum_{i=1}^{p} B_{ij}^{(0)}(T) \cdot \widehat{w}_i^{(n)}(T) + \sum_{\{i:j \in I_i^{(B)}\}} \widehat{B}_{ij}(T) \cdot \widehat{w}_i^{(n)}(T) \\
&= \sum_{i=1}^{p} B_{nij}^{(n)} \cdot \bar{w}_{ni}^{(n)} + \widehat{b}_j^{(n)} \geq a_{nj}^{(n)} + \widehat{b}_j^{(n)} \text{ (by the feasibility of } \bar{\mathbf{w}}^{(n)}) \\
&\geq \left\{ \begin{array}{ll} a_j(T) & \text{for } j \notin I^{(\mathbf{a})} \\ a_j(T) - \widehat{a}_j(T) & \text{for } j \in I^{(\mathbf{a})}. \end{array} \right\} \text{ (using (55))}
\end{aligned}$$

Therefore, we conclude that $\widehat{\mathbf{w}}^{(n)}$ is indeed a feasible solution of problem (DRCLP3). This completes this proof. □

For each $i = 1, \cdots, p$ and $j = 1, \cdots, q$, in order to obtain the approximate solutions, we need to define the step functions $\bar{a}_j^{(n)} : [0, T] \to \mathbb{R}$ and $\bar{c}_i^{(n)} : [0, T] \to \mathbb{R}$ as follows:

$$\bar{a}_j^{(n)}(t) = \left\{ \begin{array}{ll} a_{lj}^{(n)} & \text{if } t \in F_l^{(n)} \text{ for } l = 1, \cdots, n \\ a_{nj}^{(n)} & \text{if } t = T \end{array} \right.$$

and

$$\bar{c}_i^{(n)}(t) = \left\{ \begin{array}{ll} c_{li}^{(n)} & \text{if } t \in F_l^{(n)} \text{ for } l = 1, \cdots, n \\ c_{ni}^{(n)} & \text{if } t = T, \end{array} \right.$$

respectively. For each $i = 1, \cdots, p$, we also define the step function $\bar{w}_i^{(n)} : [0, T] \to \mathbb{R}$ by

$$\bar{w}_i^{(n)}(t) = \left\{ \begin{array}{ll} \bar{w}_{li}^{(n)} & \text{if } t \in F_l^{(n)} \text{ for } l = 1, \cdots, n \\ \bar{w}_{ni}^{(n)} & \text{if } t = T. \end{array} \right.$$

Lemma 5. *For $i = 1, \cdots, p$ and $j = 1, \cdots, q$, we have*

$$\left\{ \begin{array}{ll} \displaystyle\int_0^T \left[a_j^{(0)}(t) - \bar{a}_j^{(n)}(t) \right] \cdot \widehat{z}_j^{(n)}(t) dt \to 0 & \textit{for } j \notin I^{(\mathbf{a})} \\ \displaystyle\int_0^T \left[a_j^{(0)}(t) - \widehat{a}_j(t) - \bar{a}_j^{(n)}(t) \right] \cdot \widehat{z}_j^{(n)}(t) dt \to 0 & \textit{for } j \in I^{(\mathbf{a})} \end{array} \right\} \textit{as } n \to \infty \tag{56}$$

and

$$\left\{ \begin{array}{ll} \displaystyle\int_0^T \left[c_i^{(0)}(t) - \bar{c}_i^{(n)}(t) \right] \cdot \bar{w}_i^{(n)}(t) dt \to 0 & \textit{for } i \notin I^{(\mathbf{c})} \\ \displaystyle\int_0^T \left[c_i^{(0)}(t) - \widehat{c}_i(t) - \bar{c}_i^{(n)}(t) \right] \cdot \bar{w}_i^{(n)}(t) dt \to 0 & \textit{for } i \in I^{(\mathbf{c})} \end{array} \right\} \textit{as } n \to \infty. \tag{57}$$

Proof. We first observe that the following functions

$$\left\{ \begin{array}{ll} \left[a_j^{(0)}(t) - \bar{a}_j^{(n)}(t) \right] \cdot \widehat{z}_j^{(n)}(t) & \text{for } j \notin I^{(\mathbf{a})} \\ \left[a_j^{(0)}(t) - \widehat{a}_j(t) - \bar{a}_j^{(n)}(t) \right] \cdot \widehat{z}_j^{(n)}(t) & \text{for } j \in I^{(\mathbf{a})} \end{array} \right\}$$

and

$$\left\{ \begin{array}{ll} \left[c_i^{(0)}(t) - \bar{c}_i^{(n)}(t) \right] \cdot \bar{w}_i^{(n)}(t) & \text{for } i \notin I^{(\mathbf{c})} \\ \left[c_i^{(0)}(t) - \widehat{c}_i(t) - \bar{c}_i^{(n)}(t) \right] \cdot \bar{w}_i^{(n)}(t) & \text{for } i \in I^{(\mathbf{c})} \end{array} \right\}$$

are continuous a.e. on $[0, T]$, i.e., they are Riemann-integrable on $[0, T]$. In other words, their Riemann integral and Lebesgue integral are identical. Lemma 1 says that

$$\left\{ \begin{array}{ll} a_j^{(0)}(t) - \bar{a}_j^{(n)}(t) & \text{for } j \notin I^{(\mathfrak{a})} \\ a_j^{(0)}(t) - \widehat{a}_j(t) - \bar{a}_j^{(n)}(t) & \text{for } j \in I^{(\mathfrak{a})} \end{array} \right\} \to 0 \text{ as } n \to \infty \text{ a.e. on } [0, T]$$

and

$$\left\{ \begin{array}{ll} c_i^{(0)}(t) - \bar{c}_i^{(n)}(t) & \text{for } i \notin I^{(\mathfrak{c})} \\ c_i^{(0)}(t) - \widehat{c}_i(t) - \bar{c}_i^{(n)}(t) & \text{for } i \in I^{(\mathfrak{c})} \end{array} \right\} \to 0 \text{ as } n \to \infty \text{ a.e. on } [0, T].$$

Since $\widehat{z}_j^{(n)}$ is uniformly bounded by Proposition 2, using the Lebesgue bounded convergence theorem, we can obtain (56). On the other hand, since $\bar{w}_i^{(n)}$ is uniformly bounded using Proposition 1 and the Lebesgue bounded convergence theorem, we can also obtain (57), and the proof is complete □

Theorem 1. *The following statements hold true.*

(i) *We have*

$$\limsup_{n\to\infty} V(\mathrm{D}_n) = V(\mathrm{DRCLP3}) \textit{ and } 0 \le V(\mathrm{DRCLP3}) - V(\mathrm{D}_n) \le \varepsilon_n,$$

where

$$\begin{aligned} \varepsilon_n = & -V(\mathrm{D}_n) + \sum_{i=1}^{p} \sum_{l=1}^{n} \int_{\bar{E}_l^{(n)}} c_i^{(0)}(t) \cdot \bar{w}_{li}^{(n)} dt - \sum_{i \in I^{(\mathfrak{c})}} \sum_{l=1}^{n} \int_{\bar{E}_l^{(n)}} \widehat{c}_i(t) \cdot \bar{w}_{li}^{(n)} dt \\ & + \sum_{i=1}^{p} \sum_{l=1}^{n} \int_{\bar{E}_l^{(n)}} \frac{\pi_l^{(n)}}{\mathfrak{b}_l^{(n)}} \cdot \exp\left[\frac{\mathfrak{k}_l^{(n)} \cdot (T-t)}{\mathfrak{b}_l^{(n)}} \right] \cdot c_i^{(0)}(t) dt \\ & - \sum_{i \in I^{(\mathfrak{c})}} \sum_{l=1}^{n} \int_{\bar{E}_l^{(n)}} \frac{\pi_l^{(n)}}{\mathfrak{b}_l^{(n)}} \cdot \exp\left[\frac{\mathfrak{k}_l^{(n)} \cdot (T-t)}{\mathfrak{b}_l^{(n)}} \right] \cdot \widehat{c}_i(t) dt \end{aligned} \tag{58}$$

satisfying $\varepsilon_n \to 0$ *as* $n \to \infty$*. Moreover, there exists a convergent subsequence* $\{V(\mathrm{D}_{n_k})\}_{k=1}^{\infty}$ *of* $\{V(\mathrm{D}_n)\}_{n=1}^{\infty}$ *satisfying*

$$\lim_{k\to\infty} V(\mathrm{D}_{n_k}) = V(\mathrm{DRCLP3}). \tag{59}$$

(ii) (**No Duality Gap**). *Suppose that the primal problem* (P_n) *is feasible. Then, we have*

$$V(\mathrm{DRCLP3}) = V(\mathrm{RCLP3}) = \limsup_{n\to\infty} V(\mathrm{D}_n) = \limsup_{n\to\infty} V(\mathrm{P}_n)$$

and

$$0 \le V(\mathrm{RCLP3}) - V(\mathrm{P}_n) = V(\mathrm{DRCLP3}) - V(\mathrm{D}_n) \le \varepsilon_n.$$

Proof. To prove part (i), we have

$$
\begin{aligned}
0 &\leq V(\text{DRCLP3}) - V(\text{D}_n) \text{ (by (30))} \\
&= V(\text{DRCLP3}) - \sum_{l=1}^{n}\sum_{i=1}^{p}\int_{\bar{E}_l^{(n)}} c_{li}^{(n)} \cdot \bar{w}_{li}^{(n)} dt \\
&\leq \sum_{i=1}^{p}\int_0^T c_i^{(0)}(t) \cdot \widehat{w}_i(t)dt - \sum_{i\in I^{(\mathbf{c})}}\int_0^T \widehat{c}_i(t) \cdot \widehat{w}_i(t)dt - \sum_{l=1}^{n}\sum_{i=1}^{p}\int_{\bar{E}_l^{(n)}} c_{li}^{(n)} \cdot \bar{w}_{li}^{(n)} dt \qquad (60) \\
&\qquad \text{(by Proposition 4)} \\
&= \sum_{i\notin I^{(\mathbf{c})}}\int_0^T c_i^{(0)}(t) \cdot \widehat{w}_i(t)dt + \sum_{i\in I^{(\mathbf{c})}}\int_0^T \left(c_i^{(0)}(t) - \widehat{c}_i(t)\right) \cdot \widehat{w}_i(t)dt - \sum_{l=1}^{n}\sum_{i=1}^{p}\int_{\bar{E}_l^{(n)}} c_{li}^{(n)} \cdot \bar{w}_{li}^{(n)} dt \\
&= \sum_{i\notin I^{(\mathbf{c})}}\sum_{l=1}^{n}\int_{\bar{E}_l^{(n)}}\left(c_i^{(0)}(t) - c_{li}^{(n)}\right) \cdot \bar{w}_{li}^{(n)} dt + \sum_{i\in I^{(\mathbf{c})}}\sum_{l=1}^{n}\int_{\bar{E}_l^{(n)}}\left(c_i^{(0)}(t) - \widehat{c}_i(t) - c_{li}^{(n)}\right) \cdot \bar{w}_{li}^{(n)} dt \\
&\quad + \sum_{i\notin I^{(\mathbf{c})}}\int_0^T \mathfrak{f}^{(n)}(t) \cdot c_i^{(0)}(t)dt + \sum_{i\in I^{(\mathbf{c})}}\int_0^T \mathfrak{f}^{(n)}(t) \cdot \left(c_i^{(0)}(t) - \widehat{c}_i(t)\right)dt \qquad (61) \\
&= \sum_{i\notin I^{(\mathbf{c})}}\int_0^T \left[c_i^{(0)}(t) - \bar{c}_i^{(n)}(t)\right] \cdot \bar{w}_i^{(n)}(t)dt + \sum_{i\in I^{(\mathbf{c})}}\int_0^T \left[c_i^{(0)}(t) - \widehat{c}_i(t) - \bar{c}_i^{(n)}(t)\right] \cdot \bar{w}_i^{(n)}(t)dt \\
&\quad + \sum_{i\notin I^{(\mathbf{c})}}\int_0^T \mathfrak{f}^{(n)}(t) \cdot c_i^{(0)}(t)dt + \sum_{i\in I^{(\mathbf{c})}}\int_0^T \mathfrak{f}^{(n)}(t) \cdot \left(c_i^{(0)}(t) - \widehat{c}_i(t)\right)dt \\
&\equiv \varepsilon_n
\end{aligned}
$$

Lemma 3 states that $\pi_l^{(n)} \to 0$ as $n \to \infty$, which implies $\mathfrak{p}^{(n)} \to 0$ as $n \to \infty$ a.e. on $[0, T]$. Therefore, we obtain $\mathfrak{f}^{(n)} \to 0$ as $n \to \infty$ a.e. on $[0, T]$. Using the Lebesgue bounded convergence theorem for integrals, we also obtain

$$
\left\{\begin{array}{ll}
\displaystyle\int_0^T \mathfrak{f}^{(n)}(t) \cdot c_i^{(0)}(t)dt \to 0 \text{ as } n \to \infty & \text{for } i \notin I^{(\mathbf{c})} \\
\displaystyle\int_0^T \mathfrak{f}^{(n)}(t) \cdot \left(c_i^{(0)}(t) - \widehat{c}_i(t)\right)dt \to 0 \text{ as } n \to \infty & \text{for } i \in I^{(\mathbf{c})}
\end{array}\right\}. \qquad (62)
$$

From Lemma 5, we conclude that $\varepsilon_n \to 0$ as $n \to \infty$. From (61), we also obtain

$$
V(\text{D}_n) \leq V(\text{DRCLP3}) \leq V(\text{D}_n) + \varepsilon_n,
$$

which implies

$$
\begin{aligned}
\limsup_{n\to\infty} V(\text{D}_n) \leq V(\text{DRCLP3}) &\leq \limsup_{n\to\infty}[V(\text{D}_n) + \varepsilon_n] \\
&\leq \limsup_{n\to\infty} V(\text{D}_n) + \limsup_{n\to\infty} \varepsilon_n = \limsup_{n\to\infty} V(\text{D}_n).
\end{aligned}
$$

Part (ii) of Proposition 1 states that $\{V(\text{D}_n)\}_{n=1}^{\infty}$ is a bounded sequence. Therefore, there exists a convergent subsequence $\{V(\text{D}_{n_k})\}_{k=1}^{\infty}$ of $\{V(\text{D}_n)\}_{n=1}^{\infty}$. Using (61), we obtain the equality (59). It is easy to see that ε_n can be written as (58), which proves part (i).

To prove part (ii), using part (i) and inequality (30), we obtain

$$
V(\text{DRCLP3}) \geq V(\text{RCLP3}) \geq \limsup_{n\to\infty} V(\text{D}_n) = V(\text{DRCLP3}).
$$

Since $V(\text{D}_n) = V(\text{P}_n)$ for each $n \in \mathbb{N}$, we also have

$$
V(\text{DRCLP3}) = V(\text{RCLP3}) = \limsup_{n\to\infty} V(\text{D}_n) = \limsup_{n\to\infty} V(\text{P}_n)
$$

and

$$
0 \leq V(\text{RCLP3}) - V(\text{P}_n) = V(\text{DRCLP3}) - V(\text{D}_n) \leq \varepsilon_n.
$$

This completes the proof. □

From Remark 2 and Theorem 1, if the vector-valued function **c** is nonnegative, i.e., the primal problem (P_n) is feasible, then the strong duality holds for the primal and dual pair of continuous-time linear programming problems (RCLP3) and (DRCLP3).

Proposition 5. *The following statements hold true.*

(i) *Suppose that the primal problem (P_n) is feasible. Let $\widehat{z}_j^{(n)}$ be defined in (26) for $j = 1, \cdots, q$. Then, the error between $V(\mathrm{RCLP3})$ and the objective value of $(\widehat{z}_1^{(n)}, \cdots, \widehat{z}_q^{(n)})$ is less than or equal to ε_n defined in (58). In other words, we have*

$$0 \leq V(\mathrm{RCLP3}) - \left(\sum_{j=1}^{q} \int_0^T a_j^{(0)}(t) \cdot \widehat{z}_j(t)dt - \sum_{j \in I^{(\mathbf{a})}} \int_0^T \widehat{a}_j(t) \cdot \widehat{z}_j(t)dt \right) \leq \varepsilon_n.$$

(ii) *Let $\widehat{w}_i^{(n)}$ be defined in (53) for $i = 1, \cdots, p$. Then, the error between $V(\mathrm{DRCLP3})$ and the objective value of $(\widehat{w}_1^{(n)}, \cdots, \widehat{w}_p^{(n)})$ is less than or equal to ε_n. In other words, we have*

$$0 \leq \left(\sum_{i=1}^{p} \int_0^T c_i^{(0)}(t) \cdot \widehat{w}_i(t)dt - \sum_{i \in I^{(\mathbf{c})}} \int_0^T \widehat{c}_i(t) \cdot \widehat{w}_i(t)dt \right) - V(\mathrm{DRCLP3}) \leq \varepsilon_n$$

Proof. To prove part (i), Proposition 3 states that $(\widehat{z}_1^{(n)}, \cdots, \widehat{z}_q^{(n)})$ is a feasible solution of problem (RCLP3). Since

$$\sum_{l=1}^{n} \sum_{j=1}^{q} \int_{\bar{E}_l^{(n)}} a_{lj}^{(n)} \cdot \widehat{z}_j^{(n)}(t)dt = \sum_{l=1}^{n} \sum_{j=1}^{q} \mathfrak{d}_l^{(n)} a_{lj}^{(n)} \cdot \bar{z}_{lj}^{(n)} = V(\mathrm{P}_n) = V(\mathrm{D}_n) \tag{63}$$

and

$$a_{lj}^{(n)} \leq \left\{ \begin{array}{ll} a_j^{(0)}(t) & \text{for } j \notin I^{(\mathbf{a})} \\ a_j^{(0)}(t) - \widehat{a}_j(t) & \text{for } j \in I^{(\mathbf{a})} \end{array} \right\}$$

for all $t \in \bar{E}_l^{(n)}$ and $l = 1, \cdots, n$, it follows that

$$\begin{aligned} 0 &\leq V(\mathrm{RCLP3}) - \left(\sum_{j=1}^{q} \int_0^T a_j^{(0)}(t) \cdot \widehat{z}_j(t)dt - \sum_{j \in I^{(\mathbf{a})}} \int_0^T \widehat{a}_j(t) \cdot \widehat{z}_j(t)dt \right) \\ &\leq V(\mathrm{RCLP3}) - \sum_{l=1}^{n} \sum_{j=1}^{q} \int_{\bar{E}_l^{(n)}} a_{lj}^{(n)} \cdot \widehat{z}_j^{(n)}(t)dt \\ &= V(\mathrm{DRCLP3}) - V(\mathrm{D}_n) \text{ (by (63) and part (ii) of Theorem 1)} \\ &\leq \varepsilon_n \text{ (by part (i) of Theorem 1).} \end{aligned}$$

To prove part (ii), we have

$$\begin{aligned} 0 &\leq \left(\sum_{i=1}^{p} \int_0^T c_i^{(0)}(t) \cdot \widehat{w}_i(t)dt - \sum_{i \in I^{(\mathbf{c})}} \int_0^T \widehat{c}_i(t) \cdot \widehat{w}_i(t)dt \right) - V(\mathrm{DRCLP3}) \text{ (by Proposition 4)} \\ &\leq \left(\sum_{i=1}^{p} \int_0^T c_i^{(0)}(t) \cdot \widehat{w}_i(t)dt - \sum_{i \in I^{(\mathbf{c})}} \int_0^T \widehat{c}_i(t) \cdot \widehat{w}_i(t)dt \right) - V(\mathrm{D}_n) \\ &\quad \text{(since } V(\mathrm{D}_n) \leq V(\mathrm{DRCLP3}) \text{ by part (i) of Theorem 1)} \\ &= \varepsilon_n \text{ (by (60) and (61))} \end{aligned}$$

This completes the proof. □

Definition 1. Given any $\epsilon > 0$, we say that the feasible solution $(z_1^{(\epsilon)}, \cdots, z_q^{(\epsilon)})$ of primal problem (RCLP3) is an *ϵ-optimal solution* when

$$0 \leq V(\text{RCLP3}) - \left(\sum_{j=1}^{q} \int_0^T a_j^{(0)}(t) \cdot z_j^{(\epsilon)}(t)dt - \sum_{j \in I^{(\mathbf{a})}} \int_0^T \widehat{a}_j(t) \cdot z_j^{(\epsilon)}(t)dt\right) < \epsilon.$$

We say that the feasible solution $(w_1^{(\epsilon)}, \cdots, w_p^{(\epsilon)})$ of dual problem (DRCLP3) is an *ϵ-optimal solution* when

$$0 \leq \left(\sum_{i=1}^{p} \int_0^T c_i^{(0)}(t) \cdot w_i^{(\epsilon)}(t)dt - \sum_{i \in I^{(\mathbf{c})}} \int_0^T \widehat{c}_i(t) \cdot w_i^{(\epsilon)}(t)dt\right) - V(\text{DRCLP3}) < \epsilon.$$

Theorem 2. *Given any $\epsilon > 0$, the following statements hold true.*

(i) *The ϵ-optimal solution of primal problem* (RCLP3) *exists in the sense that there exists $n \in \mathbb{N}$ satisfying $(z_1^{(\epsilon)}, \cdots, z_q^{(\epsilon)}) = (\widehat{z}_1^{(n)}, \cdots, \widehat{z}_q^{(n)})$, where $(\widehat{z}_1^{(n)}, \cdots, \widehat{z}_q^{(n)})$ is obtained from Proposition 5 satisfying $\varepsilon_n < \epsilon$.*

(ii) *The ϵ-optimal solution of dual problem* (DRCLP3) *exists in the sense that there exists $n \in \mathbb{N}$ satisfying $(w_1^{(\epsilon)}, \cdots, w_p^{(\epsilon)}) = (\widehat{w}_1^{(n)}, \cdots, \widehat{w}_p^{(n)})$, where $(\widehat{w}_1^{(n)}, \cdots, \widehat{w}_p^{(n)})$ is obtained from Proposition 5 satisfying $\varepsilon_n < \epsilon$.*

Proof. Given any $\epsilon > 0$, from Proposition 5, since $\varepsilon_n \to 0$ as $n \to \infty$, there exists $n \in \mathbb{N}$ such that $\varepsilon_n < \epsilon$. Then, the results follow immediately. □

5. Convergence of Approximate Solutions

By referring to (26) and (53), we are interested in obtaining the convergent properties of the sequences $\{\widehat{\mathbf{z}}^{(n)}\}_{n=1}^{\infty}$ and $\{\widehat{\mathbf{w}}^{(n)}\}_{n=1}^{\infty}$ that are constructed from the optimal solutions $\bar{\mathbf{z}}^{(n)}$ of primal problem (P_n) and the optimal solution $\bar{\mathbf{w}}^{(n)}$ of dual problem (D_n), respectively. We need a useful lemma.

Lemma 6. *We define a real-valued function η on $[0, T]$ by*

$$\eta(t) = \frac{\tau}{\sigma} \cdot \exp\left[\frac{\nu \cdot (T - t)}{\sigma}\right]. \tag{64}$$

Let $\mathbf{w}^{(0)}$ be a feasible solution of dual problem (DRCLP3)*. We also define*

$$w_i^{(1)}(t) = \min\left\{w_i^{(0)}(t), \eta(t)\right\} \text{ for all } i = 1, \cdots, p \text{ and } t \in [0, T]. \tag{65}$$

Then, $\mathbf{w}^{(1)}$ is a feasible solution of dual problem (DRCLP3) *satisfying $\mathbf{w}^{(1)}(t) \leq \mathbf{w}^{(0)}(t)$ and $w_i^{(1)}(t) \leq \eta(t)$ for all $i = 1, \cdots, p$ and $t \in [0, T]$.*

Proof. The feasibility of $\mathbf{w}^{(0)}$ for problem (DRCLP3) states that

$$\sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot w_i^{(0)}(t) + \sum_{\{i: j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot w_i^{(0)}(t) \tag{66}$$
$$\geq \begin{cases} a_j^{(0)}(t) - \widehat{a}_j(t) + \sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot w_i^{(0)}(s)ds - \sum_{\{i: j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot w_i^{(0)}(s)ds & \text{for } j \in I^{(\mathbf{a})} \\ a_j^{(0)}(t) + \sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot w_i^{(0)}(s)ds - \sum_{\{i: j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot w_i^{(0)}(s)ds & \text{for } j \notin I^{(\mathbf{a})} \end{cases}$$

Since $K_{ij}(s,t) \geq 0$ and $w_i^{(1)}(t) \leq w_i^{(0)}(t)$, from (66), we obtain

$$\sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot w_i^{(0)}(t) + \sum_{\{i:j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot w_i^{(0)}(t) \geq \left\{ \begin{array}{ll} a_j^{(0)}(t) - \widehat{a}_j(t) + \sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot w_i^{(1)}(s) ds - \sum_{\{i:j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot w_i^{(1)}(s) ds & \text{for } j \in I^{(\mathbf{a})} \\ a_j^{(0)}(t) + \sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot w_i^{(1)}(s) ds - \sum_{\{i:j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot w_i^{(1)}(s) ds & \text{for } j \notin I^{(\mathbf{a})} \end{array} \right\} \tag{67}$$

For any fixed $t \in [0,T]$, we define the index sets

$$I_{\leq} = \left\{ i : w_i^{(0)}(t) \leq \eta(t) \right\} \text{ and } I_{>} = \left\{ i : w_i^{(0)}(t) > \eta(t) \right\}.$$

Then,

$$w_i^{(1)}(t) = \left\{ \begin{array}{ll} w_i^{(0)}(t) & \text{if } i \in I_{\leq} \\ \eta(t) & \text{if } i \in I_{>}. \end{array} \right.$$

For each fixed j, three cases are considered.

- Suppose that $I_{>} = \emptyset$ (i.e., the second sum is zero). Then, $w_i^{(0)}(t) = w_i^{(1)}(t)$ for all i. Therefore, from (67), we have

$$\sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot w_i^{(1)}(t) + \sum_{\{i:j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot w_i^{(1)}(t) = \sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot w_i^{(0)}(t) + \sum_{\{i:j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot w_i^{(0)}(t)$$
$$\geq \left\{ \begin{array}{ll} a_j^{(0)}(t) - \widehat{a}_j(t) + \sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot w_i^{(1)}(s) ds - \sum_{\{i:j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot w_i^{(1)}(s) ds & \text{for } j \in I^{(\mathbf{a})} \\ a_j^{(0)}(t) + \sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot w_i^{(1)}(s) ds - \sum_{\{i:j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot w_i^{(1)}(s) ds & \text{for } j \notin I^{(\mathbf{a})} \end{array} \right\}$$

- Suppose that $I_{>} \neq \emptyset$ and that

$$\left\{ \begin{array}{l} B_{ij}^{(0)}(t) = 0 \text{ for all } i \in I_{>} \text{ and } j \notin I_i^{(B)} \\ B_{ij}^{(0)}(t) + \widehat{B}_{ij}(t) = 0 \text{ for all } i \in I_{>} \text{ and } j \in I_i^{(B)}. \end{array} \right.$$

Then, we obtain

$$\begin{aligned} &\sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot w_i^{(1)}(t) + \sum_{\{i:j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot w_i^{(1)}(t) \\ &\quad = \sum_{i \in I_{\leq}} B_{ij}^{(0)}(t) \cdot w_i^{(1)}(t) + \sum_{\{i:j \in I_i^{(B)}, i \in I_{\leq}\}} \widehat{B}_{ij}(t) \cdot w_i^{(1)}(t) \\ &\quad = \sum_{i \in I_{\leq}} B_{ij}^{(0)}(t) \cdot w_i^{(0)}(t) + \sum_{\{i:j \in I_i^{(B)}, i \in I_{\leq}\}} \widehat{B}_{ij}(t) \cdot w_i^{(0)}(t) \\ &\quad = \sum_{i \in I_{\leq}} B_{ij}^{(0)}(t) \cdot w_i^{(0)}(t) + \sum_{i \in I_{>}} B_{ij}^{(0)}(t) \cdot w_i^{(0)}(t) \\ &\qquad + \sum_{\{i:j \in I_i^{(B)}, i \in I_{\leq}\}} \widehat{B}_{ij}(t) \cdot w_i^{(0)}(t) + \sum_{\{i:j \in I_i^{(B)}, i \in I_{>}\}} \widehat{B}_{ij}(t) \cdot w_i^{(0)}(t) \\ &\quad = \sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot w_i^{(0)}(t) + \sum_{\{i:j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot w_i^{(0)}(t), \end{aligned}$$

Using (67), we also obtain

$$\sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot w_i^{(1)}(t) + \sum_{\{i:j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot w_i^{(1)}(t)$$
$$\geq \left\{ \begin{array}{ll} a_j^{(0)}(t) - \widehat{a}_j(t) + \sum_{i=1}^{p} \int_t^T K_{ij}(s,t) \cdot w_i^{(1)}(s)ds - \sum_{\{i:j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot w_i^{(1)}(s)ds & \text{for } j \in I^{(\mathbf{a})} \\ a_j^{(0)}(t) + \sum_{i=1}^{p} \int_t^T K_{ij}(s,t) \cdot w_i^{(1)}(s)ds - \sum_{\{i:j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot w_i^{(1)}(s)ds & \text{for } j \notin I^{(\mathbf{a})} \end{array} \right\}.$$

- Suppose that $I_> \neq \emptyset$ and that there exists $i^* \in I_>$ such that $B_{i^*j}^{(0)}(t) \neq 0$ for $j \notin I_{i^*}^{(B)}$ or $B_{i^*j}^{(0)}(t) + \widehat{B}_{i^*j}(t) \neq 0$ for $j \in I_{i^*}^{(B)}$, i.e., $B_{i^*j}^{(0)}(t) \geq \sigma$ for $j \notin I_{i^*}^{(B)}$ or $B_{i^*j}^{(0)}(t) + \widehat{B}_{i^*j}(t) \geq \sigma$ for $j \in I_{i^*}^{(B)}$ by (17). Therefore, we obtain

$$\begin{aligned} &\sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot w_i^{(1)}(t) + \sum_{\{i:j \in I_i^{(B)}\}} \widehat{B}_{ij} \cdot w_i^{(1)}(t) \\ &\quad \geq \sum_{i \in I_>} B_{ij}^{(0)}(t) \cdot w_i^{(1)}(t) + \sum_{\{i:j \in I_i^{(B)}, i \in I_>\}} \widehat{B}_{ij} \cdot w_i^{(1)}(t) \\ &\quad = \sum_{i \in I_>} B_{ij}^{(0)}(t) \cdot \eta(t) + \sum_{\{i:j \in I_i^{(B)}, i \in I_>\}} \widehat{B}_{ij} \cdot \eta(t) \geq \sigma \cdot \eta(t). \end{aligned} \tag{68}$$

From (64), we see that

$$\sigma \cdot \eta(t) = \tau + \nu \cdot \int_t^T \eta(s)ds,$$

which implies

$$\sigma \cdot \eta(t) \geq \left\{ \begin{array}{ll} a_j^{(0)}(t) + \sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot \eta(s)ds - \sum_{\{i:j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot \eta(s)ds & \text{for } j \notin I^{(\mathbf{a})} \\ a_j^{(0)}(t) - \widehat{a}_j(t) + \sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot \eta(s)ds - \sum_{\{i:j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot \eta(s)ds & \text{for } j \in I^{(\mathbf{a})}. \end{array} \right\} \tag{69}$$

for all $t \in [0,T]$. Using (68) and (69), and the facts that $w_i^{(1)}(t) \leq \eta(t)$ and $K_{ij}^{(0)}(s,t) - \widehat{K}_{ij}(s,t) \geq 0$ for $j \in I_i^{(K)}$, we also have

$$\sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot w_i^{(1)}(t) + \sum_{\{i:j \in I_i^{(B)}\}} \widehat{B}_{ij} \cdot w_i^{(1)}(t)$$
$$\geq \left\{ \begin{array}{ll} a_j^{(0)}(t) + \sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot \eta(s)ds - \sum_{\{i:j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot \eta(s)ds & \text{for } j \notin I^{(\mathbf{a})} \\ a_j^{(0)}(t) - \widehat{a}_j(t) + \sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot \eta(s)ds - \sum_{\{i:j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot \eta(s)ds & \text{for } j \in I^{(\mathbf{a})}. \end{array} \right\}$$
$$\geq \left\{ \begin{array}{ll} a_j^{(0)}(t) + \sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot w_i^{(1)}(s)ds - \sum_{\{i:j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot w_i^{(1)}(s)ds & \text{for } j \notin I^{(\mathbf{a})} \\ a_j^{(0)}(t) - \widehat{a}_j(t) + \sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot w_i^{(1)}(s)ds - \sum_{\{i:j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot w_i^{(1)}(s)ds & \text{for } j \in I^{(\mathbf{a})}. \end{array} \right\}$$

This concludes that $\mathbf{w}^{(1)}$ is a feasible solution of (DRCLP3), and the proof is complete. □

Lemma 7 (Riesz and Sz.-Nagy [54] (p. 64))**.** *Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in $L^2[0,T]$. If the sequence $\{f_k\}_{k=1}^{\infty}$ is uniformly bounded with respect to $\|\cdot\|_2$, then a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ exists that weakly converges to $f \in L^2[0,T]$. In other words, for any $g \in L^2[0,T]$, we have*

$$\lim_{j\to\infty} \int_0^T f_{k_j}(t)g(t)dt = \int_0^T f(t)g(t)dt.$$

Lemma 8 (Levinson [4])**.** *If the sequence $\{f_k\}_{k=1}^{\infty}$ is uniformly bounded on $[0,T]$ with respect to $\|\cdot\|_2$ and weakly converges to $f \in L^2[0,T]$, then*

$$f(t) \le \limsup_{k\to\infty} f_k(t) \text{ and } f(t) \ge \liminf_{k\to\infty} f_k(t) \text{ a.e. on } [0,T].$$

Theorem 3. *Suppose that the primal problem (P_n) is feasible. According to (26) and (53), let $\{\widehat{\mathbf{z}}^{(n)}\}_{n=1}^{\infty}$ and $\{\widehat{\mathbf{w}}^{(n)}\}_{n=1}^{\infty}$ be the sequences that are constructed from the optimal solutions $\bar{\mathbf{z}}^{(n)}$ of primal problem (P_n) and the optimal solution $\bar{\mathbf{w}}^{(n)}$ of dual problem (D_n), respectively. Then, the following statements hold true.*

(i) *There exists a subsequence $\{\widehat{\mathbf{z}}^{(n_k)}\}_{k=1}^{\infty}$ of $\{\widehat{\mathbf{z}}^{(n)}\}_{n=1}^{\infty}$ such that $\{\widehat{\mathbf{z}}^{(n_k)}\}_{k=1}^{\infty}$ is weakly convergent to an optimal solution $\widehat{\mathbf{z}}^*$ of primal problem* (RCLP3)*.*

(ii) *For each n, we define*

$$\breve{w}_i^{(n)}(t) = \min\left\{\widehat{w}_i^{(n)}(t), \eta(t)\right\}.$$

Then, there exists a subsequence $\{\breve{\mathbf{w}}^{(n_k)}\}_{k=1}^{\infty}$ of $\{\breve{\mathbf{w}}^{(n)}\}_{n=1}^{\infty}$ such that $\{\breve{\mathbf{w}}^{(n_k)}\}_{k=1}^{\infty}$ is weakly convergent to an optimal solution $\widehat{\mathbf{w}}^*$ of dual problem (DRCLP3)*.*

Proof. Proposition 2 states that the sequence of functions $\{\widehat{\mathbf{z}}^{(n)}\}_{n=1}^{\infty}$ is uniformly bounded with respect to $\|\cdot\|_2$. We write $\widehat{z}_j^{(n)}$ to denote the jth component of $\widehat{\mathbf{z}}^{(n)}$. Lemma 7 says that there exists a subsequence $\left\{\widehat{z}_1^{(n_k^{(1)})}\right\}_{k=1}^{\infty}$ of $\left\{\widehat{z}_1^{(n)}\right\}_{n=1}^{\infty}$ that weakly converges to some $\widehat{z}_1^{(0)} \in L^2[0,T]$. Using Lemma 7 again, there exists a subsequence $\left\{\widehat{z}_2^{(n_k^{(2)})}\right\}_{k=1}^{\infty}$ of $\left\{\widehat{z}_2^{(n_k^{(1)})}\right\}_{k=1}^{\infty}$ that weakly converges to some $\widehat{z}_2^{(0)} \in L^2[0,T]$. By induction, there exists a subsequence $\left\{\widehat{z}_j^{(n_k^{(j)})}\right\}_{k=1}^{\infty}$ of $\left\{\widehat{z}_j^{(n_k^{(j-1)})}\right\}_{k=1}^{\infty}$ that weakly converges to some $\widehat{z}_j^{(0)} \in L^2[0,T]$ for each j. Therefore, we can construct a subsequence $\{\widehat{\mathbf{z}}^{(n_k)}\}_{k=1}^{\infty}$ that weakly converges to $\widehat{\mathbf{z}}^{(0)}$. Since $\widehat{\mathbf{z}}^{(n_k)}$ is a feasible solution of problem (RCLP3) for each n_k, we have $\widehat{\mathbf{z}}^{(n_k)}(t) \ge \mathbf{0}$ and

$$\begin{aligned} &\sum_{j=1}^{q} B_{ij}^{(0)}(t)\cdot \widehat{z}_j^{(n_k)}(t) + \sum_{\{j:j\in I_i^{(B)}\}} \widehat{B}_{ij}(t)\cdot \widehat{z}_j^{(n_k)}(t) \\ &\le \left\{\begin{array}{ll} c_i^{(0)}(t) - \widehat{c}_i(t) + \sum_{j=1}^{q}\int_0^t K_{ij}^{(0)}(t,s)\cdot \widehat{z}_j^{(n_k)}(s)ds - \sum_{\{j:j\in I_i^{(K)}\}}\int_0^t \widehat{K}_{ij}(t,s)\cdot \widehat{z}_j^{(n_k)}(s)ds & \text{for } i \in I^{(\mathbf{c})} \\ c_i^{(0)}(t) + \sum_{j=1}^{q}\int_0^t K_{ij}^{(0)}(t,s)\cdot \widehat{z}_j^{(n_k)}(s)ds - \sum_{\{j:j\in I_i^{(K)}\}}\int_0^t \widehat{K}_{ij}(t,s)\cdot \widehat{z}_j^{(n_k)}(s)ds & \text{for } i \notin I^{(\mathbf{c})} \end{array}\right\}. \end{aligned} \tag{70}$$

From Lemma 8, for each j, we have

$$\limsup_{k\to\infty} \widehat{z}_j^{(n_k)}(t) \ge \widehat{z}_j^{(0)}(t) \ge \liminf_{k\to\infty} \widehat{z}_j^{(n_k)}(t) \ge 0 \text{ a.e. in } [0,T]. \tag{71}$$

Therefore, we obtain $\widehat{\mathbf{z}}^{(0)}(t) \ge \mathbf{0}$ a.e. in $[0,T]$. Since $B_{ij}^{(0)} \ge 0$ and $\widehat{B}_{ij} \ge 0$, for $i \in I^{(\mathbf{c})}$, by taking the limit superior on both sides of (70), we obtain

$$\begin{aligned}
&\sum_{j=1}^{q} B_{ij}^{(0)}(t)\cdot\widehat{z}_j^{(0)}(t)+\sum_{\{j:j\in I_i^{(B)}\}}\widehat{B}_{ij}(t)\cdot\widehat{z}_j^{(0)}(t)\\
&\le \limsup_{k\to\infty}\left[\sum_{j=1}^{q} B_{ij}^{(0)}(t)\cdot\widehat{z}_j^{(n_k)}(t)+\sum_{\{j:j\in I_i^{(B)}\}}\widehat{B}_{ij}(t)\cdot\widehat{z}_j^{(n_k)}(t)\right] \text{ (by (71))}\\
&\le c_i^{(0)}(t)-\widehat{c}_i(t)+\limsup_{k\to\infty}\left[\sum_{j=1}^{q}\int_0^t K_{ij}^{(0)}(t,s)\cdot\widehat{z}_j^{(n_k)}(s)ds-\sum_{\{j:j\in I_i^{(K)}\}}\int_0^t \widehat{K}_{ij}(t,s)\cdot\widehat{z}_j^{(n_k)}(s)ds\right] \text{ (by (70))}\\
&= c_i^{(0)}(t)-\widehat{c}_i(t)+\sum_{j=1}^{q}\int_0^t K_{ij}^{(0)}(t,s)\cdot\widehat{z}_j^{(0)}(s)ds-\sum_{\{j:j\in I_i^{(K)}\}}\int_0^t \widehat{K}_{ij}(t,s)\cdot\widehat{z}_j^{(0)}(s)ds \text{ a.e. in } [0,T] \qquad (72)
\end{aligned}$$

(by the weak convergence)

For $i \notin I^{(\mathbf{c})}$, we can similarly obtain

$$\begin{aligned}
&\sum_{j=1}^{q} B_{ij}^{(0)}(t)\cdot\widehat{z}_j^{(0)}(t)+\sum_{\{j:j\in I_i^{(B)}\}}\widehat{B}_{ij}(t)\cdot\widehat{z}_j^{(0)}(t)\\
&\le c_i^{(0)}(t)+\sum_{j=1}^{q}\int_0^t K_{ij}^{(0)}(t,s)\cdot\widehat{z}_j^{(0)}(s)ds-\sum_{\{j:j\in I_i^{(K)}\}}\int_0^t \widehat{K}_{ij}(t,s)\cdot\widehat{z}_j^{(0)}(s)ds. \qquad (73)
\end{aligned}$$

Let $\mathcal{N}_0$ be a subset of $[0,T]$ on which inequalities (72) and (73) are violated, let $\mathcal{N}_1$ be a subset of $[0,T]$ on which $\widehat{\mathbf{z}}^{(0)}(t) \not\geq \mathbf{0}$, and let $\mathcal{N}=\mathcal{N}_0\cup\mathcal{N}_1$. We define

$$\widehat{\mathbf{z}}^*(t)=\begin{cases}\widehat{\mathbf{z}}^{(0)}(t) & \text{if } t\notin\mathcal{N}\\ \mathbf{0} & \text{if } t\in\mathcal{N},\end{cases}$$

where the set $\mathcal{N}$ has a measure of zero. It is clear that $\widehat{\mathbf{z}}^*(t)\geq\mathbf{0}$ for all $t\in[0,T]$ and that $\widehat{\mathbf{z}}^*(t)=\widehat{\mathbf{z}}^{(0)}(t)$ a.e. on $[0,T]$, i.e., $\widehat{z}_j^*(t)=\widehat{z}_j^{(0)}(t)$ a.e. on $[0,T]$ for each j. We also see that $\widehat{z}_j^*\in L^2[0,T]$ for each j. Now, we consider two cases.

- For $t\notin\mathcal{N}$, from (72), we have

$$\begin{aligned}
&\sum_{j=1}^{q} B_{ij}^{(0)}(t)\cdot\widehat{z}_j^{*}(t)+\sum_{\{j:j\in I_i^{(B)}\}}\widehat{B}_{ij}(t)\cdot\widehat{z}_j^{*}(t)=\sum_{j=1}^{q} B_{ij}^{(0)}(t)\cdot\widehat{z}_j^{(0)}(t)+\sum_{\{j:j\in I_i^{(B)}\}}\widehat{B}_{ij}(t)\cdot\widehat{z}_j^{(0)}(t)\\
&\le c_i^{(0)}(t)-\widehat{c}_i(t)+\sum_{j=1}^{q}\int_0^t K_{ij}^{(0)}(t,s)\cdot\widehat{z}_j^{(0)}(s)ds-\sum_{\{j:j\in I_i^{(K)}\}}\int_0^t \widehat{K}_{ij}(t,s)\cdot\widehat{z}_j^{(0)}(s)ds\\
&\le c_i^{(0)}(t)-\widehat{c}_i(t)+\sum_{j=1}^{q}\int_0^t K_{ij}^{(0)}(t,s)\cdot\widehat{z}_j^{*}(s)ds-\sum_{\{j:j\in I_i^{(K)}\}}\int_0^t \widehat{K}_{ij}(t,s)\cdot\widehat{z}_j^{*}(s)ds.
\end{aligned}$$

  From (73), we can similarly obtain

$$\begin{aligned}
&\sum_{j=1}^{q} B_{ij}^{(0)}(t)\cdot\widehat{z}_j^{*}(t)+\sum_{\{j:j\in I_i^{(B)}\}}\widehat{B}_{ij}(t)\cdot\widehat{z}_j^{*}(t)\\
&\le c_i^{(0)}(t)+\sum_{j=1}^{q}\int_0^t K_{ij}^{(0)}(t,s)\cdot\widehat{z}_j^{*}(s)ds-\sum_{\{j:j\in I_i^{(K)}\}}\int_0^t \widehat{K}_{ij}(t,s)\cdot\widehat{z}_j^{*}(s)ds.
\end{aligned}$$

- For $t \in \mathcal{N}$, from (72), we have

$$\begin{aligned}
&\sum_{j=1}^{q} B_{ij}^{(0)}(t) \cdot \widehat{z}_j^*(t) + \sum_{\{j: j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot \widehat{z}_j^*(t) = 0 \\
&\quad \leq c_i^{(0)}(t) - \widehat{c}_i(t) + \sum_{j=1}^{q} \int_0^t K_{ij}^{(0)}(t,s) \cdot \widehat{z}_j^{(0)}(s)ds - \sum_{\{j: j \in I_i^{(K)}\}} \int_0^t \widehat{K}_{ij}(t,s) \cdot \widehat{z}_j^{(0)}(s)ds \\
&\quad \leq c_i^{(0)}(t) - \widehat{c}_i(t) + \sum_{j=1}^{q} \int_0^t K_{ij}^{(0)}(t,s) \cdot \widehat{z}_j^*(s)ds - \sum_{\{j: j \in I_i^{(K)}\}} \int_0^t \widehat{K}_{ij}(t,s) \cdot \widehat{z}_j^*(s)ds.
\end{aligned}$$

From (73), we can similarly obtain

$$\begin{aligned}
&\sum_{j=1}^{q} B_{ij}^{(0)}(t) \cdot \widehat{z}_j^*(t) + \sum_{\{j: j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot \widehat{z}_j^*(t) \\
&\quad \leq c_i^{(0)}(t) + \sum_{j=1}^{q} \int_0^t K_{ij}^{(0)}(t,s) \cdot \widehat{z}_j^*(s)ds - \sum_{\{j: j \in I_i^{(K)}\}} \int_0^t \widehat{K}_{ij}(t,s) \cdot \widehat{z}_j^*(s)ds.
\end{aligned}$$

This shows that $\widehat{\mathbf{z}}^*$ is a feasible solution of problem (RCLP3). Since $\widehat{\mathbf{z}}^*(t) = \widehat{\mathbf{z}}^{(0)}(t)$ a.e. on $[0,T]$, it follows that the subsequence $\{\widehat{\mathbf{z}}^{(n_k)}\}_{k=1}^{\infty}$ is also weakly convergent to $\widehat{\mathbf{z}}^*$.

Using the feasibility of $\widehat{\mathbf{w}}^{(n)}$ for problem (DRCLP3), Lemma 6 states that $\breve{\mathbf{w}}^{(n)}$ is also a feasible solution of problem (DRCLP3) for each n satisfying $\breve{w}_i^{(n)}(t) \leq \widehat{w}_i^{(n)}(t)$ for each $i = 1, \cdots, p$ and $t \in [0,T]$. Remark 3 states that the sequence $\{\widehat{\mathbf{w}}^{(n)}\}_{n=1}^{\infty}$ is uniformly bounded. Since

$$\eta(t) = \frac{\tau}{\sigma} \cdot \exp\left[\frac{\nu \cdot (T-t)}{\sigma}\right] \leq \frac{\tau}{\sigma} \cdot \exp\left(\frac{\nu \cdot T}{\sigma}\right) \text{ for all } t \in [0,T],$$

it follows that the sequence $\{\breve{\mathbf{w}}^{(n)}\}_{n=1}^{\infty}$ is also uniformly bounded, which implies that the sequence $\{\breve{\mathbf{w}}^{(n)}\}_{n=1}^{\infty}$ is uniformly bounded with respect to $\|\cdot\|_2$. Using Lemma 7, we can similarly show that there is a subsequence $\{\breve{\mathbf{w}}^{(n_k)}\}_{k=1}^{\infty}$ of $\{\breve{\mathbf{w}}^{(n)}\}_{n=1}^{\infty}$ that weakly converges to some $\breve{\mathbf{w}}^{(0)}$. The feasibility of $\breve{\mathbf{w}}^{(n_k)}$ for problem (DRCLP3) states that $\breve{\mathbf{w}}^{(n_k)}(t) \geq \mathbf{0}$ and

$$\begin{aligned}
&\sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot \breve{w}_i^{(n_k)}(t) + \sum_{\{i: j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot \breve{w}_i^{(n_k)}(t) \\
&\geq \left\{ \begin{array}{ll}
a_j^{(0)}(t) - \widehat{a}_j(t) + \sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot \breve{w}_i^{(n_k)}(s)ds - \sum_{\{i: j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot \breve{w}_i^{(n_k)}(s)ds & \text{for } j \in I^{(\mathbf{a})} \\
a_j^{(0)}(t) + \sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot \breve{w}_i^{(n_k)}(s)ds - \sum_{\{i: j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot \breve{w}_i^{(n_k)}(s)ds & \text{for } j \notin I^{(\mathbf{a})}
\end{array} \right\}.
\end{aligned} \tag{74}$$

From Lemma 8, for each i, we have

$$\limsup_{k \to \infty} \breve{w}_i^{(n_k)}(t) \geq \breve{w}_i^{(0)}(t) \geq \liminf_{k \to \infty} \breve{w}_i^{(n_k)}(t) \geq 0 \text{ a.e. in } [0,T], \tag{75}$$

which suggests that $\breve{\mathbf{w}}^{(0)}(t) \geq \mathbf{0}$ a.e. in $[0,T]$. Since $B_{ij}^{(0)} \geq 0$ and $\widehat{B}_{ij} \geq 0$, for $j \in I^{(\mathbf{a})}$, by taking the limit inferior on both sides of (74), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{p} B_{ij}^{(0)}(t)\cdot \breve{w}_i^{(0)}(t) + \sum_{\{i:j\in I_i^{(B)}\}} \widehat{B}_{ij}(t)\cdot \breve{w}_i^{(0)}(t) \\
&\quad \geq \liminf_{n_k\to\infty}\left[\sum_{i=1}^{p} B_{ij}^{(0)}(t)\cdot \breve{w}_i^{(n_k)}(t) + \sum_{\{i:j\in I_i^{(B)}\}} \widehat{B}_{ij}(t)\cdot \breve{w}_i^{(n_k)}(t)\right] \\
&\quad \geq a_j^{(0)}(t) - \widehat{a}_j(t) + \liminf_{n_k\to\infty}\left[\sum_{i=1}^{p}\int_t^T K_{ij}^{(0)}(s,t)\cdot \breve{w}_i^{(n_k)}(s)ds - \sum_{\{i:j\in I_i^{(K)}\}}\int_t^T \widehat{K}_{ij}(s,t)\cdot \breve{w}_i^{(n_k)}(s)ds\right] \\
&\quad = a_j^{(0)}(t) - \widehat{a}_j(t) + \sum_{i=1}^{p}\int_t^T K_{ij}^{(0)}(s,t)\cdot \breve{w}_i^{(0)}(s)ds - \sum_{\{i:j\in I_i^{(K)}\}}\int_t^T \widehat{K}_{ij}(s,t)\cdot \breve{w}_i^{(0)}(s)ds \text{ a.e. in } [0,T] \\
&\quad \text{(by the weak convergence)} \qquad (76)
\end{aligned}
$$

For $j \notin I^{(\mathbf{a})}$, we can similarly obtain

$$
\begin{aligned}
&\sum_{i=1}^{p} B_{ij}^{(0)}(t)\cdot \breve{w}_i^{(0)}(t) + \sum_{\{i:j\in I_i^{(B)}\}} \widehat{B}_{ij}(t)\cdot \breve{w}_i^{(0)}(t) \\
&\quad \geq a_j^{(0)}(t) + \sum_{i=1}^{p}\int_t^T K_{ij}^{(0)}(s,t)\cdot \breve{w}_i^{(0)}(s)ds - \sum_{\{i:j\in I_i^{(K)}\}}\int_t^T \widehat{K}_{ij}(s,t)\cdot \breve{w}_i^{0)}(s)ds \qquad (77)
\end{aligned}
$$

We define $\boldsymbol{\eta}(t) = \eta(t)\mathbf{1}_p$. Then, we see that $\breve{\mathbf{w}}^{(n_k)}(t) \leq \boldsymbol{\eta}(t)$ for each k and for all $t \in [0,T]$.

Let $\widehat{\mathcal{N}}_0$ be a subset of $[0,T]$ on which the inequalities (76) and (77) are violated, let $\widehat{\mathcal{N}}_1$ be a subset of $[0,T]$ on which $\breve{\mathbf{w}}^{(0)}(t) \not\geq \mathbf{0}$, and let $\widehat{\mathcal{N}} = \widehat{\mathcal{N}}_0 \cup \widehat{\mathcal{N}}_1$. We define

$$
\widehat{\mathbf{w}}^*(t) = \begin{cases} \breve{\mathbf{w}}^{(0)}(t) & \text{if } t \notin \widehat{\mathcal{N}} \\ \boldsymbol{\eta}(t) & \text{if } t \in \widehat{\mathcal{N}}, \end{cases}
$$

where the set $\widehat{\mathcal{N}}$ has a measure of zero. Then, we see that $\widehat{\mathbf{w}}^*(t) \geq \mathbf{0}$ for all $t \in [0,T]$ and that $\widehat{\mathbf{w}}^*(t) = \breve{\mathbf{w}}^{(0)}(t)$ a.e. on $[0,T]$. Now, we claim that $\widehat{\mathbf{w}}^*$ is a feasible solution of (DRCLP3). We consider two cases.

- Suppose that $t \notin \widehat{\mathcal{N}}$. For $j \in I^{(\mathbf{a})}$, from (76), we have

$$
\begin{aligned}
&\sum_{i=1}^{p} B_{ij}^{(0)}(t)\cdot \widehat{w}_i^*(t) + \sum_{\{i:j\in I_i^{(B)}\}} \widehat{B}_{ij}(s,t)\cdot \widehat{w}_i^*(t) = \sum_{i=1}^{p} B_{ij}^{(0)}(t)\cdot \breve{w}_i^{(0)}(t) + \sum_{\{i:j\in I_i^{(B)}\}} \widehat{B}_{ij}(s,t)\cdot \breve{w}_i^{(0)}(t) \\
&\quad \geq a_j^{(0)}(t) - \widehat{a}_j(t) + \sum_{i=1}^{p}\int_t^T K_{ij}^{(0)}(s,t)\cdot \breve{w}_i^{(0)}(s)ds - \sum_{\{i:j\in I_i^{(K)}\}}\int_t^T \widehat{K}_{ij}(s,t)\cdot \breve{w}_i^{(0)}(s)ds \\
&\quad = a_j^{(0)}(t) - \widehat{a}_j(t) + \sum_{i=1}^{p}\int_t^T K_{ij}^{(0)}(s,t)\cdot \widehat{w}_i^*(s)ds - \sum_{\{i:j\in I_i^{(K)}\}}\int_t^T \widehat{K}_{ij}(s,t)\cdot \widehat{w}_i^*(s)ds.
\end{aligned}
$$

For $j \notin I^{(\mathbf{a})}$, from (77), we can similarly obtain

$$\sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot \widehat{w}_i^*(t) + \sum_{\{i: j \in I_i^{(B)}\}} \widehat{B}_{ij}(s,t) \cdot \widehat{w}_i^*(t)$$
$$\geq a_j^{(0)}(t) + \sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot \widehat{w}_i^*(s) ds - \sum_{\{i: j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot \widehat{w}_i^*(s) ds.$$

- Suppose that $t \in \widehat{\mathcal{N}}$. For each $i = 1, \cdots, p$, since $\breve{w}_i^{(n_k)}(t) \leq \eta(t)$ for all $t \in [0,T]$, using Lemma 8, we have

$$\breve{w}_i^{(0)}(t) \leq \limsup_{n_k \to \infty} \breve{w}_i^{(n_k)}(t) \leq \eta(t) \text{ a.e. on } [0,T].$$

For each $i = 1, \cdots, p$, since $\widehat{w}_i^*(t) = \breve{w}_i^{(0)}(t)$ a.e. on $[0,T]$, it follows that

$$\widehat{w}_i^*(t) \leq \eta(t) \text{ a.e. on } [0,T]. \tag{78}$$

Therefore, for $j \in I^{(\mathbf{a})}$, we obtain

$$\sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot \widehat{w}_i^*(t) + \sum_{\{i: j \in I_i^{(B)}\}} \widehat{B}_{ij}(s,t) \cdot \widehat{w}_i^*(t) = \sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot \eta(t) + \sum_{\{i: j \in I_i^{(B)}\}} \widehat{B}_{ij}(s,t) \cdot \eta(t)$$
$$\geq \sigma \cdot \eta(t) \text{ (by (68))}$$
$$\geq a_j^{(0)}(t) - \widehat{a}_j(t) + \sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot \eta(s) ds - \sum_{\{i: j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot \eta(s) ds \text{ (by (69))}$$
$$\geq a_j^{(0)}(t) - \widehat{a}_j(t) + \sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot \widehat{w}_i^*(s) ds - \sum_{\{i: j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot \widehat{w}_i^*(s) ds \text{ (by (78))}.$$

For $j \notin I^{(\mathbf{a})}$, we can similarly obtain

$$\sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot \widehat{w}_i^*(t) + \sum_{\{i: j \in I_i^{(B)}\}} \widehat{B}_{ij}(s,t) \cdot \widehat{w}_i^*(t)$$
$$\geq a_j^{(0)}(t) + \sum_{i=1}^{p} \int_t^T K_{ij}^{(0)}(s,t) \cdot \widehat{w}_i^*(s) ds - \sum_{\{i: j \in I_i^{(K)}\}} \int_t^T \widehat{K}_{ij}(s,t) \cdot \widehat{w}_i^*(s) ds.$$

The above two cases conclude that $\widehat{\mathbf{w}}^*$ is indeed a feasible solution of (DRCLP3). Since $\widehat{\mathbf{w}}^*(t) = \breve{\mathbf{w}}^{(0)}(t)$ a.e. on $[0,T]$, it follows that the subsequence $\{\breve{\mathbf{w}}^{(n_k)}\}_{k=1}^{\infty}$ is also weakly convergent to $\widehat{\mathbf{w}}^*$.

Finally, we prove the optimality. Now, we have

$$
\begin{aligned}
&\sum_{j=1}^{q}\int_0^T a_j^{(0)}(t)\cdot\widehat{z}_j^{(n_k)}(t)dt-\sum_{j\in I^{(\mathfrak{a})}}\int_0^T\widehat{a}_j(t)\cdot\widehat{z}_j^{(n_k)}(t)dt\\
&=\sum_{j\notin I^{(\mathfrak{a})}}\int_0^T a_j^{(0)}(t)\cdot\widehat{z}_j^{(n_k)}(t)dt+\sum_{j\in I^{(\mathfrak{a})}}\int_0^T\left(a_j^{(0)}(t)-\widehat{a}_j(t)\right)\cdot\widehat{z}_j^{(n_k)}(t)dt\\
&=\sum_{j\notin I^{(\mathfrak{a})}}\sum_{l=1}^{n_k}\int_{\bar{E}_l^{(n_k)}}\left(a_j^{(0)}(t)-a_{lj}^{(n_k)}\right)\cdot\widehat{z}_j^{(n_k)}(t)dt\\
&\quad+\sum_{j\in I^{(\mathfrak{a})}}\sum_{l=1}^{n_k}\int_{\bar{E}_l^{(n_k)}}\left(a_j^{(0)}(t)-\widehat{a}_j(t)-a_{lj}^{(n_k)}\right)\cdot\widehat{z}_j^{(n_k)}(t)dt+\sum_{j=1}^{q}\sum_{l=1}^{n_k}\int_{\bar{E}_l^{(n_k)}}a_{lj}^{(n_k)}\cdot\widehat{z}_j^{(n_k)}(t)dt\\
&=\sum_{j\notin I^{(\mathfrak{a})}}\sum_{l=1}^{n_k}\int_{\bar{E}_l^{(n_k)}}\left(a_j^{(0)}(t)-a_{lj}^{(n_k)}\right)\cdot\bar{z}_{lj}^{(n_k)}dt\\
&\quad+\sum_{j\in I^{(\mathfrak{a})}}\sum_{l=1}^{n_k}\int_{\bar{E}_l^{(n_k)}}\left(a_j^{(0)}(t)-\widehat{a}_j(t)-a_{lj}^{(n_k)}\right)\cdot\bar{z}_{lj}^{(n_k)}dt+\sum_{j=1}^{q}\sum_{l=1}^{n_k}\mathfrak{d}_l^{(n_k)}\cdot a_{lj}^{(n_k)}\cdot\bar{z}_{lj}^{(n_k)}dt\\
&=\sum_{j\notin I^{(\mathfrak{a})}}\sum_{l=1}^{n_k}\int_{\bar{E}_l^{(n_k)}}\left(a_j^{(0)}(t)-a_{lj}^{(n_k)}\right)\cdot\bar{z}_{lj}^{(n_k)}dt\\
&\quad+\sum_{j\in I^{(\mathfrak{a})}}\sum_{l=1}^{n_k}\int_{\bar{E}_l^{(n_k)}}\left(a_j^{(0)}(t)-\widehat{a}_j(t)-a_{lj}^{(n_k)}\right)\cdot\bar{z}_{lj}^{(n_k)}dt+V(\mathrm{P}_{n_k})
\end{aligned}\tag{79}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{p}\int_0^T c_i^{(0)}(t)\cdot\widehat{w}_i^{(n_k)}(t)dt-\sum_{i\in I^{(\mathfrak{c})}}\int_0^T\widehat{c}_i(t)\cdot\widehat{w}_i^{(n_k)}(t)dt\\
&=\sum_{i\notin I^{(\mathfrak{c})}}\int_0^T c_i^{(0)}(t)\cdot\widehat{w}_i^{(n_k)}(t)dt+\sum_{i\in I^{(\mathfrak{c})}}\int_0^T\left(c_i^{(0)}(t)-\widehat{c}_i(t)\right)\cdot\widehat{w}_i^{(n_k)}(t)dt\\
&=\sum_{i\notin I^{(\mathfrak{c})}}\sum_{l=1}^{n_k}\int_{\bar{E}_l^{(n_k)}}c_i^{(0)}(t)\cdot\bar{w}_{li}^{(n_k)}dt\;+\sum_{i\in I^{(\mathfrak{c})}}\sum_{l=1}^{n_k}\int_{\bar{E}_l^{(n_k)}}\left(c_i^{(0)}(t)-\widehat{c}_i(t)\right)\cdot\bar{w}_{li}^{(n_k)}dt\\
&\quad+\sum_{i\notin I^{(\mathfrak{c})}}\int_0^T c_i^{(0)}(t)\cdot\mathfrak{f}^{(n_k)}(t)dt+\sum_{i\in I^{(\mathfrak{c})}}\int_0^T\left(c_i^{(0)}(t)-\widehat{c}_i(t)\right)\cdot\mathfrak{f}^{(n_k)}(t)dt\\
&=\sum_{i\notin I^{(\mathfrak{c})}}\sum_{l=1}^{n_k}\int_{\bar{E}_l^{(n_k)}}\left(c_i^{(0)}(t)-c_{li}^{(n_k)}\right)\cdot\bar{w}_{li}^{(n_k)}dt\\
&\quad+\sum_{i\in I^{(\mathfrak{c})}}\sum_{l=1}^{n_k}\int_{\bar{E}_l^{(n_k)}}\left(c_i^{(0)}(t)-\widehat{c}_i(t)-c_{li}^{(n_k)}\right)\cdot\bar{w}_{li}^{(n_k)}dt+\sum_{i=1}^{p}\sum_{l=1}^{n_k}\mathfrak{d}_l^{(n_k)}\cdot c_{li}^{(n_k)}\cdot\bar{w}_{li}^{(n_k)}dt\\
&\quad+\sum_{i\notin I^{(\mathfrak{c})}}\int_0^T c_i^{(0)}(t)\cdot\mathfrak{f}^{(n_k)}(t)dt+\sum_{i\in I^{(\mathfrak{c})}}\int_0^T\left(c_i^{(0)}(t)-\widehat{c}_i(t)\right)\cdot\mathfrak{f}^{(n_k)}(t)dt\\
&=\sum_{i\notin I^{(\mathfrak{c})}}\sum_{l=1}^{n_k}\int_{\bar{E}_l^{(n_k)}}\left(c_i^{(0)}(t)-c_{li}^{(n_k)}\right)\cdot\bar{w}_{li}^{(n_k)}dt\\
&\quad+\sum_{i\in I^{(\mathfrak{c})}}\sum_{l=1}^{n_k}\int_{\bar{E}_l^{(n_k)}}\left(c_i^{(0)}(t)-\widehat{c}_i(t)-c_{li}^{(n_k)}\right)\cdot\bar{w}_{li}^{(n_k)}dt+V(\mathrm{D}_{n_k})\\
&\quad+\sum_{i\notin I^{(\mathfrak{c})}}\int_0^T c_i^{(0)}(t)\cdot\mathfrak{f}^{(n_k)}(t)dt+\sum_{i\in I^{(\mathfrak{c})}}\int_0^T\left(c_i^{(0)}(t)-\widehat{c}_i(t)\right)\cdot\mathfrak{f}^{(n_k)}(t)dt.
\end{aligned}\tag{80}
$$

Since $V(\mathrm{P}_{n_k}) = V(\mathrm{D}_{n_k})$ and $\breve{w}_i^{(n_k)} \leq \widehat{w}_i^{(n_k)}$ for each i and n_k, from (79) and (80), we have

$$\begin{aligned}
&\sum_{j=1}^{q} \int_0^T a_j^{(0)}(t) \cdot \widehat{z}_j^{(n_k)}(t)dt - \sum_{j \in I^{(\mathbf{a})}} \int_0^T \widehat{a}_j(t) \cdot \widehat{z}_j^{(n_k)}(t)dt \\
&\quad - \sum_{j \notin I^{(\mathbf{a})}} \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left(a_j^{(0)}(t) - a_{lj}^{(n_k)} \right) \cdot \bar{z}_{lj}^{(n_k)} dt \\
&\quad - \sum_{j \in I^{(\mathbf{a})}} \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left(a_j^{(0)}(t) - \widehat{a}_j(t) - a_{lj}^{(n_k)} \right) \cdot \bar{z}_{lj}^{(n_k)} dt \\
&\geq \sum_{i=1}^{p} \int_0^T c_i^{(0)}(t) \cdot \breve{w}_i^{(n_k)}(t)dt - \sum_{i \in I^{(\mathbf{c})}} \int_0^T \widehat{c}_i(t) \cdot \breve{w}_i^{(n_k)}(t)dt \\
&\quad - \sum_{i \notin I^{(\mathbf{c})}} \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left(c_i^{(0)}(t) - c_{li}^{(n_k)} \right) \cdot \bar{w}_{li}^{(n_k)} dt \\
&\quad - \sum_{i \in I^{(\mathbf{c})}} \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left(c_i^{(0)}(t) - \widehat{c}_i(t) - c_{li}^{(n_k)} \right) \cdot \bar{w}_{li}^{(n_k)} dt \\
&\quad - \sum_{i \notin I^{(\mathbf{c})}} \int_0^T c_i^{(0)}(t) \cdot \mathfrak{f}^{(n_k)}(t)dt - \sum_{i \in I^{(\mathbf{c})}} \int_0^T \left(c_i^{(0)}(t) - \widehat{c}_i(t) \right) \cdot \mathfrak{f}^{(n_k)}(t)dt. \qquad (81)
\end{aligned}$$

Using Lemma 5, for $j \notin I^{(\mathbf{a})}$, we have

$$\begin{aligned}
0 &\leq \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left(a_j^{(0)}(t) - a_{lj}^{(n_k)} \right) \cdot \bar{z}_{lj}^{(n_k)} dt \\
&= \int_0^T \left(a_j^{(0)}(t) - \bar{a}_j^{(n_k)}(t) \right) \cdot \widehat{z}_j^{(n_k)}(t)dt \to 0 \text{ as } k \to \infty \qquad (82)
\end{aligned}$$

and, for $j \in I^{(\mathbf{a})}$, we have

$$\begin{aligned}
0 &\leq \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left(a_j^{(0)}(t) - \widehat{a}_j(t) - a_{lj}^{(n_k)} \right) \cdot \bar{z}_{lj}^{(n_k)} dt \\
&= \int_0^T \left(a_j^{(0)}(t) - \widehat{a}_j(t) - \bar{a}_j^{(n_k)}(t) \right) \cdot \widehat{z}_j^{(n_k)}(t)dt \to 0 \text{ as } k \to \infty. \qquad (83)
\end{aligned}$$

Additionally, for $i \notin I^{(\mathbf{c})}$, we have

$$\begin{aligned}
0 &\leq \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left(c_i^{(0)}(t) - c_{li}^{(n_k)} \right) \cdot \bar{w}_{li}^{(n_k)} dt \\
&= \int_0^T \left(c_i^{(0)}(t) - \bar{c}_i^{(n_k)}(t) \right) \cdot \bar{w}_i^{(n_k)}(t)dt \to 0 \text{ as } k \to \infty \qquad (84)
\end{aligned}$$

and, for $i \in I^{(\mathbf{c})}$, we have

$$\begin{aligned}
0 &\leq \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left(c_i^{(0)}(t) - \widehat{c}_i(t) - c_{li}^{(n_k)} \right) \cdot \bar{w}_{li}^{(n_k)} dt \\
&= \int_0^T \left(c_i^{(0)}(t) - \widehat{c}_i(t) - \bar{c}_i^{(n_k)}(t) \right) \cdot \bar{w}_i^{(n_k)}(t)dt \to 0 \text{ as } k \to \infty. \qquad (85)
\end{aligned}$$

By taking limit on both sides of (81) and by using (62) and (82)–(85), we obtain

$$\lim_{k\to\infty}\left[\sum_{j=1}^{q}\int_0^T a_j^{(0)}(t)\cdot\widehat{z}_j^{(n_k)}(t)dt-\sum_{j\in I^{(\mathfrak{a})}}\int_0^T\widehat{a}_j(t)\cdot\widehat{z}_j^{(n_k)}(t)dt\right]$$
$$\geq\lim_{k\to\infty}\left[\sum_{i=1}^{p}\int_0^T c_i^{(0)}(t)\cdot\breve{w}_i^{(n_k)}(t)dt-\sum_{i\in I^{(\mathfrak{c})}}\int_0^T\widehat{c}_i(t)\cdot\breve{w}_i^{(n_k)}(t)dt\right].$$

Using the weak convergence, we also obtain

$$\sum_{j=1}^{q}\int_0^T a_j^{(0)}(t)\cdot\widehat{z}_j^*(t)dt-\sum_{j\in I^{(\mathfrak{a})}}\int_0^T\widehat{a}_j(t)\cdot\widehat{z}_j^*(t)dt$$
$$\geq\sum_{i=1}^{p}\int_0^T c_i^{(0)}(t)\cdot\widehat{w}_i^*(t)dt-\sum_{i\in I^{(\mathfrak{c})}}\int_0^T\widehat{c}_i(t)\cdot\widehat{w}_i^*(t)dt.$$

According to the weak duality theorem between problems (RCLP3) and (DRCLP3), we have

$$\sum_{j=1}^{q}\int_0^T a_j^{(0)}(t)\cdot\widehat{z}_j^*(t)dt-\sum_{j\in I^{(\mathfrak{a})}}\int_0^T\widehat{a}_j(t)\cdot\widehat{z}_j^*(t)dt$$
$$=\sum_{i=1}^{p}\int_0^T c_i^{(0)}(t)\cdot\widehat{w}_i^*(t)dt-\sum_{i\in I^{(\mathfrak{c})}}\int_0^T\widehat{c}_i(t)\cdot\widehat{w}_i^*(t)dt,$$

which also suggests that $(\widehat{z}_1^*,\cdots,\widehat{z}_q^*)$ and $(\widehat{w}_1^*,\cdots,\widehat{w}_p^*)$ are the optimal solutions of problems (RCLP3) and (DRCLP3), respectively. Theorem 1 also states that $V(\text{DRCLP3})=V(\text{RCLP3})$, and the proof is complete. $\square$

6. Computational Procedure and Numerical Example

In order to obtain the approximate solutions of the continuous-time linear programming problem (RCLP3), we use Proposition 5 by considering the limit situation, which can be used to design the computational procedure. It is natural to see that the approximate solutions are the step functions. Proposition 5 shows that it is possible to obtain the appropriate step functions so that the corresponding objective function value is close enough to the optimal objective function value when n is taken to be sufficiently large.

Theorem 1 and Proposition 5 state that the error between the approximate objective value and the optimal objective value is given by

$$\varepsilon_n=-V(\mathrm{D}_n)+\sum_{i=1}^{p}\sum_{l=1}^{n}\left[\int_{\bar{E}_l^{(n)}}c_i^{(0)}(t)\cdot\bar{w}_{li}^{(n)}dt+\int_{\bar{E}_l^{(n)}}\frac{\pi_l^{(n)}}{\mathfrak{b}_l^{(n)}}\cdot\exp\left[\frac{\mathfrak{k}_l^{(n)}\cdot(T-t)}{\mathfrak{b}_l^{(n)}}\right]\cdot c_i^{(0)}(t)dt\right]$$
$$-\sum_{i\in I^{(\mathfrak{c})}}\sum_{l=1}^{n}\left[\int_{\bar{E}_l^{(n)}}\widehat{c}_i(t)\cdot\bar{w}_{li}^{(n)}dt+\int_{\bar{E}_l^{(n)}}\frac{\pi_l^{(n)}}{\mathfrak{b}_l^{(n)}}\cdot\exp\left[\frac{\mathfrak{k}_l^{(n)}\cdot(T-t)}{\mathfrak{b}_l^{(n)}}\right]\cdot\widehat{c}_i(t)dt\right]$$

In order to obtain $\pi_l^{(n)}$, by referring to (34), we need to solve the following problem:

$$\sup_{t\in E_l^{(n)}}\left\{\bar{h}_{lj}^{(n)}(t)+a_j(t)\right\},\tag{86}$$

where the real-valued function a_j is given by

$$a_j(t) = \left\{ \begin{array}{ll} a_j^{(0)}(t) & \text{for } j \notin I^{(\mathbf{a})} \\ a_j^{(0)}(t) - \widehat{a}_j(t) & \text{for } j \in I^{(\mathbf{a})}. \end{array} \right\}$$

We rewrite the real-valued function $\bar{h}_{lj}^{(n)}$ as follows:

$$\begin{aligned} \bar{h}_{lj}^{(n)}(t) &= \sum_{i=1}^{p} B_{lij}^{(n)} \cdot \bar{w}_{li}^{(n)} - \left(\sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot \bar{w}_{li}^{(n)} + \sum_{\{i:j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot \bar{w}_{li}^{(n)} \right) \\ &+ \int_t^{e_l^{(n)}} \left(\sum_{i=1}^{p} K_{ij}^{(0)}(s,t) \cdot \bar{w}_{li}^{(n)} - \sum_{\{i:j \in I_i^{(K)}\}} \widehat{K}_{ij}(s,t) \cdot \bar{w}_{li}^{(n)} \right) ds \\ &+ \sum_{k=l+1}^{n} \int_{\bar{E}_k^{(n)}} \left[\left(\sum_{i=1}^{p} K_{ij}^{(0)}(s,t) \cdot \bar{w}_{ki}^{(n)} - \sum_{\{i:j \in I_i^{(K)}\}} \widehat{K}_{ij}(s,t) \cdot \bar{w}_{ki}^{(n)} \right) - \sum_{i=1}^{p} K_{klij}^{(n)} \cdot \bar{w}_{ki}^{(n)} \right] ds. \end{aligned}$$

For $t \in F_l^{(n)}$ and $l = 1, \cdots, n$, we define the constant

$$\widehat{h}_{lj}^{(n)} = \sum_{i=1}^{p} B_{lij}^{(n)} \cdot \bar{w}_{li}^{(n)} - \sum_{k=l+1}^{n} \int_{\bar{E}_k^{(n)}} \sum_{i=1}^{p} K_{klij}^{(n)} \cdot \bar{w}_{ki}^{(n)} ds$$

and the real-valued function

$$\begin{aligned} \widetilde{h}_{lj}^{(n)}(t) = &- \left(\sum_{i=1}^{p} B_{ij}^{(0)}(t) \cdot \bar{w}_{li}^{(n)} + \sum_{\{i:j \in I_i^{(B)}\}} \widehat{B}_{ij}(t) \cdot \bar{w}_{li}^{(n)} \right) \\ &+ \int_t^{e_l^{(n)}} \left(\sum_{i=1}^{p} K_{ij}^{(0)}(s,t) \cdot \bar{w}_{li}^{(n)} - \sum_{\{i:j \in I_i^{(K)}\}} \widehat{K}_{ij}(s,t) \cdot \bar{w}_{li}^{(n)} \right) ds \\ &+ \sum_{k=l+1}^{n} \int_{\bar{E}_k^{(n)}} \left(\sum_{i=1}^{p} K_{ij}^{(0)}(s,t) \cdot \bar{w}_{ki}^{(n)} - \sum_{\{i:j \in I_i^{(K)}\}} \widehat{K}_{ij}(s,t) \cdot \bar{w}_{ki}^{(n)} \right) ds. \end{aligned} \tag{87}$$

Then, the real-valued function $\bar{h}_{lj}^{(n)}$ is given by

$$\bar{h}_{lj}^{(n)}(t) = \widehat{h}_{lj}^{(n)} + \widetilde{h}_{lj}^{(n)}(t) \text{ for } t \in F_l^{(n)}. \tag{88}$$

Now, we define the real-valued function $h_{lj}^{(n)}$ on $\bar{E}_l^{(n)}$ by

$$h_{lj}^{(n)}(t) = \begin{cases} \bar{h}_{lj}^{(n)}(t) + a_j(t), & \text{if } t \in E_l^{(n)} \\ \lim\limits_{t \to e_{l-1}^{(n)}+} \left(\bar{h}_{lj}^{(n)}(t) + a_j(t) \right), & \text{if } t = e_{l-1}^{(n)} \\ \lim\limits_{t \to e_l^{(n)}-} \left(\bar{h}_{lj}^{(n)}(t) + a_j(t) \right), & \text{if } t = e_l^{(n)} \end{cases}$$

Since $a_j^{(0)}$, $\widehat{a}_j$, $B_{ij}^{(0)}$, and $\widehat{B}_{ij}$ are continuous on $E_l^{(n)}$ and since $K_{ij}^{(0)}$ and $\widehat{K}_{ij}$ are continuous on $E_k^{(n)} \times E_l^{(n)}$ for all $l, k = 1, \cdots, n$, it follows that $\bar{h}_{lj}^{(n)} + a_j$ is also continuous on $E_l^{(n)}$.

This also suggests that $h_{lj}^{(n)}$ is continuous on the compact interval $\bar{E}_l^{(n)}$. In other words, the supremum in (86) can be obtained below

$$\sup_{t\in E_l^{(n)}}\left\{\bar{h}_{lj}^{(n)}(t)+a_j(t)\right\}=\sup_{t\in\bar{E}_l^{(n)}} h_{lj}^{(n)}(t)=\max_{t\in\bar{E}_l^{(n)}} h_{lj}^{(n)}(t). \tag{89}$$

In order to further design the computational procedure, we need to assume that $a_j^{(0)}$, $\widehat{a}_j$, $B_{ij}^{(0)}$, $K_{ij}^{(0)}$, and $\widehat{K}_{ij}$ are twice-differentiable on $[0,T]$ and $[0,T]\times[0,T]$, respectively, for the purpose of applying the Newton's method, which also suggests that $a_j^{(0)}$, $\widehat{a}_j$, $B_{ij}^{(0)}$, $K_{ij}^{(0)}$, and $\widehat{K}_{ij}$ are twice-differentiable on the open interval $E_l^{(n)}$ and open rectangle $E_k^{(n)}\times E_l^{(n)}$, respectively, for all $l,k=1,\cdots,n$. From (89), we need to solve the following simple type of optimization problem:

$$\max_{e_{l-1}^{(n)}\leq t\leq e_l^{(n)}} h_{lj}^{(n)}(t). \tag{90}$$

Then, we can see that the optimal solution is

$$t^*=e_{l-1}^{(n)}\text{ or }t^*=e_l^{(n)}\text{ or satisfying }\frac{d}{dt}\left(h_{lj}^{(n)}(t)\right)\Big|_{t=t^*}=0.$$

According to (88), it follows that the optimal solution of problem (90) is

$$t^*=e_{l-1}^{(n)}\text{ or }t^*=e_l^{(n)}\text{ or satisfying }\frac{d}{dt}\left(\widetilde{h}_{lj}^{(n)}(t)+a_j(t)\right)\Big|_{t=t^*}=0.$$

Let $Z_{lj}^{(n)}$ denote the set of all zeros of the real-valued function $\frac{d}{dt}(\widetilde{h}_{lj}^{(n)}(t)+a_j(t))$. Then,

$$\max_{t\in\bar{E}_l^{(n)}} h_{lj}^{(n)}(t)=\begin{cases}\max\left\{h_{lj}^{(n)}\left(e_{l-1}^{(n)}\right),h_{lj}^{(n)}\left(e_l^{(n)}\right),\max\limits_{t^*\in Z_{lj}^{(n)}} h_{lj}^{(n)}(t^*)\right\}, & \text{if } Z_{lj}^{(n)}\neq\emptyset\\ \max\left\{h_{lj}^{(n)}\left(e_{l-1}^{(n)}\right),h_{lj}^{(n)}\left(e_l^{(n)}\right)\right\}, & \text{if } Z_{lj}^{(n)}=\emptyset.\end{cases} \tag{91}$$

Therefore, using (89) and (91), we can obtain the desired supremum (86).

From (87), for $t\in E_l^{(n)}$, we have

$$\begin{aligned}\frac{d}{dt}\left(\widetilde{h}_{lj}^{(n)}(t)\right)=&-\left(\sum_{i=1}^{p}\frac{d}{dt}B_{ij}^{(0)}(t)\cdot\bar{w}_{li}^{(n)}+\sum_{\{i:j\in I_i^{(B)}\}}\frac{d}{dt}\widehat{B}_{ij}(t)\cdot\bar{w}_{li}^{(n)}\right)\\&+\sum_{i=1}^{p}\left[\int_t^{e_l^{(n)}}\frac{\partial}{\partial t}K_{ij}^{(0)}(s,t)\cdot\bar{w}_{li}^{(n)}ds-K_{ij}^{(0)}(t,t)\cdot\bar{w}_{li}^{(n)}\right]\\&-\sum_{\{i:j\in I_i^{(K)}\}}\left[\int_t^{e_l^{(n)}}\frac{\partial}{\partial t}\widehat{K}_{ij}(s,t)\cdot\bar{w}_{li}^{(n)}ds-\widehat{K}_{ij}(t,t)\cdot\bar{w}_{li}^{(n)}\right]\\&+\sum_{k=l+1}^{n}\int_{\bar{E}_k^{(n)}}\left(\sum_{i=1}^{p}\frac{\partial}{\partial t}K_{ij}^{(0)}(s,t)\cdot\bar{w}_{ki}^{(n)}-\sum_{\{i:j\in I_i^{(K)}\}}\frac{\partial}{\partial t}\widehat{K}_{ij}(s,t)\cdot\bar{w}_{ki}^{(n)}\right)ds.\end{aligned} \tag{92}$$

and

$$\frac{d^2}{dt^2}\left(\widetilde{h}_{lj}^{(n)}(t)\right) = -\left(\sum_{i=1}^{p}\frac{d^2}{dt^2}B_{ij}^{(0)}(t)\cdot\bar{w}_{li}^{(n)} + \sum_{\{i:j\in I_i^{(B)}\}}\frac{d^2}{dt^2}\widehat{B}_{ij}(t)\cdot\bar{w}_{li}^{(n)}\right)$$
$$-\sum_{i=1}^{p}\frac{d}{dt}\left(K_{ij}^{(0)}(t,t)\right)\cdot\bar{w}_{li}^{(n)} + \sum_{i=1}^{p}\frac{d}{dt}\left(\widehat{K}_{ij}(t,t)\right)\cdot\bar{w}_{li}^{(n)}$$
$$+\sum_{i=1}^{p}\left[\int_t^{e_l^{(n)}}\frac{\partial^2}{\partial t^2}K_{ij}^{(0)}(s,t)\cdot\bar{w}_{li}^{(n)}ds - \frac{d}{dt}K_{ij}^{(0)}(t,t)\cdot\bar{w}_{li}^{(n)}\right]$$
$$-\sum_{\{i:j\in I_i^{(K)}\}}\left[\int_t^{e_l^{(n)}}\frac{\partial^2}{\partial t^2}\widehat{K}_{ij}(s,t)\cdot\bar{w}_{li}^{(n)}ds - \frac{d}{dt}\widehat{K}_{ij}(t,t)\cdot\bar{w}_{li}^{(n)}\right]$$
$$+\sum_{k=l+1}^{n}\int_{\bar{E}_k^{(n)}}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial t^2}K_{ij}^{(0)}(s,t)\cdot\bar{w}_{ki}^{(n)} - \sum_{\{i:j\in I_i^{(K)}\}}\frac{\partial^2}{\partial t^2}\widehat{K}_{ij}(s,t)\cdot\bar{w}_{ki}^{(n)}\right)ds.$$

We consider the following cases.

- Suppose that $\widetilde{h}_{lj}^{(n)} + a_j$ is a linear function of t on $E_l^{(n)}$ assumed by

$$\widetilde{h}_{lj}^{(n)}(t) + a_j(t) = \mathfrak{a}_j\cdot t + \mathfrak{b}_j$$

for $j = 1, \cdots, q$. Using (89), we obtain

$$\max_{t\in\bar{E}_l^{(n)}} h_{lj}^{(n)}(t) = \begin{cases} \max\left\{h_{lj}^{(n)}\left(e_{l-1}^{(n)}\right), h_{lj}^{(n)}\left(e_l^{(n)}\right), \mathfrak{b}_j\right\}, & \text{if } \mathfrak{a}_j = 0 \\ \max\left\{h_{lj}^{(n)}\left(e_{l-1}^{(n)}\right), h_{lj}^{(n)}\left(e_l^{(n)}\right), \mathfrak{a}_j\cdot e_l^{(n)} + \mathfrak{b}_j\right\}, & \text{if } \mathfrak{a}_j > 0 \\ \max\left\{h_{lj}^{(n)}\left(e_{l-1}^{(n)}\right), h_{lj}^{(n)}\left(e_l^{(n)}\right), \mathfrak{a}_j\cdot e_{l-1}^{(n)} + \mathfrak{b}_j\right\}, & \text{if } \mathfrak{a}_j < 0 \end{cases} \tag{93}$$

- Suppose that $\widetilde{h}_{lj}^{(n)} + a_j$ is not a linear function of t. In order to obtain the zero t^* of $\frac{d}{dt}(\widetilde{h}_{lj}^{(n)}(t) + a_j(t))$, we can apply the Newton's method to generate a sequence $\{t_m\}_{m=1}^{\infty}$ such that $t_m \to t^*$ as $m \to \infty$. The iteration is given by

$$t_{m+1} = t_m - \frac{\frac{d}{dt}\left(\widetilde{h}_{lj}^{(n)}(t)\right)\Big|_{t=t_m} + \frac{d}{dt}\left(a_j(t)\right)\big|_{t=t_m}}{\frac{d^2}{dt^2}\left(\widetilde{h}_{lj}^{(n)}(t)\right)\Big|_{t=t_m} + \frac{d^2}{dt^2}\left(a_j(t)\right)\big|_{t=t_m}} \tag{94}$$

for $m = 0, 1, 2, \cdots$. The initial guess is t_0. Since the real-valued function $\frac{d}{dt}(\widetilde{h}_{lj}^{(n)}(t) + a_j(t))$ may have more than one zero, we need to apply Newton's method by using as many as possible for the initial guess t_0.

Now, the computational procedure is given below.

- **Step 1**. First, set the error tolerance ϵ and the initial value of natural number $n \in \mathbb{N}$ regarding the iterations.
- **Step 2**. Solve the dual problem (D_n) to obtain the optimal objective value $V(\mathrm{D}_n)$ and optimal solution $\bar{\mathbf{w}}$.
- **Step 3**. Use Newton's method given in (94) to find the set $Z_{lj}^{(n)}$ of all zeros of the real-valued function $\frac{d}{dt}(\widetilde{h}_{lj}^{(n)}(t) + a_j(t))$.
- **Step 4**. Use (91) to calculate the maximum (90), and use (89) to calculate the supremum (86).

- **Step 5**. Use (34) and the supremum obtained in Step 4 to obtain $\bar{\pi}_l^{(n)}$. According to (35), use the values of $\bar{\pi}_l^{(n)}$ to obtain $\pi_l^{(n)}$.
- **Step 6**. Use (58) to calculate the error bound ε_n. If $\varepsilon_n < \epsilon$, then go to Step 7. Otherwise, consider one more subdivision of each closed subinterval and set $n \leftarrow n + \widehat{n}$ for some integer $\widehat{n}$, and go to Step 2, where $\widehat{n}$ is the number of new points of subdivisions for all the closed subintervals.
- **Step 7**. Solve the primal problem (P_n) to obtain the optimal solution $\bar{\mathbf{z}}^{(n)}$.
- **Step 8**. Use (26) to set the step function $\widehat{\mathbf{z}}^{(n)}(t)$, which is the approximate solution of problem (RCLP3). Proposition 5 states that the actual error between $V(\text{RCLP3})$ and the objective value of $\widehat{\mathbf{z}}^{(n)}(t)$ is less than ε_n, where the error tolerance ϵ is reached for this partition $\mathcal{P}_n$.

In the sequel, we present a numerical example that considers the piecewise continuous functions on the time interval $[0, T]$. We consider $T = 1$ and the following problem:

$$\begin{aligned}
\text{maximize} \quad & \int_0^1 [a_1(t) \cdot z_1(t) + a_2(t) \cdot z_2(t)]dt \\
\text{subject to} \quad & b_1(t) \cdot z_1(t) \leq c_1(t) + \int_0^t [k_1(t,s) \cdot z_1(s) + k_2(t,s) \cdot z_2(s)]ds \text{ for all } t \in [0,1] \\
& b_2(t) \cdot z_2(t) \leq c_2(t) + \int_0^t [k_3(t,s) \cdot z_1(s) + k_4(t,s) \cdot z_2(s)]ds \text{ for all } t \in [0,1] \\
& \mathbf{z} = (z_1, z_2)^\top \in L_2^2[0,1].
\end{aligned}$$

The data a_1 and a_2 are assumed to be uncertain with the nominal data

$$a_1^{(0)}(t) = \begin{cases} e^t, & \text{if } 0 \leq t \leq 0.2 \\ \sin t, & \text{if } 0.2 < t \leq 0.6 \\ t^2, & \text{if } 0.6 < t \leq 1 \end{cases} \text{ and } a_2^{(0)}(t) = \begin{cases} 2t, & \text{if } 0 \leq t \leq 0.5 \\ t, & \text{if } 0.5 < t \leq 0.7 \\ t^2, & \text{if } 0.7 < t \leq 1 \end{cases}$$

and the uncertainties

$$\widehat{a}_1(t) = \begin{cases} e^{0.01t}, & \text{if } 0 \leq t \leq 0.2 \\ \sin(0.01t), & \text{if } 0.2 < t \leq 0.6 \\ (0.02t)^2, & \text{if } 0.6 < t \leq 1 \end{cases} \text{ and } \widehat{a}_2(t) = \begin{cases} 0.02t, & \text{if } 0 \leq t \leq 0.5 \\ 0.01t, & \text{if } 0.5 < t \leq 0.7 \\ (0.02t)^2, & \text{if } 0.7 < t \leq 1, \end{cases}$$

respectively. The data c_1 and c_2 are assumed to be uncertain with the nominal data

$$c_1^{(0)}(t) = \begin{cases} t^3, & \text{if } 0 \leq t \leq 0.3 \\ (\ln t)^2, & \text{if } 0.3 < t \leq 0.5 \\ t^2, & \text{if } 0.5 < t \leq 0.8 \\ \cos t, & \text{if } 0.8 < t \leq 1 \end{cases} \text{ and } c_2^{(0)}(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 0.4 \\ 5t, & \text{if } 0.4 < t \leq 0.5 \\ t^3, & \text{if } 0.5 < t \leq 0.8 \\ t^2, & \text{if } 0.8 < t \leq 1. \end{cases}$$

and the uncertainties

$$\widehat{c}_1(t) = \begin{cases} (0.01t)^3, & \text{if } 0 \leq t \leq 0.3 \\ 0, & \text{if } 0.3 < t \leq 0.5 \\ (0.03t)^2, & \text{if } 0.5 < t \leq 0.8 \\ 0, & \text{if } 0.8 < t \leq 1 \end{cases} \text{ and } \widehat{c}_2(t) = \begin{cases} 0.01t, & \text{if } 0 \leq t \leq 0.4 \\ 0.02t, & \text{if } 0.4 < t \leq 0.5 \\ (0.01t)^3, & \text{if } 0.5 < t \leq 0.8 \\ (0.02t)^2, & \text{if } 0.8 < t \leq 1. \end{cases}$$

The uncertain time-dependent matrices $B(t)$ and $K(t,s)$ are given below:

$$B(t) = \begin{bmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{bmatrix} = \begin{bmatrix} b_1(t) & 0 \\ 0 & b_2(t) \end{bmatrix}$$

and

$$K(t,s) = \begin{bmatrix} K_{11}(t,s) & K_{12}(t,s) \\ K_{21}(t,s) & K_{22}(t,s) \end{bmatrix} = \begin{bmatrix} k_1(t,s) & k_2(t,s) \\ k_3(t,s) & k_4(t,s) \end{bmatrix}.$$

The data $b_1 = B_{11}$ and $b_2 = B_{22}$ are assumed to be uncertain with the nominal data

$$B_{11}^{(0)}(t) = b_1^{(0)}(t) = \begin{cases} 20\cos t, & \text{if } 0 \leq t \leq 0.2 \\ 25\sin t, & \text{if } 0.2 < t \leq 0.6 \\ 27t^2, & \text{if } 0.6 < t \leq 1 \end{cases}$$

and

$$B_{22}^{(0)}(t) = b_2^{(0)}(t) = \begin{cases} 25\cos t, & \text{if } 0 \leq t \leq 0.5 \\ 22t, & \text{if } 0.5 < t \leq 0.7 \\ 25t^2, & \text{if } 0.7 < t \leq 1 \end{cases}$$

and the uncertainties

$$\widehat{B}_{11}(t) = \widehat{b}_1(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq 0.2 \\ \sin(0.01t), & \text{if } 0.2 < t \leq 0.6 \\ (0.03t)^2, & \text{if } 0.6 < t \leq 1 \end{cases}$$

and

$$\widehat{B}_{11}(t) = \widehat{b}_2(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq 0.5 \\ 0.01t, & \text{if } 0.5 < t \leq 0.7 \\ (0.02t)^2, & \text{if } 0.7 < t \leq 1 \end{cases}$$

The data $k_1 = K_{11}$, $k_2 = K_{12}$, $k_3 = K_{21}$, and $k_4 = K_{22}$ are assumed to be uncertain with the nominal data

$$K_{11}^{(0)}(t,s) = k_1^{(0)}(t,s) = \begin{cases} t^3 + s^2, & \text{if } 0 \leq t \leq 0.8 \text{ and } 0 \leq s \leq 0.5 \\ t^2 + \sin s, & \text{if } 0 \leq t \leq 0.8 \text{ and } 0.5 < s \leq 1 \\ (\ln t)^2 + 3e^{-s}, & \text{if } 0.8 < t \leq 1 \text{ and } 0 \leq s \leq 0.5 \\ \cos t + 5e^{-s}, & \text{if } 0.8 < t \leq 1 \text{ and } 0.5 < s \leq 1 \end{cases}$$

$$K_{12}^{(0)}(t,s) = k_2^{(0)}(t,s) = \begin{cases} t^3 \cdot s^2, & \text{if } 0 \leq t \leq 0.6 \text{ and } 0 \leq s \leq 0.7 \\ t^2 \cdot \sin s, & \text{if } 0 \leq t \leq 0.6 \text{ and } 0.7 < s \leq 1 \\ (\ln t)^2 \cdot e^{-s}, & \text{if } 0.6 < t \leq 1 \text{ and } 0 \leq s \leq 0.7 \\ 3t^2 \cdot \sin s, & \text{if } 0.6 < t \leq 1 \text{ and } 0.7 < s \leq 1 \end{cases}$$

$$K_{21}^{(0)}(t,s) = k_3^{(0)}(t,s) = \begin{cases} 3t^2 \cdot \sin s, & \text{if } 0 \leq t \leq 0.3 \text{ and } 0 \leq s \leq 0.6 \\ 2t \cdot s^2, & \text{if } 0 \leq t \leq 0.3 \text{ and } 0.6 < s \leq 1 \\ (\ln t)^2 + (\cos s)^2, & \text{if } 0.3 < t \leq 1 \text{ and } 0 \leq s \leq 0.6 \\ t^3 \cdot s^2, & \text{if } 0.3 < t \leq 1 \text{ and } 0.6 < s \leq 1 \end{cases}$$

$$K_{22}^{(0)}(t,s) = k_4^{(0)}(t,s) = \begin{cases} t^2 + s^2, & \text{if } 0 \leq t \leq 0.5 \text{ and } 0 \leq s \leq 0.3 \\ \sin t + s^2, & \text{if } 0 \leq t \leq 0.5 \text{ and } 0.3 < s \leq 1 \\ (\cos t)^2 + 3e^{-s}, & \text{if } 0.5 < t \leq 1 \text{ and } 0 \leq s \leq 0.3 \\ 2t^3 \cdot s^2, & \text{if } 0.5 < t \leq 1 \text{ and } 0.3 < s \leq 1. \end{cases}$$

and the uncertainties

$$\widehat{K}_{11}(t,s) = \widehat{k}_1(t,s) = \begin{cases} (0.05t)^3 + (0.02s)^2, & \text{if } 0 \le t \le 0.8 \text{ and } 0 \le s \le 0.5 \\ (0.03t)^2 + \sin(0.02s), & \text{if } 0 \le t \le 0.8 \text{ and } 0.5 < s \le 1 \\ e^{-0.01s}, & \text{if } 0.8 < t \le 1 \text{ and } 0 \le s \le 0.5 \\ e^{-0.01s}, & \text{if } 0.8 < t \le 1 \text{ and } 0.5 < s \le 1 \end{cases}$$

$$\widehat{K}_{12}(t,s) = \widehat{k}_2(t,s) = \begin{cases} (0.02t)^3 \cdot (0.05s)^2, & \text{if } 0 \le t \le 0.6 \text{ and } 0 \le s \le 0.7 \\ (0.03t)^2 \cdot \sin(0.05s), & \text{if } 0 \le t \le 0.6 \text{ and } 0.7 < s \le 1 \\ e^{-0.01s}, & \text{if } 0.6 < t \le 1 \text{ and } 0 \le s \le 0.7 \\ (0.02t)^2 \cdot \sin(0.02s), & \text{if } 0.6 < t \le 1 \text{ and } 0.7 < s \le 1 \end{cases}$$

$$\widehat{K}_{21}(t,s) = \widehat{k}_3(t,s) = \begin{cases} (0.03t)^2 \cdot \sin(0.01s), & \text{if } 0 \le t \le 0.3 \text{ and } 0 \le s \le 0.6 \\ (0.04t) \cdot (0.02s)^2, & \text{if } 0 \le t \le 0.3 \text{ and } 0.6 < s \le 1 \\ 0, & \text{if } 0.3 < t \le 1 \text{ and } 0 \le s \le 0.6 \\ (0.01t)^3 \cdot (0.05s)^2, & \text{if } 0.3 < t \le 1 \text{ and } 0.6 < s \le 1 \end{cases}$$

$$\widehat{K}_{22}(t,s) = \widehat{k}_4(t,s) = \begin{cases} (0.01t)^2 + (0.02s)^2, & \text{if } 0 \le t \le 0.5 \text{ and } 0 \le s \le 0.3 \\ \sin(0.01t) + (0.02s)^2, & \text{if } 0 \le t \le 0.5 \text{ and } 0.3 < s \le 1 \\ e^{-0.03s}, & \text{if } 0.5 < t \le 1 \text{ and } 0 \le s \le 0.3 \\ (0.02t)^3 \cdot (0.03s)^2, & \text{if } 0.5 < t \le 1 \text{ and } 0.3 < s \le 1. \end{cases}$$

We see that $B_{ij}^{(0)}(t)$ and $\widehat{B}_{ij}(t)$ satisfy the conditions (6) and (7). From the discontinuities of $a_1, a_2, c_1, c_2, b_1, b_2, k_1, k_2, k_3$, and k_4, according to the setting of partition $\mathcal{P}_n$, we see that $r = 8$ and

$$\mathcal{D} = \{d_0 = 0, d_1 = 0.2, d_2 = 0.3, d_3 = 0.4, d_4 = 0.5, d_5 = 0.6, d_6 = 0.7, d_7 = 0.8, d_8 = 1\}.$$

For $n^* = 2$, this means that each closed interval $[d_v, d_{v+1}]$ is equally divided by two subintervals for $v = 0, 1, \cdots 7$. In this case, we have $n = 2 \cdot 8 = 16$. Therefore, we obtain a partition $\mathcal{P}_{16}$.

We denote by

$$V(\text{RCLP3}_n) = \int_0^T \mathbf{a}(t)^\top \widehat{\mathbf{z}}^{(n)}(t)dt$$

the approximate optimal objective value of problem (RCLP3). Theorem 1 and Proposition 5 state that

$$0 \le V(\text{RCLP3}) - V(\text{RCLP3}_n) \le \varepsilon_n$$

and

$$0 \le V(\text{RCLP3}_n) - V(\text{P}_n) \le V(\text{RCLP3}) - V(\text{P}_n) \le \varepsilon_n.$$

The numerical results are shown in the following Table 1.

Table 1. Numerical Results.

n^*	$n = n^* \cdot 8$	ε_n	$V(\text{P}_n)$	$V(\text{RCLP3}_n)$
2	16	0.0261958	0.0303016	0.0327564
10	80	0.0053931	0.0367996	0.0373742
50	400	0.0011151	0.0382602	0.0383788
100	800	0.0005599	0.0384469	0.0385064
200	1600	0.0002805	0.0385406	0.0385704
300	2400	0.0001871	0.0385719	0.0385918
400	3200	0.0001404	0.0385875	0.0386025
500	4000	0.0001124	0.0385969	0.0386089

The decision-maker tolerating the error $\epsilon = 0.0005$ suggests that $n^* = 100$ is sufficient to achieve this error ϵ by referring to the error bound $\varepsilon_n = 0.0005599$. We use the active set method in MATLAB to solve the primal and dual linear programming problems (P_n)

and (D_n), respectively, to obtain the numerical results. We need to mention that using the simplex method in MATLAB to solve the problems (P_n) and (D_n) for large n encounters a little bug. There is a warning message from MATLAB when the simplex method is used to solve the dual problem (D_n) for large n. However, it is fine when the simplex method is used to solve the primal problem (P_n).

7. Conclusions

Solving the continuous-time linear programming problem is indeed difficult. Especially, when the time-dependent matrices are involved in the problem, more efforts should put into taking care of the time factor through the time interval $[0, T]$. In this paper, a more complicated problem was studied when the uncertainty was assumed to be considered in the continuous-time linear programming problem with time-dependent matrices. In this case, a robust counter part was established and solved.

The main essence for solving the continuous-time linear programming problem is to formulate the discretization problem by considering n time points that divide the whole time interval $[0, T]$ into n time subintervals. In this case, we can formulate a large-scale conventional linear programming problem that can be solved to obtain the approximate optimal solution. In other words, when the scale is large enough, the error between the actual optimal solution and approximate optimal solution is small. The main purpose of this paper is to obtain an analytic formula for the upper bound of error as shown in Theorem 1. The limitation of the approach proposed in this paper is that the large-scale linear programming problem should be solved, which will consume huge computer resources, which the personal computer sometimes lacks. In other words, the high-level computer will increase the efficiency of the methodology proposed in this paper. Alternatively, a new computational procedure such as parallel computing may be proposed, which can be future research.

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