

## Article

# The Influence of Fear Effect to a Discrete-Time Predator-Prey System with Predator Has Other Food Resource

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**Abstract:** A discrete-time predator–prey system incorporating fear effect of the prey with the predator has other food resource is proposed in this paper. The trivial equilibrium and the predator free equilibrium are both unstable. A set of sufficient conditions for the global attractivity of prey free equilibrium and interior equilibrium are established by using iteration scheme and the comparison principle of difference equations. Our study shows that due to the fear of predation, the prey species will be driven to extinction while the predator species tends to be stable since it has other food resource, i.e., the prey free equilibrium may be globally stable under some suitable conditions. Numeric simulations are provided to illustrate the feasibility of the main results.

**Keywords:** discrete predator-prey system; fear effect; global attractivity; other food resource



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## 1. Introduction

In ecology, predator-prey relationship is one of the most important relationship. The interaction between predator and prey is the core of evolutionary biology. In the development of biomathematics, many scholars have studied the dynamic behaviors of predators and prey in continuous models. However, in the past few decades, some scholars have also argued that discrete models are more appropriate than continuous ones when the populations have non-overlapping generations [1,2]. Scholars investigated the persistence, periodic solutions, probabilistic periodic solutions, etc. of non-autonomous discrete systems [3–7]. In addition, many scholars have successively studied the dynamical behaviors of autonomous discrete-time systems [8–15]. For example, Chen and Teng [8] discussed the stability of positive equilibrium of a two-species discrete competition system. Chen [9] investigated the global attractivity property of a discrete competition model. In [10], a discrete May type cooperative model incorporating Michaelis-Menten type harvesting is studied by Zhu et al. Stability, bifurcation and chaos of discrete predator-prey systems are investigated in [11–15]. The above studies show that discrete systems often have more complex dynamic behaviors than the corresponding continuous ones.

However, most of the discrete models mentioned above only considered the case where the predator kills the prey directly. In recent years, scholars have gradually noting that besides the direct hunting, the presence of predators would make the prey afraid, which affects their habitat, reproduction rate, foraging behavior, etc. Further, some scholars pointed out that fear effect has an influence on prey populations even more powerful than direct hunting [16–29].

Wang, Zanette and Zou [30] first time proposed a predator-prey system incorporating fear effect in 2016:

$$\begin{aligned}\frac{dx}{dt} &= \frac{r_0x}{1+ky} - dx - ax^2 - pxy, \\ \frac{dy}{dt} &= cpxy - my.\end{aligned}\tag{1}$$

In [30], Wang, Zanette and Zou obtained that the fear effect has no influence on the dynamic behaviors of the system with linear functional response. Both predator and prey species will go to extinction if  $r_0 < d$ .

Discrete systems usually exhibit more complex dynamics than their continuous counterparts. To investigate the impact of fear effect in discrete predator-prey system, Kundu, Pal, Samanta, et al. [31] have discretized the continuous model proposed by Wang, which leads to the following system:

$$\begin{aligned}x(n+1) &= x(n) \exp\left[\frac{r_0}{1+ky(n)} - d - ax(n) - py(n)\right], \\y(n+1) &= y(n) \exp[cp x(n) - m].\end{aligned}\quad (2)$$

In [31], the authors observed that system (2) becomes stable from chaotic dynamics when the fear parameter  $k$  increasing, which is quite different from the property of system (1). The results of [31] illustrated that discrete systems do have more complex dynamical behaviors.

On the other hand, Wang, Zanette and Zou in [30] found that predators and prey would become extinct simultaneously under some suitable conditions. The reason is that the predator species take prey species as its unique food resource, however, generally speaking, predators are omnivorous and do not hunt for only one kind of prey, they look for other food sources to ensure their survival if one of food source is scarce. Thus, Zhu, Wu, Lai, et al. [32] proposed the continuous system (3):

$$\begin{aligned}\frac{dx}{dt} &= \frac{r_0 x}{1+ky} - dx - ax^2 - bxy, \\ \frac{dy}{dt} &= cbxy + my - hy^2.\end{aligned}\quad (3)$$

In [32], Zhu et al. showed that the trivial equilibrium and the predator free equilibrium are both unstable. And the following results are obtained (see Theorems 2.2 and 3.1 in [32]):

**Theorem 1.** *The prey free equilibrium is globally asymptotically stable if*

$$r_0 < d + \frac{dhkm + bhm + bkm^2}{h^2}.$$

*The positive equilibrium of system (3) is globally asymptotically stable if*

$$r_0 > d + \frac{dhkm + bhm + bkm^2}{h^2}.$$

That is, the prey free equilibrium is globally asymptotically stable under some suitable condition. This affirms the fact that predators may still be permanent even if the prey populations become extinct, which differs from the result of Wang, Zanette and Zou.

To further analyze the complex dynamical behaviors of the discrete system, corresponding to the continuous system (3) we propose the following system:

$$\begin{aligned}x(n+1) &= x(n) \exp\left[\frac{r_0}{1+ky(n)} - d - ax(n) - by(n)\right], \\y(n+1) &= y(n) \exp[m - hy(n) + cbx(n)],\end{aligned}\quad (4)$$

where  $x(n)$ ,  $y(n)$  are the population density of prey and predator at  $n$ -generation, respectively.  $r_0$ ,  $d$  denote the birth rate and natural death rate of the prey, respectively.  $a$ ,  $h$  represent the death rate due to intraspecific competition of the prey and the predator respectively.  $b$  is the maximum predation rate of the predator,  $m$  denotes the intrinsic growth rate of the predator,  $c$  is the conversion rate from prey's bioenergy to predators' bioenergy,  $k$  describes the level of the fear effect.

The rest of the paper is arranged as follows. We investigate the existence and local stability of the equilibria in Sections 2 and 3, respectively. In Sections 4 and 5, some sufficient conditions to guarantee the global stability of the interior equilibrium point and prey free equilibrium are given, respectively. In Section 6, some numerical simulations are provided to show the feasibility of the main results in this paper. Finally, a brief conclusion is presented in Section 7.

## 2. The Existence of Equilibria

To determine the equilibrium of system (4), we solve the following equations:

$$\begin{aligned}x &= x \exp\left(\frac{r_0}{1+ky} - d - ax - by\right), \\y &= y \exp(m - hy + cbx).\end{aligned}\quad (5)$$

By simple computation, we obtain the following results.

**Theorem 2.** System (4) always has the trivial equilibrium  $E_0(0,0)$  and the prey free equilibrium  $E_1(0, \frac{m}{h})$ ; if  $r_0 > d$  holds, the predator free equilibrium  $E_2(\frac{r_0-d}{a}, 0)$  exists.

**Theorem 3.** The positive equilibrium  $E^*(x^*, y^*)$  exists if

$$r_0 > d + \frac{dhkm + bhm + bkm^2}{h^2} \quad (6)$$

holds, where  $y^* = \frac{m+cbx^*}{h}$  and  $x^*$  is the unique positive solution of the equation

$$A_1x^2 + A_2x + A_3 = 0, \quad (7)$$

with

$$\begin{aligned}A_1 &= b^3c^2k + abchk, \\A_2 &= 2b^2ckm + bcdhk + ahkm + b^2ch + ah^2, \\A_3 &= dhkm + bhm + dh^2 - h^2r_0 + bkm^2.\end{aligned}\quad (8)$$

**Proof.** The positive equilibrium of system (4) satisfies the equations

$$\begin{aligned}\frac{r_0}{1+ky} - d - ax - by &= 0, \\m - hy + cbx &= 0.\end{aligned}\quad (9)$$

We get  $y = \frac{m+cbx}{h}$  from the second equation. Substituting  $y$  to the first equation of (9) leads to

$$A_1x^2 + A_2x + A_3 = 0, \quad (10)$$

where  $A_1, A_2, A_3$  is defined in (8).

Obviously,  $A_1$  and  $A_2$  are both positive, so the axis of symmetry of Equation (10) is on the left half axel. Under the assumption of (6), one could easily see that  $A_3 < 0$ , hence, (10) has a unique positive solution  $x^*$ . Consequently, system (4) has a unique positive equilibrium  $E^*(x^*, y^*)$ , where  $y^* = \frac{m+cbx^*}{h}$ .

This ends the proof of Theorem 3.  $\square$

### 3. The Local Stability of Equilibria

The Jacobian matrix of system (4) is given by

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}, \quad (11)$$

where

$$\begin{aligned} J_{11} &= (1 - ax) \exp\left(\frac{r_0}{1 + ky} - d - ax - by\right), \\ J_{12} &= x\left(-\frac{r_0 k}{(1 + ky)^2} - b\right) \exp\left(\frac{r_0}{1 + ky} - d - ax - by\right), \\ J_{21} &= cby \exp(m - hy + cbx), \\ J_{22} &= (1 - hy) \exp(m - hy + cbx). \end{aligned}$$

We first state the results about how to determine the nature of equilibrium points.

**Lemma 1** ([8]). Let the characteristic equation of  $J$  is  $F(\lambda) = \lambda^2 + B\lambda + C = 0$ . Suppose  $\lambda_1$  and  $\lambda_2$  are two roots of  $F(\lambda) = 0$ . Then there are the following definitions.

- (1) If  $|\lambda_1| < 1$  and  $|\lambda_2| < 1$ , then the equilibrium point is called a sink and is locally asymptotically stable.
- (2) If  $|\lambda_1| > 1$  and  $|\lambda_2| > 1$ , then the equilibrium point is called a source and is unstable.
- (3) If  $|\lambda_1| > 1$  and  $|\lambda_2| < 1$ , then the equilibrium point is called a saddle.
- (4) If  $|\lambda_1| = 1$  or  $|\lambda_2| = 1$ , then the equilibrium point is called non-hyperbolic.

**Lemma 2** ([8]). Let  $F(\lambda) = \lambda^2 + B\lambda + C$ , where  $B$  and  $C$  are constants. Suppose  $F(1) > 0$  and  $\lambda_1, \lambda_2$  are two roots of  $F(\lambda) = 0$ . Then

- (1)  $|\lambda_1| < 1$  and  $|\lambda_2| < 1$  if and only if  $F(-1) > 0$  and  $C < 1$ ;
- (2)  $|\lambda_1| > 1$  and  $|\lambda_2| > 1$  if and only if  $F(-1) > 0$  and  $C > 1$ ;
- (3)  $|\lambda_1| < 1$  and  $|\lambda_2| > 1$  if and only if  $F(-1) < 0$ ;
- (4)  $\lambda_1 = -1$  and  $|\lambda_2| \neq 1$  if and only if  $F(-1) = 0$  and  $B \neq 0, 2$ ;
- (5)  $\lambda_1$  and  $\lambda_2$  are the conjugate complex roots and  $|\lambda_1| = |\lambda_2| = 1$  if and only if  $B^2 - 4C < 0$  and  $C = 1$ .

**Theorem 4.** For all positive parameters, system (4) has the trivial equilibrium  $E_0(0, 0)$ , then:

- (1)  $E_0(0, 0)$  is a source if  $r_0 > d$ .
  - (2)  $E_0(0, 0)$  is a saddle if  $r_0 < d$ .
  - (3)  $E_0(0, 0)$  is non-hyperbolic if  $r_0 = d$ .
- $E_0$  is always unstable.

**Proof.** The Jacobian matrix of the system (4) about the trivial equilibrium  $E_0(0, 0)$  is

$$J(E_0) = \begin{pmatrix} \exp(r_0 - d) & 0 \\ 0 & \exp(m) \end{pmatrix}. \quad (12)$$

The eigenvalues of  $J(E_0)$  are  $\lambda_1 = \exp(r_0 - d)$ ,  $\lambda_2 = \exp(m) > 1$  since  $m > 0$ . Hence, if  $r_0 > d$ , i.e.,  $\lambda_1 > 1$ , then  $E_0$  is a source; if  $r_0 < d$ , i.e.,  $0 < \lambda_1 < 1$ , then  $E_0$  is a saddle; if  $r_0 = d$ , i.e.,  $\lambda_1 = 1$ , then  $E_0$  is non-hyperbolic. This completes the proof.  $\square$

**Theorem 5.** System (4) always has the prey free equilibrium  $E_1(0, \frac{m}{h})$ .  $E_1$  is

- (1) sink if  $m < 2$  and  $r_0 < d + \frac{dhkm + bhm + bkm^2}{h^2}$ , then  $E_1$  is stable.
- (2) saddle if one of the following conditions holds:

- (a)  $m > 2$  and  $r_0 < d + \frac{dhkm + bhm + bkm^2}{h^2}$ ;
- (b)  $m < 2$  and  $r_0 > d + \frac{dhkm + bhm + bkm^2}{h^2}$ .
- (3) source if  $m > 2$  and  $r_0 > d + \frac{dhkm + bhm + bkm^2}{h^2}$ , then  $E_1$  is unstable.
- (4) non-hyperbolic if  $m = 2$  or  $r_0 = d + \frac{dhkm + bhm + bkm^2}{h^2}$ .

**Proof.** The Jacobian matrix of the system (4) about  $E_1(0, \frac{m}{h})$  is

$$J(E_1) = \begin{pmatrix} \exp\left(\frac{r_0 h}{h + km} - d - \frac{bm}{h}\right) & 0 \\ \frac{cbm}{h} & 1 - m \end{pmatrix}. \quad (13)$$

The eigenvalues of  $J(E_1)$  are  $\lambda_1 = \exp\left(\frac{r_0 h}{h + km} - d - \frac{bm}{h}\right)$ ,  $\lambda_2 = 1 - m$ . Similar to the proof in Theorem 4, the above results can be easily obtained.  $\square$

**Theorem 6.** If  $r_0 > d$ , system (4) has the predator free equilibrium  $E_2\left(\frac{r_0 - d}{a}, 0\right)$ , then  $E_2$  is

- (1) source if  $r_0 - d > 2$ .
- (2) saddle if  $r_0 - d < 2$ .
- (3) non-hyperbolic if  $r_0 - d = 2$ .
- $E_2$  is always unstable.

**Proof.** The Jacobian matrix of the system (4) about  $E_2$  is

$$J(E_2) = \begin{pmatrix} 1 - (r_0 - d) & \frac{-(r_0 - d)(kr_0 + b)}{\exp\left[\frac{cb}{a}(r_0 - d) + m\right]} \\ 0 & \exp\left[\frac{cb}{a}(r_0 - d) + m\right] \end{pmatrix}. \quad (14)$$

The eigenvalues of  $J(E_2)$  are  $\lambda_1 = 1 - (r_0 - d)$ ,  $\lambda_2 = \exp\left[\frac{cb}{a}(r_0 - d) + m\right]$ . A similar method of proof as Theorem 4, the above results can be easily obtained.  $\square$

**Theorem 7.** The interior equilibrium  $E^*(x^*, y^*)$  exists if  $r_0 > d + \frac{dhkm + bhm + bkm^2}{h^2}$  holds, then  $E^*(x^*, y^*)$  is

- (1) sink if  $S < T \leq 2$ , or  $T > 2$  and  $2T - 4 < S < T$ ;
- (2) source if  $T \leq 2$  and  $S > T$ , or  $T > 2$  and  $S > \max(T, 2T - 4)$ ;
- (3) saddle if  $T > 2$  and  $S < 2T - 4$ ;
- (4) non-hyperbolic if  $S = 2T - 4$ ;

where

$$\begin{aligned} S &= \left(ah + \frac{cbr_0k}{(1 + ky^*)^2} + cb^2\right)x^*y^*, \\ T &= ax^* + hy^*. \end{aligned} \quad (15)$$

**Proof.** Calculating Jacobian matrix  $J(E^*)$  of system (4) at  $E^*$ , we have

$$J(E^*) = \begin{pmatrix} 1 - ax & -x\left(\frac{r_0k}{(1 + ky)^2} + b\right) \\ cby & 1 - hy \end{pmatrix}. \quad (16)$$

So the characteristic equation of the above matrix can be written as

$$F(\lambda) = \lambda^2 + B\lambda + C = 0,$$

where

$$B = -2 + ax^* + hy^* \quad \text{and} \quad C = (1 - ax^*)(1 - hy^*) + cbx^*y^*\left(\frac{r_0k}{(1 + ky^*)^2} + b\right).$$

By simple computation, we obtain

$$\begin{aligned} F(1) &= 1 + B + C = \left(ah + \frac{cbr_0k}{(1 + ky^*)^2} + cb^2\right)x^*y^* > 0, \\ F(-1) &= 1 - B + C = 4 - 2ax^* - 2hy^* + \left(ah + \frac{cbr_0k}{(1 + ky^*)^2} + cb^2\right)x^*y^*, \\ C - 1 &= -ax^* - hy^* + \left(ah + \frac{cbr_0k}{(1 + ky^*)^2} + cb^2\right)x^*y^*, \end{aligned} \quad (17)$$

where  $x^* = \frac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1}$ ,  $y^* = \frac{m + cbx^*}{h}$  and  $A_1, A_2, A_3$  as (8) shows. Let

$$\begin{aligned} S &:= \left(ah + \frac{cbr_0k}{(1 + ky^*)^2} + cb^2\right)x^*y^*, \\ T &:= ax^* + hy^*. \end{aligned}$$

Then we can easily obtain  $F(-1) > 0$  if  $T \leq 2$  or  $T > 2$  and  $S > 2T - 4$ ,  $F(-1) < 0$  if  $T > 2$  and  $S < 2T - 4$ ,  $C - 1 < 0$  if  $S < T$ . According to Lemmas 1 and 2,  $E^*$  is a sink and it's stable, if the conclusion (1) of Theorem 7 holds.

Next, if the other conditions of Theorem 7 hold separately,  $E^*$  is a source, saddle and non-hyperbolic, respectively, at which  $E^*$  is unstable.

This completes the proof.  $\square$

#### 4. Global Stability of Interior Equilibrium

In this section, we discuss the global stability of interior equilibrium  $E^*(x^*, y^*)$  by using the method of iteration scheme. Firstly, we introduce the following lemmas:

**Lemma 3** ([8]). Let  $f(u) = u \exp(\alpha - \beta u)$ , where  $\alpha$  and  $\beta$  are positive constants, then  $f(u)$  is nondecreasing for  $u \in (0, \frac{1}{\beta}]$ .

**Lemma 4** ([8]). Assume that sequence  $\{u_n\}$  satisfy  $u_{n+1} = u_n \exp(\alpha - \beta u_n)$ ,  $n \in \mathbb{N}$ , where  $\alpha$  and  $\beta$  are positive constants and  $u(0) > 0$ . Then

- (i) If  $\alpha < 2$ , then  $\lim_{n \rightarrow \infty} u_n = \frac{\alpha}{\beta}$ .
- (ii) If  $\alpha \leq 1$ , then  $u_n \leq \frac{1}{\beta}$ ,  $n = 2, 3, \dots$

**Lemma 5** ([33]). Suppose that functions  $f, g : \mathbb{Z}_+ \times [0, \infty) \rightarrow [0, \infty)$  satisfy  $f(n, x) \leq g(n, x)$  ( $f(n, x) \geq g(n, x)$ ) for  $n \in \mathbb{Z}_+$  and  $x \in [0, \infty)$  and  $g(n, x)$  is nondecreasing with respect to  $x$ . If  $\{x(n)\}$  and  $\{u(n)\}$  are the solutions of the following difference equations

$$x(n+1) = f(n, x(n)), \quad u(n+1) = g(n, u(n)),$$

respectively, and  $x(0) \leq u(0)$  ( $x(0) \geq u(0)$ ), then

$$x(n) \leq u(n) \quad (x(n) \geq u(n)), \quad \text{for all } n \geq 0.$$

**Theorem 8.** Assume that

$$r_0 \leq 1, \quad m + cbM_1^x \leq 1, \quad (18)$$

$$r_0 > (1 + kM_1^y)(d + bM_1^y), \quad (19)$$

$$m < \frac{h(ah - bcdk)}{k(b^2c - ah)}, \quad bcdk < ah < b^2c, \quad (20)$$

where

$$M_1^y = \frac{m + cbM_1^x}{h}, \quad M_1^x = \frac{r_0}{a}.$$

Then equilibrium  $E^*$  is globally attractive, that is,

$$\lim_{n \rightarrow +\infty} x(n) = x^*, \quad \lim_{n \rightarrow +\infty} y(n) = y^*.$$

**Proof.** Let  $(x(n), y(n))$  be any positive solution of system (4) with  $x(0) > 0$  and  $y(0) > 0$ . Let

$$\begin{aligned} U_1 &= \limsup_{n \rightarrow +\infty} x(n), & V_1 &= \liminf_{n \rightarrow +\infty} x(n), \\ U_2 &= \limsup_{n \rightarrow +\infty} y(n), & V_2 &= \liminf_{n \rightarrow +\infty} y(n). \end{aligned}$$

We now claim that  $U_1 = V_1 = x^*$  and  $U_2 = V_2 = y^*$ .

For the first iteration, we aim to show that  $U_1 \leq M_1^x$  and  $U_2 \leq M_1^y$ .

From the first equation of system (4) we obtain

$$x(n+1) \leq x(n) \exp[r_0 - ax(n)], \quad n = 0, 1, 2, \dots \quad (21)$$

Consider the auxiliary equation

$$u(n+1) = u(n) \exp[r_0 - au(n)].$$

Since  $r_0 \leq 1$ , by conclusion (ii) of Lemma 4, we have  $u(n) \leq \frac{1}{a}$  for all  $n \geq 2$ , where  $u(n)$  is any solution of system (4) with  $u(0) > 0$ . From Lemma 3,  $f(u) = u \exp(r_0 - au)$  is nondecreasing for  $u \in (0, \frac{1}{a}]$ , from Lemma 5, we have  $x(n) \leq u(n)$  for all  $n \geq 2$ , where  $u(n)$  is the solution of system (4) with  $u(2) = x(2)$ . By conclusion (i) of Lemma 4, we have

$$U_1 = \limsup_{n \rightarrow +\infty} x(n) \leq \lim_{n \rightarrow +\infty} u(n) = \frac{r_0}{a}.$$

Hence, for any sufficiently small  $\epsilon > 0$ , there exists an integer  $N_1 > 2$  such that if  $n \geq N_1$ , then  $x(n) \leq \frac{r_0}{a} + \epsilon := M_1^x$ .

From the second equation of system (4) we have

$$y(n+1) \leq y(n) \exp[m - hy(n) + cbM_1^x], \quad n \geq N_1. \quad (22)$$

Consider the auxiliary equation

$$u(n+1) = u(n) \exp[m - hu(n) + cbM_1^x].$$

Since (18) holds, then we have  $m + cbM_1^x \leq 1$ . By conclusion (ii) of Lemma 4, we have  $u(n) \leq \frac{1}{h}$  for all  $n \geq N_1$ . From Lemma 3,  $f(u) = u \exp(m + cbM_1^x - hu)$  is nondecreasing for  $u \in (0, \frac{1}{h}]$ , from Lemma 5 we have  $y(n) \leq u(n)$  for all  $n \geq N_1$ . By conclusion (i) of Lemma 4, we have

$$U_2 = \limsup_{n \rightarrow +\infty} y(n) \leq \lim_{n \rightarrow +\infty} u(n) = \frac{m + cbM_1^x}{h}.$$

Hence, for any sufficiently small  $\epsilon > 0$ , there exists an integer  $N_2 > N_1$  such that if  $n \geq N_2$ , then  $y(n) \leq \frac{m + cbM_1^x}{h} + \epsilon := M_1^y$ .

For the second iteration, we aim to show that  $V_1 \geq N_1^x$  and  $V_2 \geq N_1^y$ . From the first equation of system (4) we have

$$x(n+1) \geq x(n) \exp \left[ \frac{r_0}{1 + kM_1^y} - d - ax(n) - bM_1^y \right]. \quad (23)$$

Consider the auxiliary equation

$$u(n+1) = u(n) \exp \left[ \frac{r_0}{1 + kM_1^y} - d - au(n) - bM_1^y \right].$$

Since (19) holds, we have

$$0 < \frac{r_0}{1 + kM_1^y} - d - bM_1^y \leq r_0 \leq 1.$$

By conclusion (ii) of Lemma 4, we have  $u(n) \leq \frac{1}{a}$  for all  $n \geq N_2$ . From Lemma 3,  $f(u) = u \exp(\frac{r_0}{1 + kM_1^y} - d - bM_1^y - au)$  is nondecreasing for  $u \in (0, \frac{1}{a}]$ , from Lemma 5 we have  $x(n) \geq u(n)$  for all  $n \geq N_2$ . By conclusion (i) of Lemma 4, we have

$$V_1 = \liminf_{n \rightarrow +\infty} x(n) \geq \lim_{n \rightarrow +\infty} u(n) = \frac{1}{a} \left( \frac{r_0}{1 + kM_1^y} - d - bM_1^y \right).$$

Hence, for any sufficiently small  $\epsilon > 0$ , there exists an integer  $N_3 > N_2$  such that if  $n \geq N_3$ , then  $x(n) \geq \frac{1}{a} \left[ \frac{r_0}{1 + kM_1^y} - d - bM_1^y \right] - \epsilon := N_1^x$ .

From the second equation of system (4), we further have

$$y(n+1) \geq y(n) \exp[m - hy(n) + cbN_1^x], \quad n \geq N_3. \quad (24)$$

Since  $m + cbN_1^x \leq m + cbM_1^x \leq 1$ , a similar argument as above, we can obtain

$$V_2 = \liminf_{n \rightarrow +\infty} y(n) \geq \frac{m + cbN_1^x}{h}.$$

Hence, for any sufficiently small  $\epsilon > 0$ , there exists an integer  $N_4 > N_3$  such that if  $n \geq N_4$ , then  $y(n) \geq \frac{m + cbN_1^x}{h} - \epsilon := N_1^y$ .

For the third iteration, we aim to show that  $U_1 \leq M_2^x$ ,  $U_2 \leq M_2^y$ , where  $M_2^x \leq M_1^x$ ,  $M_2^y \leq M_1^y$ .

From the first equation of system (4) for  $n \geq N_4$  we have

$$x(n+1) \leq x(n) \exp \left[ \frac{r_0}{1 + kN_1^y} - d - ax(n) - bN_1^y \right], \quad n \geq N_4.$$

Since  $M_1^y > N_1^y$ , we get

$$0 < \frac{r_0}{1 + kM_1^y} - d - bM_1^y < \frac{r_0}{1 + kN_1^y} - d - bN_1^y \leq r_0 \leq 1.$$

A similar argument as above, we can obtain

$$U_1 = \limsup_{n \rightarrow +\infty} x(n) \leq \frac{1}{a} \left( \frac{r_0}{1 + kN_1^y} - d - bN_1^y \right).$$

Hence, for any sufficiently small  $\epsilon > 0$ , there exists an integer  $N_5 > N_4$  such that if  $n \geq N_5$ , then  $x(n) \leq \frac{1}{a} \left[ \frac{r_0}{1 + kN_1^y} - d - bN_1^y \right] + \frac{\epsilon}{2} := M_2^x \leq M_1^x$ .

From the second equation of system (4) for  $n \geq N_5$ , we have

$$y(n+1) \leq y(n) \exp[m - hy(n) + cbM_2^x], \quad n \geq N_5.$$

Since  $m + cbM_2^x \leq m + cbM_1^x \leq 1$ , a similar argument as above, we can obtain

$$U_2 = \limsup_{n \rightarrow +\infty} y(n) \leq \frac{m + cbM_2^x}{h}.$$

Hence, for any sufficiently small  $\epsilon > 0$ , there exists an integer  $N_6 > N_5$  such that if  $n \geq N_6$ , then  $y(n) \leq \frac{m + cbM_2^x}{h} + \frac{\epsilon}{2} := M_2^y \leq M_1^y$ .

For the fourth iteration, we aim to show that  $V_1 \geq N_2^x$  and  $V_2 \geq N_2^y$ .

From the first equation of system (4) we have

$$x(n+1) \geq x(n) \exp \left[ \frac{r_0}{1 + kM_2^y} - d - ax(n) - bM_2^y \right], \quad n \geq N_6.$$

Since  $M_1^y \geq M_2^y$ , we have

$$0 < \frac{r_0}{1 + kM_1^y} - d - bM_1^y \leq \frac{r_0}{1 + kM_2^y} - d - bM_2^y \leq r_0 \leq 1.$$

A similar argument as above, we can obtain

$$V_1 = \liminf_{n \rightarrow +\infty} x(n) \geq \frac{1}{a} \left( \frac{r_0}{1 + kM_2^y} - d - bM_2^y \right).$$

Hence, for any sufficiently small  $\epsilon > 0$ , there exists an integer  $N_7 > N_6$  such that if  $n \geq N_7$ , then  $x(n) \geq \frac{1}{a} \left[ \frac{r_0}{1 + kM_2^y} - d - bM_2^y \right] - \frac{\epsilon}{2} := N_2^x$ .

From the second equation of system (4) for  $n \geq N_7$ , we have

$$y(n+1) \geq y(n) \exp[m - hy(n) + cbN_2^x], \quad n \geq N_7.$$

Since  $m + cbN_2^x \leq m + cbM_1^x \leq 1$ , a similar argument as above, we can obtain

$$V_2 = \liminf_{n \rightarrow +\infty} y(n) \geq \frac{m + cbN_2^x}{h}.$$

Hence, for any sufficiently small  $\epsilon > 0$ , there exists an integer  $N_8 > N_7$  such that if  $n \geq N_8$ , then  $y(n) \geq \frac{m + cbN_2^x}{h} - \frac{\epsilon}{2} := N_2^y$ .

Continuing the above process, we can finally obtain four sequences  $\{M_n^x\}$ ,  $\{M_n^y\}$ ,  $\{N_n^x\}$ ,  $\{N_n^y\}$  such that

$$\begin{aligned}
M_n^x &= \frac{1}{a} \left( \frac{r_0}{1 + kN_{n-1}^y} - d - bN_{n-1}^y \right) + \frac{\epsilon}{n}, \\
M_n^y &= \frac{m + cbM_n^x}{h} + \frac{\epsilon}{n}, \\
N_n^x &= \frac{1}{a} \left( \frac{r_0}{1 + kM_n^y} - d - bM_n^y \right) - \frac{\epsilon}{n}, \\
N_n^y &= \frac{m + cbN_n^x}{h} - \frac{\epsilon}{n}.
\end{aligned} \tag{25}$$

Clearly, we have that for any integer  $n > 0$ ,

$$N_n^x \leq V_1 \leq U_1 \leq M_n^x, \quad N_n^y \leq V_2 \leq U_2 \leq M_n^y.$$

Now, we prove that  $\{M_n^x\}$  and  $\{M_n^y\}$  are monotonically decreasing,  $\{N_n^x\}$ ,  $\{N_n^y\}$  are monotonically increasing, by means of inductive method.

Firstly, when  $n = 2$ , it is obvious

$$M_2^x \leq M_1^x, \quad M_2^y \leq M_1^y, \quad N_2^x \geq N_1^x, \quad N_2^y \geq N_1^y.$$

For  $n = k (k \geq 2)$ , we assume that  $M_k^x \leq M_{k-1}^x$  and  $N_k^x \geq N_{k-1}^x$  holds. Then further we have

$$\begin{aligned}
M_k^y &= \frac{m + cbM_k^x}{h} + \frac{\epsilon}{k} \leq \frac{m + cbM_{k-1}^x}{h} + \frac{\epsilon}{k-1} = M_{k-1}^y, \\
N_k^y &= \frac{m + cbN_k^x}{h} - \frac{\epsilon}{k} \leq \frac{m + cbN_{k-1}^x}{h} - \frac{\epsilon}{k-1} = N_{k-1}^y.
\end{aligned}$$

Then

$$\begin{aligned}
M_{k+1}^x &= \frac{1}{a} \left( \frac{r_0}{1 + kN_k^y} - d - bN_k^y \right) + \frac{\epsilon}{k+1} \leq \frac{1}{a} \left( \frac{r_0}{1 + kN_{k-1}^y} - d - bN_{k-1}^y \right) + \frac{\epsilon}{k} = M_k^x, \\
M_{k+1}^y &= \frac{m + cbM_{k+1}^x}{h} + \frac{\epsilon}{k+1} \leq \frac{m + cbM_k^x}{h} + \frac{\epsilon}{k} = M_k^y, \\
N_{k+1}^x &= \frac{1}{a} \left( \frac{r_0}{1 + kM_{k+1}^y} - d - bM_{k+1}^y \right) - \frac{\epsilon}{k+1} \geq \frac{1}{a} \left( \frac{r_0}{1 + kM_k^y} - d - bM_k^y \right) - \frac{\epsilon}{k} = N_k^x, \\
N_{k+1}^y &= \frac{m + cbN_{k+1}^x}{h} - \frac{\epsilon}{k+1} \geq \frac{m + cbN_k^x}{h} - \frac{\epsilon}{k} = N_k^y.
\end{aligned}$$

Therefore  $\{M_n^x\}$  and  $\{M_n^y\}$  are monotonically decreasing,  $\{N_n^x\}$ ,  $\{N_n^y\}$  are monotonically increasing. Consequently, the limits of these sequences exist. Let

$$\lim_{n \rightarrow +\infty} M_n^x = x_1, \quad \lim_{n \rightarrow +\infty} M_n^y = y_1, \quad \lim_{n \rightarrow +\infty} N_n^x = x_2, \quad \lim_{n \rightarrow +\infty} N_n^y = y_2.$$

Taking  $n \rightarrow +\infty$  in (25), we obtain

$$\begin{aligned}
x_1 &= \frac{1}{a} \left( \frac{r_0}{1 + ky_2} - d - by_2 \right), \\
y_1 &= \frac{m + cbx_1}{h}, \\
x_2 &= \frac{1}{a} \left( \frac{r_0}{1 + ky_1} - d - by_1 \right), \\
y_2 &= \frac{m + cbx_2}{h}.
\end{aligned} \tag{26}$$

It follows from (26) that

$$\begin{aligned} [ax_1 + d + \frac{b}{h}(m + cbx_2)] [1 + \frac{k}{h}(m + cbx_2)] &= r_0, \\ [ax_2 + d + \frac{b}{h}(m + cbx_1)] [1 + \frac{k}{h}(m + cbx_1)] &= r_0. \end{aligned} \quad (27)$$

Subtracting the first equation of (27) from the second equation, we obtain

$$(x_2 - x_1) [\frac{cb^2}{h} + \frac{cbdk}{h} + \frac{2cb^2km}{h^2} + \frac{c^2b^3k}{h^2}(x_1 + x_2) - a - \frac{akm}{h}] = 0. \quad (28)$$

Assume that  $x_1 \neq x_2$ , (28) gives

$$c^2b^3k(x_1 + x_2) = -cb^2h - cbdkh - 2cb^2km + ah^2 + akmh. \quad (29)$$

Substituting (29) into (27), we have

$$\begin{aligned} B_1x_1^2 + B_2x_1 + B_3 &= 0, \\ B_1x_2^2 + B_2x_2 + B_3 &= 0, \end{aligned} \quad (30)$$

where

$$\begin{aligned} B_1 &= bck(-ab^3c^2hk + b^5c^3k), \\ B_2 &= bck(-2ab^2chkm - abcdh^2k + a^2h^2km + a^2h^3 - ab^2ch^2 \\ &\quad + b^3c^2dhk + b^4c^2km) + (-ab^3c^2hk + b^5c^3k)(h + km), \\ B_3 &= (-2ab^2chkm - abcdh^2k + a^2h^2km + a^2h^3 - ab^2ch^2 \\ &\quad + b^3c^2dhk + b^4c^2km)(h + km) - b^3c^2h^2kr_0. \end{aligned}$$

Therefore,  $x_1$  and  $x_2$  are two positive solutions of the following equation

$$B_1x^2 + B_2x + B_3 = 0. \quad (31)$$

Since (20) holds,  $B_1 > 0$  and  $B_3 < 0$ . Hence, (31) only has one positive solution, that is,  $x_1 = x_2$ . And from (26), we have  $y_1 = y_2$ .

Notice that the condition for the existence of  $E^*$  is

$$r_0 > d + \frac{dhkm + bhm + bkm^2}{h^2}.$$

Due to

$$\begin{aligned} (1 + kM_1^y)(d + bM_1^y) - (d + \frac{dhkm + bhm + bkm^2}{h^2}) \\ = \frac{cb^2r_0}{ah} + \frac{cbdkr_0}{ah} + \frac{2cb^2kmr_0}{ah^2} + \frac{c^2b^3r_0^2k}{a^2h^2} > 0. \end{aligned}$$

Hence, (19) holds implies the existence of interior equilibrium  $E^*(x^*, y^*)$ , which satisfies (26). So we have  $x_1 = x_2 = x^*$  and  $y_1 = y_2 = y^*$ , then it follows  $U_1 = V_1 = x^*$  and  $U_2 = V_2 = y^*$ . Thus, we can obtain

$$\lim_{n \rightarrow +\infty} x(n) = x^*, \quad \lim_{n \rightarrow +\infty} y(n) = y^*,$$

that is, the equilibrium  $E^*(x^*, y^*)$  is globally attractive.

This completes the proof of Theorem 8.  $\square$

**Remark 1.** In this paper, we apply the comparison principle of difference equations to prove the globally asymptotic stability. However, in the process of inequality scaling, sometimes the scaling steps are too broad, leading to too narrow sufficient conditions. Therefore, a worthiness further study is whether the interior equilibrium  $E^*$  is still globally stable when  $r_0 > 1$  or  $hM_1^y > 1$ . It is an open problem which may be tested by using numerical simulation.

### 5. Global Stability of Prey Free Equilibrium

The previous section shows that populations of predators and prey can reach coexistence under certain conditions. However, a major feature of this paper is to consider that the predator has other food resource. In this section, we study the global stability of the prey free equilibrium  $E_1(0, \frac{m}{h})$ , i.e., discuss whether predators population remain stable when the prey population becomes extinct.

We first state the following lemmas.

**Lemma 6** ([34]). Assume that  $\{x(n)\}$  satisfy  $x(n) > 0$  and

$$x(n+1) \leq x(n) \exp[a - bx(n)], \quad n \in \mathbb{N},$$

where  $a$  and  $b$  are positive constants. Then

$$\limsup_{n \rightarrow +\infty} x(n) \leq \frac{\exp(a-1)}{b} := M.$$

**Lemma 7** ([34]). Assume that  $\{x(n)\}$  satisfy  $x(n) > 0$  and

$$x(n+1) \geq x(n) \exp[a - bx(n)], \quad n \in \mathbb{N},$$

where  $a$  and  $b$  are positive constants. Then

$$\liminf_{n \rightarrow +\infty} x(n) \geq \frac{a}{b} \exp(a - bM),$$

where  $M$  is given by Lemma 6.

Now we give the main theorems of this section.

**Theorem 9.** Assume that

$$r_0 < d + \frac{bm}{h} \quad (32)$$

hold.  $(x(n), y(n))$  is any positive solution of system (4), then

$$\lim_{n \rightarrow +\infty} x(n) = 0.$$

**Proof.** Since (32) we get  $\frac{r_0 - d}{m} < \frac{b}{h}$ . Take the positive constants  $\alpha$  and  $\beta$  such that

$$\frac{r_0 - d}{m} < \frac{\beta}{\alpha} < \frac{b}{h}.$$

Then we have

$$\alpha b - \beta h > 0. \quad (33)$$

And there exists  $\delta > 0$  such that

$$\alpha(r_0 - d) - \beta m < -\delta < 0. \quad (34)$$

For any  $p \in \mathbb{N}$ , we can obtain

$$\begin{aligned} \ln \frac{x(p+1)}{x(p)} &= \frac{r_0}{1 + ky(p)} - d - ax(p) - by(p) \\ &\leq r_0 - d - ax(p) - by(p), \\ \ln \frac{y(p+1)}{y(p)} &= m - hy(p) + cbx(p). \end{aligned} \quad (35)$$

Then from (33)–(35), we get

$$\begin{aligned}\alpha \ln \frac{x(p+1)}{x(p)} - \beta \ln \frac{y(p+1)}{y(p)} &\leq [\alpha(r_0 - d) - \beta m] - (\alpha a + \beta cb)x(p) - (\alpha b - \beta h)y(p) \\ &\leq \alpha(r_0 - d) - \beta m < -\delta < 0.\end{aligned}$$

Add both sides from 0 to  $n - 1$ , we have

$$\alpha \ln \frac{x(n)}{x(0)} - \beta \ln \frac{y(n)}{y(0)} < -\delta n,$$

then

$$x(n) < \left[ \left( \frac{y(n)}{y(0)} \right)^\beta (x(0))^\alpha \right]^{\frac{1}{\alpha}} \exp\left(-\frac{\delta}{\alpha} n\right). \quad (36)$$

From Lemma 6, since  $x(n+1) \leq x(n) \exp[r_0 - ax(n)]$ , we have

$$\limsup_{n \rightarrow +\infty} x(n) \leq \frac{\exp(r_0 - 1)}{a}.$$

For any sufficiently small  $\epsilon > 0$ , there exists an integer  $N_1 > 0$  such that if  $n \geq N_1$ , then

$$x(n) \leq \frac{\exp(r_0 - 1)}{a} + \epsilon := M_1. \quad (37)$$

Then from the second equation of system (4) for  $n \geq N_1$ , we have

$$y(n+1) \leq y(n) \exp[m + cbM_1 - hy(n)], \quad n \geq N_1. \quad (38)$$

From Lemma 6, we obtain

$$\limsup_{n \rightarrow +\infty} y(n) \leq \frac{\exp(m + cbM_1 - 1)}{h}. \quad (39)$$

Hence, for any sufficiently small  $\epsilon > 0$ , there exists an integer  $N_2 > N_1$  such that if  $n \geq N_2$ , then  $y(n) \leq \frac{\exp(m + cbM_1 - 1)}{h} + \epsilon := M_2$ . Then, the inequality (36) is equivalent to

$$x(n) < \left[ \left( \frac{M_2}{y(0)} \right)^\beta (x(0))^\alpha \right]^{\frac{1}{\alpha}} \exp\left(-\frac{\delta}{\alpha} n\right), \quad n \geq N_2. \quad (40)$$

(40) shows that for sufficiently small  $\epsilon > 0$ ,  $x(n) \rightarrow 0$ , when  $n \rightarrow +\infty$ .

This ends the proof of Theorem 9.  $\square$

Now we consider the following system

$$y_1(n+1) = y_1(n) \exp[m - hy_1(n)], \quad (41)$$

whose any positive solution is  $y_1(n) = \frac{m}{h}$ . We can get the following theorem.

**Theorem 10.** Assume that (32) and

$$m < \ln 2 + 1 \quad (42)$$

hold, then

$$\lim_{n \rightarrow +\infty} [y(n) - y_1(n)] = 0,$$

where  $y_1(n)$  is any positive solution of system (41).

**Proof.** Since (32) holds, we have  $\lim_{n \rightarrow +\infty} x(n) = 0$  according to Theorem 9. That is, for any sufficiently small  $\epsilon > 0$ , there exists an integer  $N_3 > N_2$  such that if  $n \geq N_3$ , then

$$x(n) < \epsilon. \quad (43)$$

In order to prove  $\lim_{n \rightarrow +\infty} [y(n) - y_1(n)] = 0$ , we assume that

$$y(n) = y_1(n) \exp[k(n)].$$

Then the second equation of system (4) is equivalent to

$$\begin{aligned} k(n+1) &= k(n) - hy_1(n) \exp[k(n)] + cbx(n) + hy_1(n) \\ &= [1 - hy_1(n) \exp(\theta(n)k(n))]k(n) + cbx(n), \end{aligned} \quad (44)$$

where  $\theta(n) \in [0, 1]$ . Hence,  $y_1(n) \exp(\theta(n)k(n))$  is between  $y_1(n)$  and  $y(n)$ . To prove that  $\lim_{n \rightarrow +\infty} [y(n) - y_1(n)] = 0$  is equivalent to proving

$$\lim_{n \rightarrow +\infty} k(n) = 0. \quad (45)$$

Since  $y(n) \exp[m - hy(n)] \leq y(n+1) \leq y(n) \exp[m + cb\epsilon - hy(n)]$ , from Lemma 6 and Lemma 7, we obtain

$$\begin{aligned} \limsup_{n \rightarrow +\infty} y_1(n) &\leq \frac{\exp(m + cb\epsilon - 1)}{h} := U_1, \\ \liminf_{n \rightarrow +\infty} y_1(n) &\geq \frac{m}{h} \exp(m - hU_1) := V_1. \end{aligned} \quad (46)$$

In addition, according to system (41) and Lemmas 6 and 7, we have

$$\begin{aligned} \limsup_{n \rightarrow +\infty} y_1(n) &\leq \frac{\exp(m - 1)}{h} \leq U_1, \\ \liminf_{n \rightarrow +\infty} y_1(n) &\geq \frac{m}{h} \exp(m - hU_1) := V_1. \end{aligned} \quad (47)$$

Hence, for any sufficiently small  $\epsilon > 0$ , there exists an integer  $N_4 > N_3$  such that if  $n \geq N_4$ , then

$$V_1 - \epsilon \leq y(n), y_1(n) \leq U_1 + \epsilon, \quad n \geq N_4. \quad (48)$$

Assume

$$\lambda = \max\{|1 - hV_1|, |1 - hU_1|\} < 1. \quad (49)$$

Then, for any sufficiently small  $\epsilon > 0$ , we assume

$$\lambda_\epsilon = \max\{|1 - h(V_1 - \epsilon)|, |1 - h(U_1 + \epsilon)|\} < 1. \quad (50)$$

From (43), (44), (48) and (50), we have

$$\begin{aligned} |k(n+1)| &\leq \max\{|1 - h(V_1 - \epsilon)|, |1 - h(U_1 + \epsilon)|\} |k(n)| + cb\epsilon \\ &= \lambda_\epsilon |k(n)| + cb\epsilon, \quad n \geq N_4. \end{aligned} \quad (51)$$

Then we can get the following equation from (51):

$$|k(n)| \leq \lambda_\epsilon^{n-N_4} |k(N_4)| + \frac{1 - \lambda_\epsilon^{n-N_4}}{1 - \lambda_\epsilon} cb\epsilon, \quad n \geq N_4. \quad (52)$$

Since  $\lambda_\epsilon < 1$  and  $\epsilon$  is sufficiently small, we can get  $\lim_{n \rightarrow +\infty} k(n) = 0$ , i.e.,  $\lim_{n \rightarrow +\infty} [y(n) - y_1(n)] = 0$  set up when  $\lambda < 1$ . Notice that

$$1 - hU_1 < 1 - hV_1 < 1,$$

then  $\lambda < 1$  is equivalent to

$$1 - hU_1 > -1,$$

i.e.,

$$\exp(m + cb\epsilon - 1) < 2. \quad (53)$$

Hence, (53) can be satisfied if (42) holds, since  $\epsilon$  is sufficiently small.  $\lim_{n \rightarrow +\infty} [y(n) - y_1(n)] = 0$  has been proved.  $\square$

According to Theorems 9 and 10, the prey free equilibrium  $E_1(0, \frac{m}{h})$  is globally attractive.

## 6. Numerical Simulations

In this section, we show the feasibility of the main results in this paper. We show the impact of the fear effect on the discrete system visually through numerical simulations. In addition, a feature of this paper is that predators have other food resource, so we explore it in this section.

**Example 1.** We show the feasibility of Theorem 7. In Figure 1a, the chosen parameters are:

$$k = 0.7, d = 0.3, a = 3, b = 0.5, m = 0.5, h = 1, c = 0.5, r_0 = 4, \quad (54)$$

with initial value  $(0.8, 0.5)$ , which satisfies the condition (1) of Theorem 7, i.e.,  $r_0 > d + (dhkm + bhm + bkm^2)/h^2 = 0.7425$ ,  $T = 2.76 > 2$  and  $2T - 4 < S = 1.61 < T$ , with  $x^* = 0.69$ ,  $y^* = 0.67$ . So that the interior equilibrium  $E^*$  exists and is locally stable. In fact, we can see that  $E^*$  is also globally stable in Figure 1a. Hence, regarding the open problem mentioned by Remark 1, we can verify that  $E^*$  may be globally stable probably in the case of  $r_0 > 1$ .

In Figure 1b, we take the parameter values as

$$k = 0, d = 0.3, a = 3, b = 0.5, m = 0.5, h = 1, c = 0.5, r_0 = 4, \quad (55)$$

with initial value  $(0.8, 0.5)$ . (55) satisfies the condition (3) of Theorem 7, that is,  $r_0 > d + (dhkm + bhm + bkm^2)/h^2 = 0.55$ ,  $T = 4.088 > 2$  and  $S = 2.677 < 2T - 4 = 4.176$ , with  $x^* = 1.104$  and  $y^* = 0.776$ . Figure 1b shows that  $E^*$  is unstable which confirm the feasibility of conclusion of Theorem 7.

**Example 2.** In this example, we show the feasibility of Section 5. We set the parameter values as

$$r_0 = 0.5, d = 0.3, a = 3, b = 0.5, m = 0.5, h = 1, c = 0.5, k = 0.7, \quad (56)$$

with initial value  $(0.8, 0.5)$ , which satisfies the conditions (32) and (42),  $E_1(0, 0.5)$  is globally stable in Figure 2.

Moreover, set the parameter values as

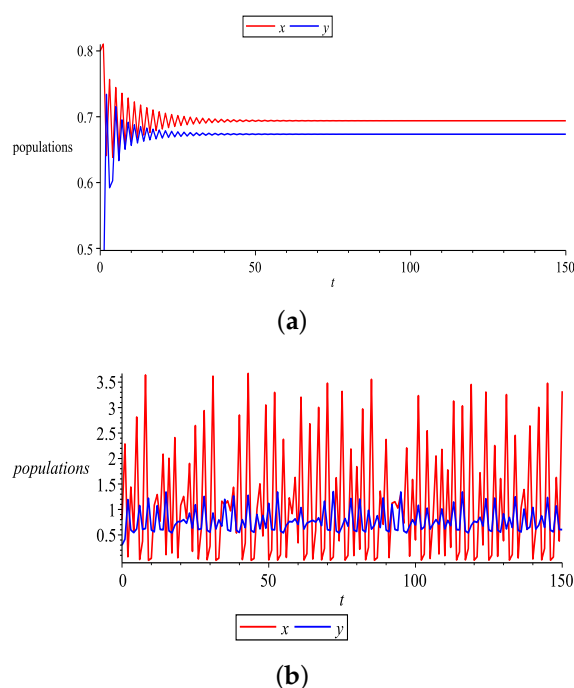
$$r_0 = 4, d = 0.3, a = 3, b = 0.5, h = 1, c = 0.5, k = 5, m > 2, \quad (57)$$

which satisfies the following conditions:

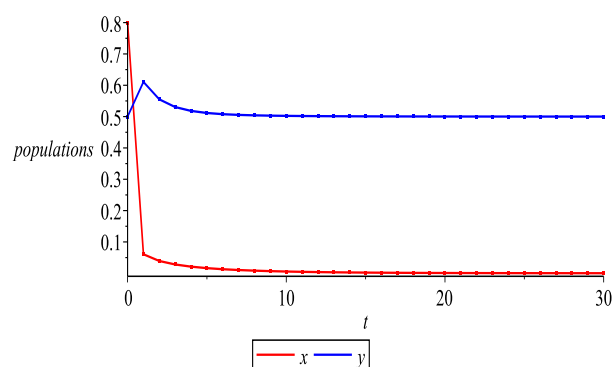
$$m > 2, \quad r_0 = 4 < d + \frac{dhkm + bhm + bkm^2}{h^2} < 14.3,$$

i.e., the condition (2) of Theorem 5. Now we choose  $m$  as the bifurcation parameter to draw the plot. In Figure 3, the prey population tends to die out when  $m$  is large enough and the predator population becomes chaos as  $m$  increases. In addition, under (57) the condition (42) is not satisfied,

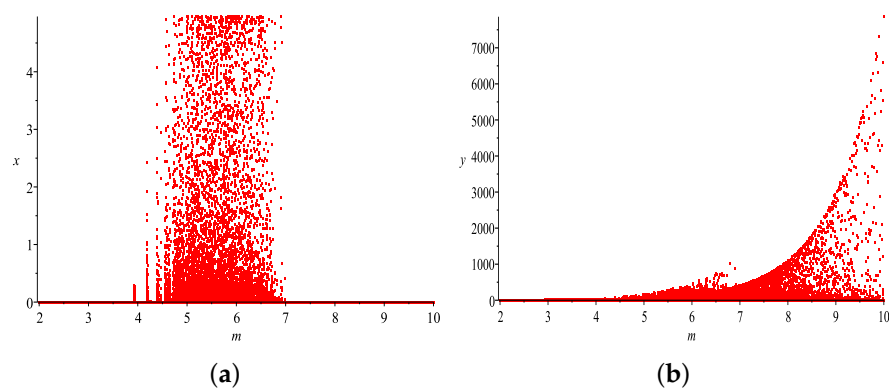
so the chaos in Figure 3 can be used as a counter example of Section 5 to further prove the condition of Section 5, that is,  $E_1$  is not globally stable if (58) holds.



**Figure 1.** (a) The stability of interior equilibrium with initial value  $(0.8, 0.5)$ . (b) The instability of interior equilibrium with initial value  $(0.8, 0.5)$ .



**Figure 2.** The global stability of prey free equilibrium  $E_1$ , with the initial values  $(0.8, 0.5)$ .

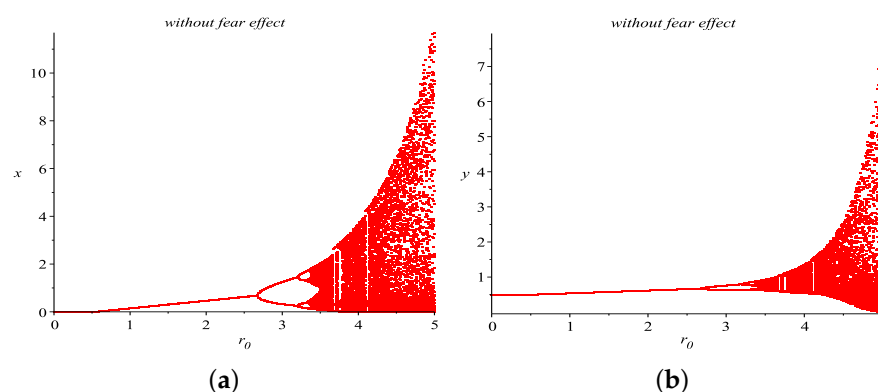


**Figure 3.** (a) The population density of prey varies with bifurcating parameter  $m$ . (b) The population density of predator varies with bifurcating parameter  $m$ .

**Example 3.** To investigate the impact of the fear effect on system (4), we set the parameter values as

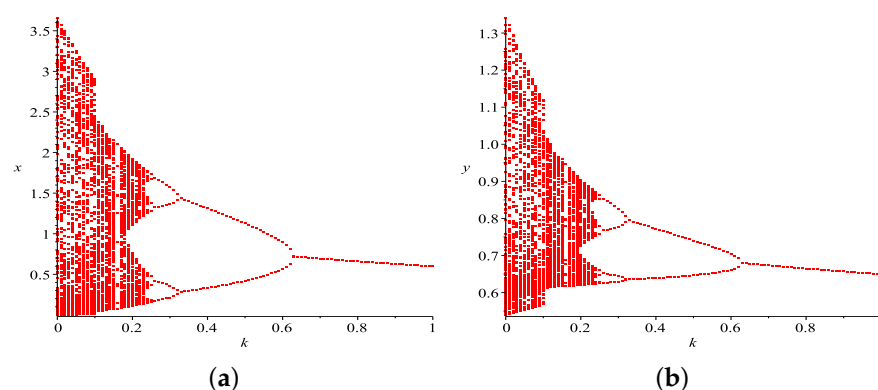
$$d = 0.3, a = 3, b = 0.5, m = 0.5, h = 1, c = 0.5, \quad (58)$$

with initial value  $(0.8, 0.5)$ . We first choose the growth rate  $r_0$  as the bifurcation parameter to draw the diagram without fear effect ( $k = 0$ ). In Figure 4, the bifurcation plot shows that there is a single stable equilibrium when  $r_0$  is less than 2.67. When  $r_0$  reaches 3.2, the single stable equilibrium bifurcates into 2 stable points with a stable period-2 cycle. When  $r_0$  reaches 3.32, the period-2 cycle bifurcates into a stable period-4 cycle. Then if  $r_0 > 3.32$  the system (4) becomes chaotic dynamics without fear effect. In this case, we observe that as  $r_0$  increases, the system (4) goes from stable to unstable.



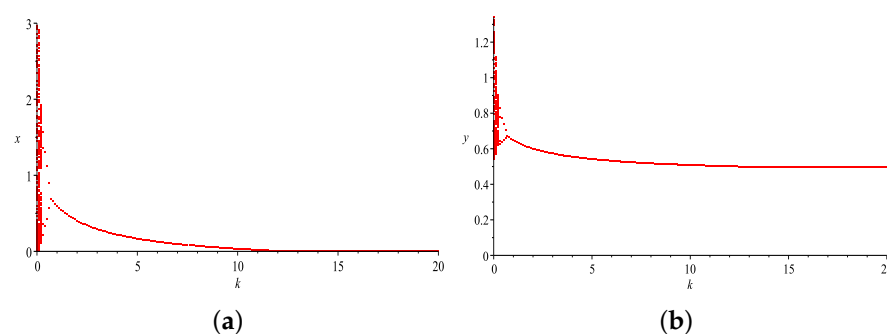
**Figure 4.** (a) The population density of prey varies with bifurcating parameter  $r_0$  without fear effect ( $k = 0$ ), where other parameters are give in (58). (b) The population density of predator varies with bifurcating parameter  $r_0$  without fear effect.

Now we set  $r_0 = 4$  to see if the original chaotic dynamics would become stable because of the presence and growth of  $k$ . In Figure 5, we observe that the system (4) changes from chaos to a stable period-4 cycle, then to a stable period-2 cycle, and finally to a stable equilibrium if we increase the value of  $k$ . Further, Figure 5 shows that when  $r_0 = 4 > 1$  the system (4) may be still globally stable which just could illustrate the feasibility of Remark 1.



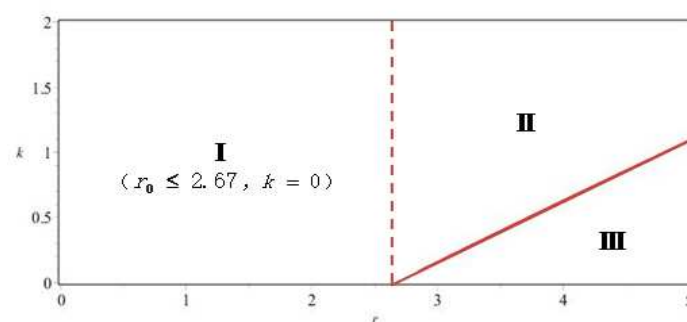
**Figure 5.** (a) The bifurcation plot of prey species corresponding to the bifurcating parameter  $k$ , where other parameters are give in (58) and  $r_0 = 4$ . (b) The bifurcation plot of predator species corresponding to the bifurcating parameter  $k$  and  $r_0 = 4$ .

In Figure 6, it is obvious that with the increase of  $k$ , the prey tends to become extinct, while the population of predators tends to a certain constant. It demonstrates the correctness of the global attractivity of prey free equilibrium in Section 5.



**Figure 6.** (a) The globally stability of the prey species population. (b) The globally stability of the predator species population, where  $r_0 = 4$  and other parameters are given in (58).

**Example 4.** We still use the parameter settings of (58) to discuss about the impact of fear effect. Figure 7 shows that in domain I the interior equilibrium of system (4) is locally stable without fear effect. In this case, the populations of prey and predators eventually tend to positive constants under certain initial conditions. In domain II, as the fear level  $k$  gradually increases,  $E^*$  changes from unstable state to asymptotically stable state, which illustrates that the cost of fear  $k$  can increase the stability of system.



**Figure 7.** The stability region of  $E^*$  in  $r_0 - k$  parametric space. Other parameters are given in (58).

## 7. Conclusions

In this paper, we have studied a discrete-time predator–prey system incorporating fear effect of the prey with the predator has other food resource. System (4) is proposed based on the continuous system (3), which was proposed by Zhu et al. [32]. It is shown that the system (4) has a trivial equilibrium for all positive parameters which is always unstable. This property is similar to the conclusion of Zhu et al. [32]. Meanwhile, there exists a prey free equilibrium for all positive parameters and it may be locally stable for some values of parameters while it becomes unstable for other parametric values. System (4) has a predator free equilibrium and a positive equilibrium for some values of parameters. We get the existence condition of positive equilibrium is  $r_0 > [d + (dhkm + bhm + bkm^2)/h^2]$  as same as the result of Zhu et al. Moreover, the rigorous mathematical proof of the global attractivity of the positive equilibrium is given by using iteration scheme and the comparison principle of difference equations. Compared with the results of continuous system (3), we obtain more strict sufficient conditions. However, since we scale the equation when using the comparison principle, it may leads to too strict for the global stability condition of positive equilibrium. In this regard, we propose Remark 1 and verify the conjecture of Remark 1 in the numerical simulation.

A major feature of this paper is to consider that the predator has other food resource. So we specifically study the stability of the prey free equilibrium. In [32], Zhu et al. studied locally and globally asymptotically stable of the prey free equilibrium of continuous

system (3). Their result (see Theorem 1) indicates that  $E_1(0, \frac{m}{h})$  remains stable if  $r_0 < [d + (dhkm + bhm + bkm^2)/h^2]$  holds. However, in this paper, we obtain Theorems 5, 9 and 10 which shows that the stability of  $E_1$  is related not only to the value of  $r_0$ , but also to the value of  $m$ , i.e., the intrinsic grow rate of the predator species. Analytical results and numerical simulation (Figure 3) show that the prey and the predator may go extinct at the same time if the intrinsic grow rate of the predator is not large enough. With the increase of  $m$ , the prey population first appears chaos and then tends to die out while the predator population becomes chaos. Although  $r_0 < [d + (dhkm + bhm + bkm^2)/h^2]$  holds,  $E_1$  is not globally stable as the intrinsic grow rate of the predator increasing. This is the difference between this paper and Zhu et al.'s. Thus, we can also confirm what we stated in the first section: although the dynamic behaviors of discrete and continuous systems are similar in general, there are some differences that discrete systems often have more complex dynamic behavior than continuous.

On the other hand, we consider the effect of the value of fear parameter  $k$  on the dynamic behavior of the discrete system (4). In Section 6, the influence of fear effect is visually demonstrated. From Figures 4–7, it is very intuitive to see that the fear effect can turn an unstable system into a stable one. In other words, the fear effect enhances the stability of the discrete system (4).

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