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On the Norm of the Abelian p-Group-Residuals

Baojun Li, Yu Han, Lü Gong * and Tong Jiang

School of Sciences, Nantong University, Nantong 226019, China; libj@ntu.edu.cn (B.L.); 2002310009@stmail.ntu.edu.cn (Y.H.); 1902310009@stmail.ntu.edu.cn (T.J.)
* Correspondence: gonglv@ntu.edu.cn

Abstract: Let G be a group. $D_p(G) = \bigcap_{H \le G} N_G(H'(p))$ is defined and, the properties of $D_p(G)$ are investigated. It is proved that $D_p(G) = P[A]$, where $P = D(P)$ is the Sylow p-subgroup and $A = N(A)$ is a Hall p'-subgroup of $D_p(G)$, respectively. Furthermore,it is proved in a group G that (1) $D_p(G) = 1$ if and only if $C_G(G'(p)) = 1$; (2) $O_{p'}(D_p(G)) \le Z_\infty(O^p(G))$ and (3) if $Z(G'(p)) = 1$, then $C_G(G'(p)) = D_p(G)$.

Keywords: finite group; abelian p-group residual; soluble group; normalizer

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1. Introduction

All groups considered in this paper are finite. The reader is referred to [1] for notation and terminology. Recall that the norm $N(G)$ of a group G, introduced by Baer in [2], is the intersection of the normalizers of all subgroups of G (cf. [2]). A closely related subgroup was introduced and studied by Wielandt in [3]. It is defined as the intersection of the normalizers of all subnormal subgroups of G and called the Wielandt subgroup of G (see [4]).

Some generalisations of Baer and Wielandt's subgroups were considered and a lot of interesting results have been obtained (see [5–9]). The idea behind these investigations is to consider a set S of subgroups of G and consider the intersection of the normalizers of all subgroups in S.

If S is the set of the commutators of all subgroups of a group G, the intersection $D(G)$ of their normalisers was studied in [5].

Our main goal in this paper is to study a local version of $D(G)$: the intersection of the normalizers of the residuals of all subgroups of G with respect to the class of all abelian p-groups, p a prime.

Given a prime p and a group G, let $G'(p)$ denote the residual of G with respect to the class of all abelian p-groups; it is know that $G'(p)$ is the unique smallest normal subgroup of G for which the corresponding factor group is an abelian p-group. There is, of course, a relationship between $G'(p)$ and $O^p(G)$ which is the unique smallest normal subgroup of G whose factor group is a (not necessarily abelian) p-group. In fact, $O^p(G) \le G'(p)$ and $G'(p)/O^p(G)$ is the commutator subgroup of $G/O^p(G)$. Therefore $G'(p) = O^p(G)G'$. This subgroup plays an important role in group theory because it is the kernel of the transfer homomorphism from G to P/P', where P is a Sylow p-subgroup of G ([10], 10.1.5).

Definition 1. *Let p a prime. The norm $D_p(G)$ of the abelian p-group-residuals is the subgroup*

$$D_p(G) = \bigcap_{H \le G} N_G(H'(p)).$$

Note that $D_p(G) \ne D(G)$ in general (it is enough to consider the alternating group of deree 4).

We prove:

Theorem 1. *Let G be a group. Then $D_p(G) = P[A]$, where P is the Sylow p-subgroup and A is a Hall p'-subgroup of $D_p(G)$. Moreover, $P = D(P)$ and $A = N(A)$. In particular, $\cap_{p||G|} D_p(G) = N(G)$.*

Theorem 2. *Let G be a group. Then*

(1) $D_p(G) = 1$ if and only if $C_G(G'(p)) = 1$;
(2) $O_{p'}(D_p(G)) \leq Z_\infty(O^p(G))$;
(3) if $Z(G'(p)) = 1$, then $C_G(G'(p)) = D_p(G)$.

2. Elementary Properties on $D_p(G)$

In this section, we list some elementary properties of $D_p(G)$ that will be used in the proofs of the main results.

Lemma 1. *Let G be a group. Then*

(1) If $H \leq G$, then $H'(p) \leq G'(p)$;
(2) if $N \trianglelefteq G$ and $N \leq H \leq G$, then $(H/N)'(p) = H'(p)N/N$;
(3) $G'(p)$ is nilpotent if and only if $(G/\Phi(G))'(p)$ is nilpotent;
(4) if $G = MN$, where $M \leq G$ and $N \trianglelefteq G$, then $G'(p) \leq M'(p)N$. In particular, $(M \times N)'(p) = M'(p) \times N'(p)$.

Proof.

(1) Let $H \leq G$. Since $H/(H \cap O^p(G)) \cong HO^p(G)/O^p(G) \leq G/O^p(G)$, $H/(H \cap O^p(G))$ is a p-group and so $H'(p) = H'O^p(H) \leq H'(H \cap O^p(G)) \leq G'O^p(G) = G'(p)$.
(2) Let $O^p(H/N) = R/N$. $O^p(H)N \leq R$ Since $(H/N)/O^p(H/N) = (H/N)/(R/N) \cong H/R$. Conversely, $H/O^p(H)N \cong (H/O^p(H)/(O^p(H)N/O^p(H))$ and $H/O^p(H)N \cong (H/N)/(O^p(H)N/N)$, so $R/N \leq O^p(H)N/N$. Hence $O^p(H/N) = O^p(H)N/N$. Then $(H/N)'(p) = (H/N)'O^p(H/N)$
$= (H'N/N)(O^p(H)N/N) = H'O^p(H)N/N = H'(p)N/N$.
(3) Clearly, $G'(p)$ is nilpotent if and only if $G'(p)\Phi(G)/\Phi(G) \cong G'(p)/G'(p) \cap \Phi(G)$ is nilpotent. So (3) follows from (2).
(4) By $G' \leq M'N$ and $O^p(G) \leq O^p(M)N$, we get $G'(p) \leq M'(p)N$. If $M \trianglelefteq G$, then $G'(p) \leq N'(p)M$. Thus $G'(p) \leq M'(p)N \cap N'(p)M = M'(p)(N \cap N'(p)M) = M'(p)N'(p)(M \cap N)$. Hence, if $G = M \times N$, then $G'(p) = M'(p) \times N'(p)$ by (1).

□

Proposition 1. *Let G be a group. Then*

(1) $N(G)C_G(G'(p)) \leq D_p(G) \leq D(G)$;
(2) $D_p(G)$ is soluble;
(3) if $M \leq G$, then $M \cap D_p(G) \leq D_p(M)$;
(4) if $N \trianglelefteq G$, then $D_p(G)N/N \leq D_p(G/N)$;
(5) if $G = A \times B$, where $A, B \leq G$ and $(|A|, |B|) = 1$, then $D_p(G) = D_p(A) \times D_p(B)$.

Proof.

(1) Since $H' \leq H'(p) \leq H$ and $H', H'(p)$ are characteristic subgroups of H, we have $N_G(H') \geq N_G(H'(p)) \geq N_G(H)$, that is, $N(G) \leq D_p(G) \leq D(G)$.
If $x \in C_G(G'(p))$, then x is a normalizer of $H'(p)$ for all $H \leq G$ by Lemma 1 (1). Hence $x \in D_p(G)$ and so, $C_G(G'(p) \leq D_p(G)$.
(2) It follows from (1) and $D(G)$ is soluble in ([9], Proposition 2.4).
(3) It is easy to see that $M \cap D_p(G) \leq \bigcap_{H \leq M} N_M(H'(p)) = D_p(M)$.
(4) If $x \in D_p(G)$, then xN normalizes $(H/N)'(p)$ for all $H/N \leq G/N$ by Lemma 1 (2). Hence $D_p(G)N/N \leq D_p(G/N)$.

(5) $H = (H \cap A) \times (H \cap B)$ for all $H \leq G$ by the hypotheses. It follows from Lemma 1 (4) that $H'(p) = (H \cap A)'(p) \times (H \cap B)'(p)$. Hence

$$\begin{aligned} N_G(H'(p)) &= N_G((H \cap A)'(p)) \cap N_G((H \cap B)'(p)) \\ &= (N_A((H \cap A)'(p)) \times B) \cap (A \times N_B((H \cap B)'(p))) \\ &= N_A((H \cap A)'(p)) \times N_B((H \cap B)'(p)), \end{aligned}$$

which implies that $D_p(G) = D_p(A) \times D_p(B)$.

□

Proposition 2. *Let $G \neq 1$ be a group. Then*

(1) If $G'(p)$ is nilpotent, then $D_p(G) > 1$.
(2) If $G'(p)$ is a minimal normal subgroup of G and $D_p(G)$ is nilpotent, then $C_G(G'(p)) = D_p(G)$.

Proof.

(1) If $G'(p) = 1$, then G is abelian p-group and $G = D_p(G) > 1$. If $G'(p) \neq 1$, then $D_p(G) \geq C_G(G'(p)) \geq Z(G'(p)) > 1$ by Proposition 1 (1).
(2) Since $G'(p)$ is a minimal normal subgroup of G, $G'(p) \cap F(G) = G'(p)$ or 1. If $G'(p) \cap F(G) = 1$, then $G'(p)F(G) = [G'(p) \times F(G)]$ and so, $F(G) \leq C_G(G'(p))$. If $G'(p) \cap F(G) \neq 1$, then $G'(p) \leq F(G)$ and $[G'(p), F(G)] \leq G'(p)$. However, $F(G)$ is nilpotent and hence $[G'(p), F(G)] < G'(p)$. Thus, $[G'(p), F(G)] = 1$ and we have $F(G) \leq C_G(G'(p))$.
By Proposition 1 (1), $F(G) \leq C_G(G'(p)) \leq D_p(G)$. The nilpotency of $D_p(G)$ implies that $F(G) = C_G(G'(p)) = D_p(G)$.

□

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. (1) By Proposition 1 (2), $D_p(G)$ is soluble. Then $D_p(G)$ has a Hall p'-subgroup, denoted by A. Let P be a Sylow p-subgroup of $D_p(G)$. Then $D_p(G) = PA$.

Firstly, A is a Dedekind group. Case 1. A is a q-group.

For a subgroup H of A. Since $A \leq D_p(G)$, we have A normalizes $H'(p)$. It follows from $H'(p) = H$ that H is normal in A, that is, $A = N(A)$ is a Dedekind group.

Case 2. A is not a q-group.

Let A_q and A_r be any Sylow q-subgroup and Sylow r-subgroup of A, respectively, $q \neq r$. Since A_q and A_r are subgroups of $D_p(G)$, A_q normalizes $A_r'(p)$ and A_r normalizes $A_q'(p)$. Then it follows from $A_r'(p) = A_r$ and $A_q = A_q'(p)$ that $[A_q, A_r] = 1$, that is, A is nilpotent. For a subgroup H of A_q, by the same argument above, A_q is Dedekind group, hence A is Dedekind group.

Secondly, $P = D(P)$ is a D-group.

For a subgroup K of P. Since $P \leq D_p(G)$, we have P normalizes $K'(p)$. It follows from $K'(p) = K'$ that K' is normal in P, that is, $P = D(P)$ is a D-group.

Finally, $D_p(G) = P[A]$.

Since P normalizes $A'(p)$ and $A'(p) = A$, we have $D_p(G) = P[A]$.

(2) By (1), the Hall p'-subgroup of $D_p(G)$ is Dedekind group for any prime $p \in \pi(G)$, then $\cap_{p||G|} D_p(G) \leq N(G)$. Hence $\cap_{p||G|} D_p(G) = N(G)$ by Proportion 1 (1). □

Suppose that a group H acts on a group G. We say that H acts hypercentrally on N if N has a subnormal series $1 = N_0 \leq N_1 \leq \cdots \leq N_s = N$ such that $[H, N_i] \leq N_{i-1}$, for all $i = 1, 2, \cdots, s$ (cf. [11]). Clearly, if N is a normal subgroup of H then H acts hypercentrally on N if and only if $N \leq Z_\infty(H)$.

Lemma 2. *Let G be a $\{p, q\}$-group. Assume that N is a normal q-subgroup of G and H is a subgroup of G with $H = O^p(H)$. If $N \leq D_p(G)$, then H acts hypercentrally on N.*

Proof. Suppose that the lemma is not true. Let G be a counterexample of minimal order. Then

(1) $G = NH$.

If $NH < G$, then NH satisfies the condition of the lemma by Proposition 1 (3) and the choice of G shows that H acts hypercentrally on N, a contradiction.

(2) *Let T be a minimal supplement of $C_G(N)$ in G, then $O^p(T) = T$.*

Since $H = O^p(H)$ and N is a normal q-subgroup, we have $G = HN = O^p(H)N = O^p(G)$. Let T be a minimal supplement of $C_G(N)$ in G. Then $G = C_G(N)T$. Assume that $O^p(T) < T$. Then $C_G(N)O^p(T) < G$ by the minimality of T. It is easy to see that $G/C_G(N)O^p(T) = C_G(N)T/C_G(N)O^p(T) \cong T/C_T(N)O^p(T)$ is a p-group, and then $O^p(G) \leq C_G(N)O^p(T) < G$, a contradiction.

(3) $G = NT$, *and* $T \trianglelefteq G$.

If $NT < G$, then NT satisfies the condition of the lemma by Proposition 1 (3). By the choice of G, T acts hypercentrally on N. Let T_p and $C_G(N)_p$ be Sylow p-subgroup of T and $C_G(N)$, respectively. Then T_p acts trivially on N, and then $G_p = C_G(N)_pT_p$ acts trivially on N. Since G is a $\{p,q\}$-group, $G/C_G(N_i/N_{i-1})$ is a q-group for each G-chief factor N_i/N_{i-1} of N. However, $O_q(G/C_G(N_i/N_{i-1})) = 1$ by ([12], Lemma 1.7.11). It follows that $G/C_G(N_i/N_{i-1}) = 1$. This shows that G acts hypercentrally on N, and so does H, a contradiction. Thus, $G = NT$

Since N normalizes $T'(p)$ and $T = T'(p)$, we have $T \trianglelefteq G = NT$.

(4) $G = RT$, *where R is a nontrivial normal subgroup in G with $R \leq N$.*

If $RT < G$, then one can see that RT satisfies the condition by Proposition 1 (3). Hence T acts hypercentrally on R by the choice of G. Since $N/R \leq D_p(RT/R)$ and $O^p(RT/R) = RT/R$, then, by the choice of G, RT/R acts hypercentrally on N/R. Then T acts hypercentrally on N, that is, $G = NT$ acts hypercentrally on N by ([12], Lemma 1.7.11), so does H, a contradiction.

(5) *Final contradiction.*

Since $G/R = TR/R$ acts hypercentrally on N/R, without generality, we can assume $R = N$ is minimal normal in G. Then, by the minimality of N and the normality of T, we have that $G = N \times T$ or $G = T$.

If $G = N \times T$, then $N \leq Z(G)$, a contradiction.

Let $G = T$. Since T is the minimal supplement of $C_G(N)$ in G, we have that $T \cap C_G(N) \leq \Phi(T)$ by ([12], Lemma 2.3.4). Thus, $C_G(N) \leq \Phi(G)$. By the minimality of N and N, $O_q(G) \leq C_G(N) \leq \Phi(T) = \Phi(G)$. It follows that $O_{q,p}(G)$ is p-closed. Choose P to be a Sylow p-subgroup in $O_{q,p}(G)$. Then $P \trianglelefteq G$ and so, $P \leq C_G(N) \leq \Phi(G)$. Therefore $O_{q,p}(G) \leq \Phi(G)$, a contradiction. □

Proof of Theorem 2. (1) Since $C_G(G'(p)) \leq D_p(G)$, the necessity is clear.

Conversely, assume that $C_G(G'(p)) = 1$ and $D_p(G) > 1$. It implies that $G'(p) \cap D_p(G) > 1$. Otherwise, $D_p(G) \leq C_G(G'(p))$ and $C_G(G'(p)) \neq 1$. By Proposition 1 (2), $D_p(G)$ is soluble. So G has a minimal normal subgroup N such that $N \leq G'(p) \cap D_p(G)$. Then N is elementary abelian.

$N \leq Z(G')$.

Assume $G' \cap D_p(G) = 1$. Since $[G, D_p(G)] \leq [G,G] = G'$ and $[G, D_p(G)] \leq D_p(G)$, $[G, D_p(G)] \leq G' \cap D_p(G) = 1$. It follows that $D_p(G) \leq Z(G)$, a contradiction and thus $G' \cap D_p(G) \neq 1$. Since $G'(p) \cap D_p(G) \geq G' \cap D_p(G)$, we can assume that $N \leq G' \cap D_p(G)$.

Now, by the ([13], Theorem 2.3 (1)), we have $N \leq G' \cap D(G) \leq Z_\infty(G')$. It follows from the minimality of N that $N \leq Z(G')$.

$N \leq C_G(O^p(G))$.

Let N be q-group for some prime q and r a prime divisor of $|G|$ different to p and q. If R is a r-group. Then $N \leq N_G(R)$ by $N \leq D_p(G)$ and hence $[N, R] \leq N \cap R = 1$. Thus, $R \leq C_G(N)$ and it follows from the choice of r that $G/C_G(N)$ is a $\{p,q\}$-group. Therefore, without generality, we can assume that G is a $\{p,q\}$-group.

If $q \neq p$, then, by Lemma 2, $N \leq Z_\infty(O^p(G))$. It follows from the minimality of N that $N \leq Z(O^p(G))$.

If N is a p-group, then $[N, Q] = [N, Q'O^p(Q)] = 1$ for any Sylow r-subgroup of G with $r \neq p$. Then $[N, O^p(G)] = 1$, and $N \leq C_G(O^p(G))$.

Hence, one can see that $N \leq C_G(G'(p))$, a contradiction.

(2) If $O_{p'}(D_p(G)) = 1$, the result is clear.

If $O_{p'}(D_p(G)) \neq 1$, then G has a minimal normal subgroup N with $N \leq O_{p'}(D_p(G))$.

For any Sylow r-subgroup R of G, we have $[N, R] = 1$. Then $G/C_G(N)$ is a $\{p, q\}$-group, hence, without loss of generality, we assume that G is a $\{p, q\}$-group.

If N is a q-group, then, by Lemma 2, $N \leq Z_\infty(O^p(G))$. It follows from the minimality of N that $N \leq Z(O^p(G))$.

If N is a p-group, then $[N, Q] = [N, Q'O^p(Q)] = 1$ for any Sylow q-subgroup of G. Then $[N, O^p(G)] = 1$, and $N \leq Z(O^p(G))$.

By induction, $O_{p'}(D_p(G)/N) \leq Z_\infty(O^p(G)/N)$, then $O_{p'}(D_p(G)) \leq Z_\infty(O^p(G))$.

(3) Note that $Z(G'(p)) = 1$ if and only if $D_p(G) \cap G'(p) = 1$ by (1). Then $[D_p(G), G'(p)] \leq D_p(G) \cap G'(p) = 1$, therefore $D_p(G) = C_G(G'(p))$ by Proposition 1 (1). □

4. Minimal Subgroups and $D_p(G)$

The main aim of this section is to to prove the following theorem.

Theorem 3. *Let q be a prime. Assume that every element of order q lies in $D_p(G)$, and in addition, if $q = 2$ and the Sylow q-subgroup of G is nonabelian, then every element of order 4 lies in $D_p(G)$. Then G is q-soluble and $l_q(G) \leq 1$.*

Proof. Let $\Omega = \langle x \in O^p(G) \mid x^q = 1 \rangle$, if $q \neq 2$ or the Sylow q-subgroup of G is abelian; $\Omega = \langle x \in O^p(G) \mid x^4 = 1 \rangle$, if $q = 2$ and the Sylow q-subgroup of G is nonabelian. Then $\Omega \leq O^p(G) \cap D_p(G)$ by hypothesis.

Assume $p \neq q$. By Theorem 1.3, Ω is a p'-group and by Theorem 1.4, $\Omega \leq Z_\infty(O^p(G))$. If $O^p(G)$ is not q-nilpotent, then there exists a minimal non-q-nilpotent subgroup H of $O^p(G)$. By the structure of the minimal non-q-nilpotent groups, we have that $H = [Q]R$, where $Q = O_q(H)$ and $\exp(Q) = q$ or 4 (if q = 2 and Q is non-abelian) and R is a cyclic r-group with $r \neq q$. However, $Q \leq \Omega \leq Z_\infty(O^p(G)$, so $Q \leq H \cap Z_\infty(O^p(G) \leq Z_\infty(H)$. It follows that H is nilpotent, a contradiction. This contradiction shows that $O^p(G)$ is q-nilpotent. Thus, G is q-soluble and $l_q(G) \leq 1$ since $G/O^p(G)$ is a p-group.

Assume $p = q$. If $O^p(G)$ is of order p' then G is p'-closed and so is p-nilpotent. In particular, G is p-soluble with $l_p(G) \leq 1$. If $O^p(G)$ is not a p'-group, then $\Omega \neq \emptyset$ and by Theorem 1.3, $O_{p'}(\Omega)$ is the Hall p'-subgroup of Ω. Let T be any p'-subgroup of G. Then $\Omega \leq N_G(R)$. Since, clearly, Ω is normal in G, we see that $[\Omega, T] \leq \Omega \cap T \leq O_{p'}(\Omega)$. Since $O^p(G) = \langle T \leq G \mid p \nmid |T| \rangle$, $[\Omega, O^p(G)] \leq O_{p'}(\Omega)$. Now, considering on the quotient $O^p(G)/O_{p'}(\Omega)$, we have that $\Omega/O_{p'}(\Omega) \leq Z(O^p(G)/O_{p'}(\Omega))$. By a same argument as above (or by Ito's theorem), it can be obtained that $O^p(G)/O_{p'}(\Omega)$ is p-nilpotent. Therefore, $O^p(G)$ is p-nilpotent and so is G. Thus, is p-soluble with $l_p(G) \leq 1$. The proof is completed. □

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