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Characterization of Dissipative Structures for First-Order Symmetric Hyperbolic System with General Relaxation

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Abstract: In this paper, we study the dissipative structure of first-order linear symmetric hyperbolic system with general relaxation and provide the algebraic characterization for the uniform dissipativity up to order 1. Our result extends the classical Shizuta–Kawashima condition for the case of symmetric relaxation, with a full generality and optimality.

Keywords: symmetric hyperbolic system with relaxation; dissipative structure; time decay with regularity-loss

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1. Introduction

In this paper, we consider the first-order linear symmetric hyperbolic system with relaxation:

$$\begin{aligned} &\partial_t u + \sum_{j=1}^{n} A_j \partial_{x_j} u + Lu = 0\,, && t > 0\,, \quad x \in \mathbb{R}^n\,, \\ &u|_{t=0} = f\,, && x \in \mathbb{R}^n\,. \end{aligned} \tag{1}$$

Here, $n \in \mathbb{N}$ and $u = u(t,x) = (u_1(t,x), \cdots, u_m(t,x))^\top$ is an unknown function with valued in $\mathbb{C}^m$, $m \in \mathbb{N}$, and $f = f(x) = (f_1(x), \cdots, f_m(x))^\top$ is a given function with valued in $\mathbb{C}^m$. We use the standard notations for derivatives; $\partial_t = \partial/\partial t$ and $\partial_{x_j} = \partial/\partial x_j$ for $x = (x_1, \cdots, x_n)$. Each A_j and L is a given $m \times m$ constant matrix with complex coefficients, and, in particular, each A_j is assumed to be an Hermitian matrix, $A_j = A_j^*$, where $M^* = \overline{M}^\top$ denotes the adjoint of a given matrix M. Here, $\overline{M}$ is the complex conjugate of M, and $\overline{M}^\top$ is the transpose of $\overline{M}$. We denote by $M^\sharp$ and $M^\flat$ the Hermitian part and the skew-Hermitian part of M, respectively:

$$M^\sharp = \frac{1}{2}(M + M^*)\,, \qquad M^\flat = \frac{1}{2}(M - M^*)\,.$$

We also use the standard notations for the kernel and the range of M as

$$\mathrm{Ker}\,(M) = \{v \in \mathbb{C}^m \,|\, Mv = 0\}\,, \qquad \mathrm{Ran}\,(M) = \{Mv \in \mathbb{C}^m \,|\, v \in \mathbb{C}^m\}\,.$$

We use similar notations if the matrix M is restricted in the invariant space X other than $\mathbb{C}^m$.

The system of the form (1) arises as the linearization of hyperbolic systems of balance laws, and its study has a long history. One of the most active themes is the dissipation mechanism underlying (1), as a result of the effective interaction between the partial

damping from the Hermitian part $L^\sharp$ and the other hyperbolic terms. To explain this in details, let us consider the system (1) in the Fourier variables with respect to x:

$$\begin{aligned} \partial_t \hat{u} + \mathcal{A}(\xi)\hat{u} = 0\,, \qquad & t > 0\,, \quad \xi \in \mathbb{R}^n\,, \\ \hat{u}|_{t=0} = \hat{f}\,, \qquad & \xi \in \mathbb{R}^n\,, \end{aligned} \tag{2}$$

where $\mathcal{A}(\xi)$ is the $m \times m$ matrix given by $\mathcal{A}(\xi) = L$ for $\xi = 0$, while, for $\xi \neq 0$,

$$\mathcal{A}(\xi) = i|\xi| A(\frac{\xi}{|\xi|}) + L\,, \qquad A(\omega) = \sum_{j=1}^{n} A_j \omega_j\,, \tag{3}$$

and $\omega = (\omega_1, \cdots, \omega_n) \in S^{n-1}$. The function $\hat{u}$ is the Fourier transform of u, i.e.,

$$\hat{u}(t,\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(t,x) e^{-ix\cdot\xi}\, dx\,.$$

Since each A_j is Hermitian, so is $A(\omega)$ for each $\omega \in S^{n-1}$:

$$A(\omega) = A(\omega)^*\,. \tag{4}$$

The one-parameter family $\{e^{-t\mathcal{A}}\}_{t\geq 0}$ given by

$$e^{-t\mathcal{A}} f = \mathcal{F}^{-1}[e^{-t\mathcal{A}(\xi)}\hat{f}(\xi)] = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-t\mathcal{A}(\xi)} \hat{f}(\xi) e^{i\xi\cdot x}\, d\xi$$

defines a C_0-semigroup acting on $L^2(\mathbb{R}^n)^m$ with the generator $-\mathcal{A}$, whose domain contains the Sobolev space $H^1(\mathbb{R}^n)^m$ and $\mathcal{A}f = \sum_{j=1}^n A_j \partial_{x_j} f + Lf$ for $f \in H^1(\mathbb{R}^n)^m$. Thus, by the Plancherel theorem, the estimate of the semigroup $e^{-t\mathcal{A}}$ in L^2 is reduced to the analysis of $e^{-t\mathcal{A}(\xi)}$ for $\xi \in \mathbb{R}^n \setminus \{0\}$.

In order for the first order ODE system (2) to be dissipative (i.e., $\lim_{t\to\infty} e^{-t\mathcal{A}(\xi)} = 0$) for each $\xi \in \mathbb{R}^n \setminus \{0\}$, the necessary and sufficient condition is

$$\{\lambda \in \mathbb{C} \,|\, \Re\lambda \geq 0\} \subset \bigcap_{\xi\in\mathbb{R}^n\setminus\{0\}} \rho(-\mathcal{A}(\xi))\,, \tag{5}$$

where $\Re\lambda$ is the real part of the complex number λ, and $\rho(-\mathcal{A}(\xi))$ is the resolvent set of the matrix $-\mathcal{A}(\xi)$. We note that the condition (5) always implies $L^\sharp \neq O$; otherwise, $-\mathcal{A}(\xi)$ becomes a skew-Hermitian matrix and thus must possess the eigenvalues on the imaginary axis. The key and common relaxation condition assumed in this study is the nonnegativity of $L^\sharp$, i.e.,

$$L^\sharp \geq 0\,, \tag{6}$$

which automatically leads to the inclusion

$$\{\lambda \in \mathbb{C} \,|\, \Re\lambda > 0\} \subset \bigcap_{\xi\in\mathbb{R}^n\setminus\{0\}} \rho(-\mathcal{A}(\xi)).$$

Thus, under the nonnegativity condition (6), the condition to ensure (5) is

$$i\mathbb{R} \subset \bigcap_{\xi\in\mathbb{R}^n\setminus\{0\}} \rho(-\mathcal{A}(\xi))\,, \tag{7}$$

where $i\mathbb{R} = \{\lambda \in \mathbb{C} \,|\, \Re\lambda = 0\}$. We note that, since $A(\omega)$ is Hermitian, the identity $\mathcal{A}(\xi)^\sharp = L^\sharp$ holds. Therefore, under the condition (6) and (7) the matrix $\mathcal{A}(\xi)$ is an m-accretive operator for each $\xi \in \mathbb{R}^n$. We come back to this important fact below. In

general, if the matrix M satisfies $M^\sharp \geq 0$, then it is not difficult to see $\mathrm{Ker}\,(i\lambda I + M) = \mathrm{Ker}\,(\lambda I - iM^\flat) \cap \mathrm{Ker}\,(M^\sharp)$ for any $\lambda \in \mathbb{R}$, where I is the identity matrix. Hence, by recalling $\mathcal{A}(\xi) = irA(\omega) + L$ with $r = |\xi|$ and $\omega = \xi/|\xi| \in S^{n-1}$ for $\xi \neq 0$, we find that the condition (6) and (7) is equivalent with (6) and

$$\bigcup_{\lambda\in\mathbb{R},\, r>0,\, \omega\in S^{n-1}} \mathrm{Ker}\,\left(\lambda I + rA(\omega) - iL^\flat\right) \cap \mathrm{Ker}\,(L^\sharp) = \{0\}\,. \tag{8}$$

The class of symmetric hyperbolic systems satisfying (8) is wide, and we call (8) the *general stability condition*.

For the systems satisfying (6) and (8), the elegant general theory was established under the additional condition

$$\mathrm{Ker}\,(L) = \mathrm{Ker}\,(L^\sharp) \tag{9}$$

by Shizuta and Kawashima [1] and Umeda, Kawashima, and Shizuta [2], which is now classical in this research field, and we call (8) and (9) the classical condition. In this case, we have $\mathrm{Ker}\,(L^\sharp) \subset \mathrm{Ker}\,(L^\flat)$ and thus the condition (8) is reduced to a simpler one:

$$\bigcup_{\lambda\in\mathbb{R},\, r>0,\, \omega\in S^{n-1}} \mathrm{Ker}\,\left(\lambda I + rA(\omega)\right) \cap \mathrm{Ker}\,(L^\sharp) = \{0\}\,. \tag{10}$$

In [1], it is proved that the validity of (6), (9), and (10) implies the existence of a suitable *energy* assumed in the work of Umeda, Kawashima, and Shizuta [2], resulting in the pointwise decay estimate of $e^{-t\mathcal{A}(\xi)}$ such as

$$\|e^{-t\mathcal{A}(\xi)}\|_{\mathbb{C}^m\to\mathbb{C}^m} \leq Ce^{-c\frac{|\xi|^2}{1+|\xi|^2}t}, \qquad t > 0\,, \tag{11}$$

where C and c are positive constants independent of ξ and t. The semigroup estimate (11) implies the condition (5), or, more strongly,

$$\Re\lambda(i\xi) \leq -c\frac{|\xi|^2}{1+|\xi|^2}, \qquad \xi \in \mathbb{R}^n \setminus \{0\} \tag{12}$$

for any eigenvalue $\lambda(i\xi)$ of $-\mathcal{A}(\xi)$. The spectral bound (12) is called the uniformly dissipative of type $(1,1)$ in [3]. As a further study from [1,2], Hanouzet and Natalini [4], Yong [5], Kawashima and Yong [6,7], Ruggeri and Serre [8], and Bianchini, Hanouzet, and Natalini [9] analyzed the nonlinear problems for the hyperbolic system with relaxation. Furthermore, in the case of (9), Beauchard and Zuazua [10] introduced the stability condition called Kalman rank condition which is equivalent to (10). We remark that the entropy condition and (10) guarantee the nonlinear stability of the equilibrium states for the hyperbolic balance laws, while it should be emphasized that the concrete decay rate such as (11) for the linearized system is important to achieve the nonlinear stability.

When (6) holds, in general, we only have

$$\mathrm{Ker}\,(L) \subset \mathrm{Ker}\,(L^\sharp)\,, \tag{13}$$

rather than (9). There are also several important examples for which (9) is not satisfied. For example, the dissipative Timoshenko system and the linearized compressible Euler–Maxwell system do not satisfy (9), and these systems were analyzed for the dissipative structure by the authors of [11–15]. The analysis for these physical models has revealed that, the system with the condition (6) will possess fruitful and more complicated dissipative structures. Several structural conditions have been proposed by to handle the important examples, while most of them are built upon the assumption on the existence of the matrix which provide an explicit source of the energy functional to achieve the dissipative estimate with the desired rate (see [3,16,17]).

Among others, a remarkable point of Shizuta–Kawashima theory [1] is that the condition (9) and (10) is purely algebraic; nevertheless, the quantitative estimate (11) is achieved with a concrete dependence on ξ. This is highly nontrivial. Indeed, since $\mathcal{A}(\xi)$ is not a normal operator, even the spectral bound (12) does not necessarily yield (11) from the abstract semigroup theory, for the constant C in (11) must be uniform in ξ. Notice that the abstract spectral mapping theorem does not give information on the prefactor constant C. Inspired by the philosophy of Shizuta [1], Ueda [18,19] tried to extend Shizuta–Kawashima theory, and partially succeeded for an extension under (13). Precisely, the author obtained the uniform dissipativity for (1) under the general stability condition (8). However, this result does not mention the optimality of the type of the uniform dissipativity, and, thus, the application to the nonlinear problem is still out of reach in this general setting. For the nonlinear problem of general hyperbolic systems, the entropy condition derived in [6,7] is not enough to cover all physical models described by the balance laws, and, thus, the theory still needs to be developed. In this context, Kawashima and Ueda [20] recently refined the entropy condition which can be applied to the compressible Euler–Maxwell system, where the key generalization is to allow the nonsymmetric relaxation. The reader is also referred to the works of Zeng [21] and Lou and Ruggeri [22] for another direction of generalization, where it is discussed even the case when the general stability condition (8) is violated but in a specific way so that the formation of shocks is prevented. However, the general theory to ensure the global existence of small smooth solutions for the nonlinear problem seems to be still open.

In this paper, we study the linear system (1) for the case of general relaxation, and our goal is to provide the algebraic characterization in achieving the uniform dissipative estimate of order 1 (see the definition in front of Theorem 1 below) without assuming (9). It is stressed here that, although the nonlinear problem is not discussed in this paper, achieving the concrete decay rate for the linear problem (1), which is nontrivial if the classical condition (9) does not hold, is a key step also for the global existence of small smooth solutions to the nonlinear problem.

As described in Theorems 1–3 in the next section, our result is optimal in the sense that the algebraic condition given in this paper is necessary for any $n \geq 1$, and is sufficient for $n = 1$, as well as for $n \geq 2$ under additional but rather mild assumptions. In particular, some important examples such as the dissipative Timoshenko system and the compressible Euler–Maxwell system are within the range of our result. We note that the finite dimensional nature of the problem (2) is the key that enables us obtaining the concrete decay rate of the semigroup only from the algebraic condition; in the infinite dimensional problem, one needs to introduce a quantitative condition at some point to achieve a concrete decay rate, as seen in the systematic work by Villani [23] in this direction.

This paper is organized as follows. In Section 2, we collect some notations and state the main results. In Section 3, we briefly refer to the idea of the proof in connection with the key general assumptions (4) and (6). In Sections 4 and 5, the dissipative structure is analyzed in detail for the low frequency part and high frequency part, respectively. The proofs of the main results are stated in Section 6. In Section 7, we show how our result is applied to the well-known examples such as the dissipative Timoshenko system and the compressible Euler–Maxwell system. In Appendix A we recall the Gearhart-Prüss type theorem for the semigroup generated by the m-accretive operator on the Hilbert space, and in Appendix B we state the elementary fact about the nonnegative matrices with the spectral parameters on the imaginary axis. These results are the key in our argument.

2. Nondegenerate Condition and Main Results

Let X_ω be a nontrivial subspace of $\mathbb{C}^m$ and $\mathcal{P}_\omega$ be the orthogonal projection to X_ω, which depend on $\omega \in S^{n-1}$. Then, we also introduce a family of nontrivial subspaces $\{X_{\omega,r}\}_{r>0}$ such that $X_{\omega,r} \subset X_\omega$, and $\mathcal{P}_{\omega,r}$ denotes the orthogonal projection from $\mathbb{C}^m$ to $X_{\omega,r}$. The spaces X_ω and $X_{\omega,r}$, rather than $\mathbb{C}^m$, are introduced in order for the application to the system with the constraint condition such as the compressible Euler–Maxwell system.

To keep the generality, let us also allow the dependence of L on $\omega \in S^{n-1}$ and write $L(\omega)$ instead of L. We assume the following invariance and continuity:

$$\begin{aligned}
&\text{(i) } A(\omega)X_\omega \subset X_\omega\,,\ L(\omega)^\sharp X_\omega \subset X_\omega\,,\ L(\omega)^\flat X_\omega \subset X_\omega \text{ for all } \omega \in S^{n-1}\,;\\
&\text{(ii) } \big(irA(\omega) + L(\omega)\big)X_{\omega,r} \subset X_{\omega,r} \ \text{ for all } \ \omega \in S^{n-1}\,, r > 0\,;\\
&\text{(iii) the map } \ S^{n-1} \times (0,\infty) \ni (\omega, r) \mapsto (A(\omega), L(\omega), \mathcal{P}_\omega, \mathcal{P}_{\omega,r}) \in (\mathbb{C}^{m\times m})^4 \ \text{ is continuous; } \textit{and}\\
&\text{(iv) the limits } \ \mathcal{P}_{\omega,0} = \lim_{r\to 0} \mathcal{P}_{\omega,r} \ \text{ and } \ \mathcal{P}_{\omega,\infty} = \lim_{r\to\infty} \mathcal{P}_{\omega,r} \ \text{ exist in the topology of } \mathbb{C}^{m\times m}\,.
\end{aligned} \tag{Inv}$$

It should be stressed that the invariant property about $X_{\omega,r}$ is not necessarily imposed on each matrix $A(\omega)$, $L(\omega)^\sharp$, or $L(\omega)^\flat$. We recall the *general stability condition* stated as follows:

$$\begin{aligned}
&A(\omega) = A(\omega)^\sharp \ \text{ and } \ L(\omega)^\sharp \geq 0 \ \text{ for any } \ \omega \in S^{n-1}\,, \ \text{ and}\\
&\bigcup_{\lambda\in\mathbb{R},\, r>0,\, \omega\in S^{n-1}} \mathrm{Ker}\left(\lambda I + rA(\omega) - iL(\omega)^\flat\right) \cap \mathrm{Ker}\left(L(\omega)^\sharp\right) \cap X_{\omega,r} = \{0\}\,.
\end{aligned} \tag{SC}$$

In particular, $A(\omega)$ is Hermitian. To state our result, we collect the notations of some orthogonal projections.

Definition 1 (Orthogonal projections)**.** *Below the notation $\mathbb{F} : Y \to Z$ denotes that $\mathbb{F}$ is the orthogonal projection from the subspace Y of $\mathbb{C}^m$ to the subspace Z of Y. We also denote by $\mathbb{F}^\perp$ the orthogonal projection $I|_Y - \mathbb{F}$, where $I|_Y$ is the identity map on Y. Each s_j is a given real number and $\omega \in S^{n-1}$.*

(0) $\mathbb{D}_{\sharp,\omega} : X_\omega \to \mathrm{Ker}\left(L(\omega)^\sharp|_{X_\omega}\right)$

(1) $\mathbb{P}_{s_0,\omega} : X_\omega \to \mathrm{Ker}\left((is_0 I + L(\omega))|_{X_\omega}\right)$

(2) $\mathbb{P}_{s_1,s_0,\omega} : \mathbb{P}_{s_0,\omega}X_\omega \to \mathrm{Ker}\left((s_1 I + \mathbb{P}_{s_0,\omega}A(\omega))|_{\mathbb{P}_{s_0,\omega}X_\omega}\right)$

(3) $\mathbb{P}_{s_2,s_1,s_0,\omega} : \mathbb{P}_{s_1,s_0,\omega}X_\omega \to \mathrm{Ker}\left(\left(is_2 I + \mathbb{P}_{s_1,s_0,\omega}L^{(1)}_{low}(s_0,\omega)\right)|_{\mathbb{P}_{s_1,s_0,\omega}X_\omega}\right)$,

where

$$L^{(1)}_{low}(s_0,\omega) = \mathbb{P}_{s_0,\omega}A(\omega)\big(is_0 I + L(\omega)\big)|^{-1}_{\mathbb{P}^\perp_{s_0,\omega}X_\omega}\mathbb{P}^\perp_{s_0,\omega}A(\omega)|_{\mathbb{P}_{s_0,\omega}X_\omega}$$

(4) $\mathbb{Q}_{s_0,\omega} : X_\omega \to \mathrm{Ker}\left((s_0 I + A(\omega))|_{X_\omega}\right)$

(5) $\mathbb{Q}_{s_1,s_0,\omega} : \mathbb{Q}_{s_0,\omega}X_\omega \to \mathrm{Ker}\left((is_1 I + \mathbb{Q}_{s_0,\omega}L(\omega))|_{\mathbb{Q}_{s_0,\omega}X_\omega}\right)$

(6) $\mathbb{Q}_{s_2,s_1,s_0,\omega} : \mathbb{Q}_{s_1,s_0,\omega}X_\omega \to \mathrm{Ker}\left(\left(s_2 I + \mathbb{Q}_{s_1,s_0,\omega}A^{(1)}(s_0,\omega)\right)|_{\mathbb{Q}_{s_1,s_0,\omega}X_\omega}\right)$,

where

$$A^{(1)}(s_0,\omega) := \mathbb{Q}_{s_0,\omega}L(\omega)K(s_0,\omega)L(\omega)^*|_{\mathbb{Q}_{s_0,\omega}X_\omega}\,, \quad K(s_0,\omega) := -\big(s_0 I + A(\omega)\big)|^{-1}_{\mathbb{Q}^\perp_{s_0,\omega}X_\omega}\mathbb{Q}^\perp_{s_0,\omega}$$

(7) $\mathbb{Q}_{s_3,s_2,s_1,s_0,\omega} : \mathbb{Q}_{s_2,s_1,s_0,\omega}X_\omega \to \mathrm{Ker}\left(\left(is_3 I + \mathbb{Q}_{s_2,s_1,s_0,\omega}L^{(1)}_{high}(s_1,s_0,\omega)\right)|_{\mathbb{Q}_{s_2,s_1,s_0,\omega}X_\omega}\right)$,

where

$$L^{(1)}_{high}(s_1,s_0,\omega) = is_1\mathbb{Q}_{s_1,s_0,\omega}F(s_0,\omega)|_{\mathbb{Q}_{s_1,s_0,\omega}X_\omega} + \mathbb{Q}_{s_1,s_0,\omega}G(s_1,s_0,\omega)|_{\mathbb{Q}_{s_1,s_0,\omega}X_\omega}$$

with

$$F(s_0,\omega) := L(\omega_*)K(s_0,\omega)^2L(\omega_*)^*\,,$$
$$G(s_1,s_0,\omega) := A^{(1)}(s_0,\omega)\big(is_1I+\mathbb{Q}_{s_0,\omega}L(\omega)\big)|^{-1}_{\mathbb{Q}^\perp_{s_1,s_0,\omega}X_\omega}\mathbb{Q}^\perp_{s_1,s_0,\omega}A^{(1)}(s_0,\omega) - L(\omega)K(s_0,\omega)L(\omega)^*K(s_0,\omega)L(\omega)^*\,.$$

Remark 1. *(1) The projections $\mathbb{P}_{s_j,\cdots,s_0,\omega}$ are used in the analysis for the low frequency part, while $\mathbb{Q}_{s_j,\cdots,s_0,\omega}$ are used for the high frequency part. From the definitions, we have*

$$\mathbb{P}_{s_{j+1},s_j,\cdots,s_0,\omega}\mathbb{P}_{s_j,\cdots,s_0,\omega} = \mathbb{P}_{s_{j+1},s_j,\cdots,s_0,\omega}\,,\qquad \mathbb{Q}_{s_{j+1},s_j,\cdots,s_0,\omega}\mathbb{Q}_{s_j,\cdots,s_0,\omega} = \mathbb{Q}_{s_{j+1},s_j,\cdots,s_0,\omega}\,.$$

(2) When $L(\omega)^\sharp \geq 0$ holds, the space $\mathrm{Ker}\,((is_0I+L(\omega))|_{X_\omega})$ *is equal to* $\mathrm{Ker}\,((s_0I-iL(\omega)^\flat)|_{X_\omega})\cap \mathrm{Ker}\,(L(\omega)^\sharp)$*; see Lemma A1.*

(3) The symmetric part of $-\mathbb{Q}_{s_2,s_1,s_0,\omega}L^{(1)}_{high}(s_1,s_0,\omega)|_{\mathbb{Q}_{s_2,s_1,s_0,\omega}X_\omega}$ is, in fact, nonnegative (see Remark 7).

Next, we define the singular sets, which consist of the parameters such that the resolvent can be singular at the limit of low/high frequencies.

Definition 2 (Singular sets in the limit)**.**
(1) *Let $s_0,s_1,s_2\in\mathbb{R}$ and $\omega\in S^{n-1}$. The spaces $V^{low,0}(s_0,\omega)$ and $V^{low,1}(s_2,s_1,s_0,\omega)$ are defined by*

$$V^{low,0}(s_0,\omega) = \mathrm{Ran}\,(\mathbb{P}_{s_0,\omega})\cap\mathrm{Ran}\,(\mathcal{P}_{\omega,0})\,,$$
$$V^{low,1}(s_2,s_1,s_0,\omega) = \mathrm{Ran}\,(\mathbb{P}_{s_2,s_1,s_0,\omega})\cap\mathrm{Ran}\,(\mathcal{P}_{\omega,0})\,,$$

and the singular sets $\mathcal{S}_{low,0}$ and $\mathcal{S}_{low,1}$ are defined as

$$\mathcal{S}_{low,0} = \big\{(s_0,\omega)\in\mathbb{R}\times S^{n-1}\,|\,V^{low,0}(s_0,\omega)\neq\{0\}\big\}\,,$$
$$\mathcal{S}_{low,1} = \big\{(s_2,s_1,s_0,\omega)\in\mathbb{R}^3\times S^{n-1}\,|\,V^{low,1}(s_2,s_1,s_0,\omega)\neq\{0\}\big\}\,.$$

We also set

$$\tilde{V}^{low,1}(s_1,s_0,\omega) = \mathrm{Ran}\,(\mathbb{P}_{s_1,s_0,\omega})\cap\mathrm{Ker}\left(L(\omega)^\sharp\big(is_0I+L(\omega)\big)|^{-1}_{\mathbb{P}^\perp_{s_0,\omega}X_\omega}\mathbb{P}^\perp_{s_0,\omega}A(\omega)|_{X_\omega}\right)\cap\mathrm{Ran}\,(\mathcal{P}_{\omega,0})\,,$$
$$\tilde{\mathcal{S}}_{low,1} = \big\{(s_1,s_0,\omega)\in\mathbb{R}^2\times S^{n-1}\,|\,\tilde{V}^{low,1}(s_1,s_0,\omega)\neq\{0\}\big\}\,,$$

(2) *Let $s_0,s_1\in\mathbb{R}$ and $\omega\in S^{n-1}$. The spaces $V^{high,0}(s_1,s_0,\omega)$ and $V^{high,1}(s_1,s_0,\omega)$ are defined by*

$$V^{high,0}(s_1,s_0,\omega) = \mathrm{Ran}\,(\mathbb{Q}_{s_1,s_0,\omega})\cap\mathrm{Ran}\,(\mathcal{P}_{\omega,\infty})\,,$$
$$V^{high,1}(s_3,s_2,s_1,s_0,\omega) = \mathrm{Ran}\,(\mathbb{Q}_{s_3,s_2,s_1,s_0,\omega})\cap\mathrm{Ran}\,(\mathcal{P}_{\omega,\infty})\,,$$

and the singular sets $\mathcal{S}_{high,0}$ and $\mathcal{S}_{high,1}$ are defined as

$$\mathcal{S}_{high,0} = \big\{(s_1,s_0,\omega)\in\mathbb{R}^2\times S^{n-1}\,|\,V^{high,0}(s_1,s_0,\omega)\neq\{0\}\big\}\,,$$
$$\mathcal{S}_{high,1} = \big\{(s_3,s_2,s_1,s_0,\omega)\in\mathbb{R}^4\times S^{n-1}\,|\,V^{high,1}(s_3,s_2,s_1,s_0,\omega)\neq\{0\}\big\}\,.$$

We also set

$$\tilde{V}^{high,0}(s_0,\omega) = \mathrm{Ran}\,(\mathbb{Q}_{s_0,\omega}) \cap \mathrm{Ker}\,(L(\omega)^\sharp|_{X_\omega}) \cap \mathrm{Ran}\,(\mathcal{P}_{\omega,\infty})\,,$$

$$V^{high,1,(1)}(s_1,s_0,\omega) = \mathrm{Ker}\left(L(\omega)^\sharp K(s_0,\omega)L(\omega)^*|_{X_\omega}\right) \cap V^{high,0}(s_1,s_0,\omega)\,,$$

$$\begin{aligned} &V^{high,1,(2)}(s_2,s_1,s_0,\omega)\\ &= \mathrm{Ran}\,(\mathbb{Q}_{s_2,s_1,s_0,\omega}) \cap \mathrm{Ran}\,(\mathcal{P}_{\omega,\infty})\\ &\quad \cap \mathrm{Ker}\left(L(\omega)^\sharp\{(is_1 I + \mathbb{Q}_{s_0,\omega}L(\omega))|^{-1}_{\mathbb{Q}^\perp_{s_1,s_0,\omega}X_\omega}\mathbb{Q}^\perp_{s_1,s_0,\omega}A^{(1)}(s_0,\omega) - K(s_0,\omega)L(\omega)^*\}\right), \end{aligned}$$

and

$$\tilde{\mathcal{S}}_{high,0} = \left\{(s_0,\omega) \in \mathbb{R}\times S^{n-1} \mid \tilde{V}^{high,0}(s_0,\omega) \neq \{0\}\right\},$$

$$\mathcal{S}^{(1)}_{high,1} = \left\{(s_1,s_0,\omega) \in \mathbb{R}^2\times S^{n-1} \mid V^{high,1,(1)}(s_1,s_0,\omega) \neq \{0\}\right\},$$

$$\mathcal{S}^{(2)}_{high,1} = \left\{(s_2,s_1,s_0,\omega) \in \mathbb{R}^3\times S^{n-1} \mid V^{high,1,(2)}(s_2,s_1,s_0,\omega) \neq \{0\}\right\}.$$

Remark 2. *The singular sets can be empty. The following statements are verified in virtue of* $L(\omega)^\sharp \geq 0$ *and Lemma A1.*

(1) $\tilde{V}^{low,1}(s_1,s_0,\omega) \subset V^{low,0}(s_0,\omega)$.
(2) $V^{low,1}(s_2,s_1,s_0,\omega) \subset \tilde{V}^{low,1}(s_1,s_0,\omega)$.
(3) $V^{high,1,(1)}(s_1,s_0,\omega) \subset V^{high,0}(s_1,s_0,\omega) \subset \tilde{V}^{high,0}(s_0,\omega)$.
(4) $V^{high,1}(s_3,s_2,s_1,s_0,\omega) \subset V^{high,1,(2)}(s_1,s_0,\omega) \subset V^{high,0}(s_1,s_0,\omega)$.
(5) If $L(\omega)^\sharp K(s_0,\omega)L(\omega)^\flat\mathbb{Q}_{s_0,\omega}X_\omega \subset \mathbb{Q}^\perp_{s_0,\omega}X_\omega$ *holds for any* $s_0 \in \mathbb{R}$ *and* $\omega \in S^{n-1}$*, then* $V^{high,1,(2)}(s_2,s_1,s_0,\omega) \subset V^{high,1,(1)}(s_1,s_0,\omega)$.

The first inclusion in Assertion (4) above is apparently nontrivial, but it follows from the formula given in Remark 7. The above inclusions imply that

(I) $\pi_0\tilde{\mathcal{S}}_{low,1} \subset \mathcal{S}_{low,0}$.
(II) $\pi_1\mathcal{S}_{low,1} \subset \tilde{\mathcal{S}}_{low,1}$.
(III) $\mathcal{S}^{(1)}_{high,1} \subset \mathcal{S}_{high,0}$ *and* $\pi_0\mathcal{S}_{high,0} \subset \tilde{\mathcal{S}}_{high,0}$.
(IV) $\pi_2\mathcal{S}_{high,1} \subset \mathcal{S}^{(2)}_{high,1} \subset \mathcal{S}_{high,0}$.
(V) If $L(\omega)^\sharp K(s_0,\omega)L(\omega)^\flat\mathbb{Q}_{s_0,\omega}X_\omega \subset \mathbb{Q}^\perp_{s_0,\omega}X_\omega$ *holds for any* $s_0 \in \mathbb{R}$ *and* $\omega \in S^{n-1}$ *then* $\pi_1\mathcal{S}^{(2)}_{high,1} \subset \mathcal{S}^{(1)}_{high,1}$.

Here, $\pi_j(s_k,\cdots,s_0,\omega) = (s_j,\cdots,s_0,\omega)$ *for* $0 \leq j \leq k$*. The introduction of the spaces* $\tilde{V}^{low,1}$*,* $\tilde{V}^{high,0}$*,* $V^{high,1,(1)}$*, and* $V^{high,1,(2)}$*, which appear in connection with* $L(\omega)^\sharp \geq 0$*, are important for actual applications, as these spaces enable us to reduce the computation of the singular sets.*

The singular sets defined above characterize the dissipation rate for the semigroup. To give a precise statement, let us introduce some terminology about the semigroup bound. Set

$$\mathcal{A}(\xi) = i|\xi|A(\frac{\xi}{|\xi|}) + L(\frac{\xi}{|\xi|}). \tag{14}$$

Let $\alpha,\beta \geq 0$. We say that $\{e^{-t\mathcal{A}(\xi)}\}_{t\geq 0}$ has the uniform dissipative bound of order α at low frequency if there exist $C,c>0$ such that $\|e^{-t\mathcal{A}(r\omega)}\|_{X_{\omega,r}\to X_{\omega,r}} \leq Ce^{-cr^{2\alpha}t}$ holds for any $t>0$, $\omega \in S^{n-1}$, and $0<r\leq 1$. Similarly, we say that $\{e^{-t\mathcal{A}(r\omega)}\}_{t\geq 0}$ has the

uniform dissipative bound of order β at high frequency if there exist $C, c > 0$ such that $\|e^{-t\mathcal{A}(r\omega)}\|_{X_{\omega,r}\to X_{\omega,r}} \le Ce^{-cr^{-2\beta}t}$ holds for any $t > 0$, $\omega \in S^{n-1}$, and $r \ge 1$.

When $n = 1$, we have the complete characterization of these dissipative structures in terms of the singular sets, as follows.

Theorem 1. *Let $\alpha, \beta \in \{0, 1\}$. Assume that (SC) and (Inv) hold. Let $n = 1$. Assume in addition that $\mathcal{A}(r\omega)\mathcal{P}^{\perp}_{\omega,r}X_\omega \subset \mathcal{P}^{\perp}_{\omega,r}X_\omega$ holds for any $r > 0$ and $\omega \in S^{n-1}$. Then, the following statements hold.*
(1) $\{e^{-t\mathcal{A}(r\omega)}\}_{t\ge0}$ has the uniform dissipative bound of order α at low frequency if and only if $\mathcal{S}_{low,\alpha} = \emptyset$.
(2) $\{e^{-t\mathcal{A}(r\omega)}\}_{t\ge0}$ has the uniform dissipative bound of order β at high frequency if and only if $\mathcal{S}_{high,\beta} = \emptyset$.

For the higher dimensional case $n \ge 2$, the following necessary condition holds.

Theorem 2. *Let $\alpha, \beta \in \{0, 1\}$. Assume that (SC) and (Inv) hold. Assume in addition that $\mathcal{A}(r\omega)\mathcal{P}^{\perp}_{\omega,r}X_\omega \subset \mathcal{P}^{\perp}_{\omega,r}X_\omega$ holds for any $r > 0$ and $\omega \in S^{n-1}$. Then, the following statements hold.*
(1) If $\mathcal{S}_{low,\alpha} \neq \emptyset$, then $\{e^{-t\mathcal{A}(r\omega)}\}_{t\ge0}$ does not have the uniform dissipative bound of order α at low frequency.
(2) If $\mathcal{S}_{high,\beta} \neq \emptyset$, then $\{e^{-t\mathcal{A}(r\omega)}\}_{t\ge0}$ does not have the uniform dissipative bound of order β at high frequency.

Even when $n \ge 2$, the absence of the singular sets is *almost* sufficient to achieve the uniform dissipative estimate, but we need a technical assumption to get rid of the difficulty related to a spectral bifurcation coming from the dependence on $\omega \in S^{n-1}$. This situation is very similar to the result in [19]. To this end, we introduce the following condition.

Definition 3 (no-splitting condition on real eigenvalues)**.** *Let $\{Z_\omega\}_{\omega\in S^{n-1}}$ be a family of the subspaces of $\mathbb{C}^m$, and let $\{M_\omega\}_{\omega\in S^{n-1}}$, $M_\omega : Z_\omega \to Z_\omega$, be a family of linear operators. We say that $\{(M_\omega, Z_\omega)\}_{\omega\in S^{n-1}}$ has no-splitting real eigenvalues if the following two conditions are satisfied.* (i) *The map $\omega \mapsto (M_\omega\mathcal{P}_{Z_\omega}, \mathcal{P}_{Z_\omega}) \in (\mathbb{C}^{m\times m})^2$ is continuous, where $\mathcal{P}_{Z_\omega}$ is the orthogonal projection from $\mathbb{C}^m$ to Z_ω.* (ii) *The numbers $\#\sigma(M_\omega)$ and $\#\big(\sigma(M_\omega)\cap\mathbb{R}\big)$, where $\sigma(M_\omega)$ is the set of the eigenvalues of M_ω, are independent of $\omega \in S^{n-1}$.*

Remark 3. *(1) Assume that $\{(M_\omega, Z_\omega)\}_{\omega\in S^{n-1}}$ has no-splitting real eigenvalues. Then, we have from the continuity of the eigenvalues about ω and, from Condition (ii),*

$$\inf_{\omega\in S^{n-1}} \operatorname{dist}\Big(\sigma(M_\omega)\cap\mathbb{R},\ \sigma(M_\omega)\setminus\mathbb{R}\Big) > 0\,.$$

Moreover, when $k := \#(\sigma(M_\omega)\cap\mathbb{R}) \ge 1$, there exist k continuous maps $\mu_j : S^{n-1} \to \mathbb{R}$ such that $\{\mu_j(\omega)\}_{j=1}^k = \sigma(M_\omega)\cap\mathbb{R}$ and also

$$\min_{\omega\in S^{n-1}, 1\le j\le k} \operatorname{dist}\big(\mu_j(\omega), \sigma(M_\omega)\setminus\{\mu_j(\omega)\}\big) > 0\,.$$

We also have the continuity of the spectral projection

$$T_{\mu_j(\omega)} = \frac{1}{2\pi i}\int_\gamma (\lambda I|_{Z_\omega} - M_\omega)^{-1}\,\mathrm{d}\lambda\,,$$

where γ is a small circle centered at $\mu_j(\omega)$ oriented counter clockwise, in the sense that $T_{\mu_j(\omega)}\mathcal{P}_{Z_\omega} : \mathbb{C}^m \to \mathbb{C}^m$ is continuous about ω. These facts, which are valid from (i) and (ii) in the definitions, are frequently used in this paper.

(2) If $n = 1$, then $S^{n-1} = \{\pm 1\}$, which is a finite set. Thus, the concept of the no-splitting condition on real eigenvalues is needed only when $n \geq 2$.

With these notations, let us give the *nondegenerate conditions*

$$(\mathrm{NDC})_{low,0}\,, \quad (\mathrm{NDC})_{low,1}\,, \quad (\mathrm{NDC})_{high,0}\,, \quad (\mathrm{NDC})_{high,1}$$

as follows.

Definition 4. *The condition* $(\mathrm{NDC})_{low,0}$ *consists of* (0) *and* (i) *stated below.*
(0) (SC) *and* (Inv) *hold.*
(i) $\mathcal{S}_{low,0} = \emptyset$.

Definition 5. *The condition* $(\mathrm{NDC})_{low,1}$ *consists of* (0) *and* (i) *stated below.*
(0) (SC) *and* (Inv) *hold and* $\{(iL(\omega)|_{X_\omega}, X_\omega)\}_{\omega \in S^{n-1}}$ *has no-splitting real eigenvalues.*
(i) *If* $\tilde{\mathcal{S}}_{low,1} \neq \emptyset$, *then both* (i-a) *and* (i-b) *hold for any* $(s_1, s_0, \omega) \in \tilde{\mathcal{S}}_{low,1}$:
(i-a) $\{(\mathbb{P}_{s_0(\omega'),\omega'} A(\omega')|_{\mathbb{P}_{s_0(\omega'),\omega'} X_{\omega'}}, \mathbb{P}_{s_0(\omega'),\omega'} X_{\omega'})\}_{\omega' \in S^{n-1}}$ *has no-splitting real eigenvalues, where* $s_0(\cdot) : S^{n-1} \to \mathbb{R}$ *is the continuous map such that* $s_0(\omega) = s_0$ *and each* $s_0(\omega)$ *is an eigenvalue of* $iL(\omega)|_{X_\omega}$.
(i-b) $V^{low,1}(s_2, s_1, s_0, \omega) = \{0\}$ *for any* $s_2 \in \mathbb{R}$, *that is,* $\mathcal{S}_{low,1} = \emptyset$.

Definition 6. *The condition* $(\mathrm{NDC})_{high,0}$ *consists of* (0) *and* (i) *stated below.*
(0) (SC) *and* (Inv) *hold.*
(i) *If* $\tilde{\mathcal{S}}_{high,0} \neq \emptyset$, *then both* (i-a) *and* (i-b) *hold:*
(i-a) $\{(A(\omega')|_{X_{\omega'}}, X_{\omega'})\}_{\omega' \in S^{n-1}}$ *has no-splitting real eigenvalues.*
(i-b) $\mathcal{S}_{high,0} = \emptyset$.

Definition 7. *The condition* $(\mathrm{NDC})_{high,1}$ *consists of* (0), (i), *and* (ii) *stated below.*
(0) (SC) *and* (Inv) *hold.*
(i) $\{(A(\omega')|_{X_{\omega'}}, X_{\omega'})\}_{\omega' \in S^{n-1}}$ *has no-splitting real eigenvalues.*
(ii) *If* $\mathcal{S}_{high,0} \neq \emptyset$, *then either* (iii′) *or* (iii) *holds.*
(iii′) *Both* (iii′-a) *and* (iii′-b) *hold for any* $(s_1, s_0, \omega) \in \mathcal{S}_{high,0}$:
(iii′-a) *For any* $\omega' \in S^{n-1}$,

$$L(\omega')^\sharp K(s_0(\omega'), \omega') L(\omega')^\flat \mathbb{Q}_{s_0(\omega'),\omega'} X_{\omega'} \subset \mathbb{Q}^\perp_{s_0(\omega'),\omega'} X_{\omega'}\,,$$

where $s_0(\cdot) : S^{n-1} \to \mathbb{R}$ *is the continuous map such that* $s_0(\omega) = s_0$ *and that each* $s_0(\omega')$ *is an eigenvalue of* $-A(\omega')|_{X_{\omega'}}$.
(iii′-b) $V^{high,1,(1)}(s_1, s_0, \omega) = \emptyset$ *for any* $s_1 \in \mathbb{C}$, *that is,* $\mathcal{S}^{(1)}_{high,1} = \emptyset$.
(iii) *Both* (iii-1) *and* (iii-2) *hold for any* $(s_1, s_0, \omega) \in \mathcal{S}_{high,0}$:
(iii-1) $\{(\mathbb{Q}_{s_0(\omega'),\omega'} iL(\omega')|_{\mathbb{Q}_{s_0(\omega'),\omega'} X_{\omega'}}, \mathbb{Q}_{s_0(\omega'),\omega'} X_{\omega'})\}_{\omega' \in S^{n-1}}$ *has no-splitting real eigenvalues, where* $s_0(\cdot) : S^{n-1} \to \mathbb{R}$ *is the continuous map such that* $s_0(\omega) = s_0$ *and each* $s_0(\omega)$ *is an eigenvalue of* $-A(\omega)|_{X_\omega}$.
(iii-2) *If* $\mathcal{S}^{(2)}_{high,1} \neq \emptyset$, *then both* (iii-2-a) *and* (iii-2-b) *hold for any* $(s_2, s_1, s_0, \omega) \in \mathcal{S}^{(2)}_{high,1}$:
(iii-2-a)

$$\{(\mathbb{Q}_{s_1(\omega'),s_0(\omega'),\omega'} A^{(1)}(s_0(\omega'), \omega')|_{\mathbb{Q}_{s_1(\omega'),s_0(\omega'),\omega'} X_{\omega'}}, \mathbb{Q}_{s_1(\omega'),s_0(\omega'),\omega'} X_{\omega'})\}_{\omega' \in S^{n-1}}$$

has no-splitting real eigenvalues, where $s_1(\cdot) : S^{n-1} \to \mathbb{R}$ *is the continuous map such that* $s_1(\omega) = s_1$ *and that each* $s_1(\omega)$ *is an eigenvalue of* $\mathbb{Q}_{s_0(\omega'),\omega'} iL(\omega')|_{\mathbb{Q}_{s_0(\omega'),\omega'} X_{\omega'}}$.

(iii-2-b) $V^{high,1}(s_3, s_2, s_1, s_0, \omega) = \{0\}$ *for any* $s_3 \in \mathbb{R}$, *that is,* $\mathcal{S}_{high,1} = \emptyset$.

Remark 4. *(1) As we see in the main theorem below, the nondegenerate condition* $(NDC)_{low,\alpha}$ *provides the decay* $e^{-c|\xi|^{2\alpha}t}$ *of the semigroup* $e^{-t\mathcal{A}(\xi)}$ *for the low frequency part* $|\xi| \le 1$, *while the nondegenerate condition* $(NDC)_{high,\beta}$ *provides the decay* $e^{-c|\xi|^{-2\beta}t}$ *of* $e^{-t\mathcal{A}(\xi)}$ *for the high frequency part* $|\xi| \ge 1$.

(2) If $\tilde{\mathcal{S}}_{low,1} = \emptyset$, *then Condition (i) in Definition 5 holds. Similarly, if* $\tilde{\mathcal{S}}_{high,0} = \emptyset$, *then Condition (i) in Definition 6 holds, while if* $\mathcal{S}^{(2)}_{high,1} = \emptyset$, *then Condition (iii-2) in Definition 7 holds. Condition* (iii') *with* $\mathcal{S}^{(1)}_{high,1}$ *is important in actual applications, as, if Condition* (iii') *holds, then one can skip checking Condition (iii), that would need lengthy computation. For the (linearized) dissipative Timoshenko system and the compressible Euler–Maxwell equations, which are well known examples for the nonclassical case, we can indeed show that* (iii') *holds. We note that the classical stability condition, in which* $\mathrm{Ker}\,(L) = \mathrm{Ker}\,(L^\sharp)$ *holds, implies* $\tilde{\mathcal{S}}_{low,1} = \tilde{\mathcal{S}}_{high,0} = \emptyset$ *(see, e.g., Remark 6 for* $\tilde{\mathcal{S}}_{low,1} = \emptyset$; *the condition* $\tilde{\mathcal{S}}_{high,0} = \emptyset$ *is trivial in the classical case), resulting in* $(\mathrm{NDC})_{low,1}$ *and* $(\mathrm{NDC})_{high,0}$. *Thus, our result covers the classical theory by Shizuta and Kawashima [1] and Umeda, Kawashima, and Shizuta [2].*

(3) *Note that* $S^{n-1} = \{\pm 1\}$ *when* $n = 1$. *Thus, if* $n = 1$, *then the condition of the no-splitting real eigenvalues is automatically satisfied. Even when* $n \ge 2$, *for actual applications, the condition of the no-splitting real eigenvalues is widely satisfied and is not a 'real' obstacle for the range of applications.*

The nondegenerate conditions stated above are sufficient to obtain the uniform dissipative estimate, as follows.

Theorem 3. *Let* $\alpha, \beta \in \{0, 1\}$. *Assume that* $(\mathrm{NDC})_{low,\alpha}$ *and* $(\mathrm{NDC})_{high,\beta}$ *hold. Then, it follows that, for any* $\xi \ne 0$, $t > 0$, *and* $\hat{f}(\xi) \in X_{\xi/|\xi|,|\xi|}$,

$$|e^{-t\mathcal{A}(\xi)}\hat{f}(\xi)| \le Ce^{-c\eta_{\alpha,\beta}(\xi)t}|\hat{f}(\xi)|, \qquad \eta_{\alpha,\beta}(\xi) = \frac{|\xi|^{2\alpha}}{1+|\xi|^{2(\alpha+\beta)}}.$$

Here, C and c are independent of ξ, t, *and* $\hat{f}(\xi)$.

Note that, if $(\mathrm{NDC})_{high,\beta}$ holds with $\beta = 0$, then the solution decays exponentially in the high frequency region. From Theorem 3 combined with the Plancherel theorem and the Hausdorff–Young inequality for the Fourier transform, we have the following corollary. For $s \ge 0$, we denote by $\mathcal{H}^s$ the closed subspace of the usual Sobolev space $H^s(\mathbb{R}^n)^m$ defined as

$$\mathcal{H}^s = \{f \in H^s(\mathbb{R}^n)^m \,|\, \hat{f}(\xi) \in X_{\xi/|\xi|,|\xi|} \text{ a.e. } \xi \in \mathbb{R}^n\}.$$

To simplify the notation, we also write L^q instead of the Lebesgue space $L^q(\mathbb{R}^n)^m$, $1 \le q \le \infty$.

Corollary 1. *Assume that* $(\mathrm{NDC})_{low,\alpha}$ *and* $(\mathrm{NDC})_{high,\beta}$ *hold. Then, it follows that, for any* $t > 0$ *and* $f \in \mathcal{H}^s \cap L^1$ *with* $k, l \ge 0$ *and* $0 \le k + l \le s$,

$$\|\partial_x^k e^{-t\mathcal{A}} f\|_{L^2} \le C(1+t)^{-n/4-k/2}e^{-c(1-\alpha)t}\|f\|_{L^1} + C(1+t)^{-l/2}e^{-c(1-\beta)t}\|\partial_x^{k+l} f\|_{L^2}.$$

Here, C and c are independent of t *and* f.

Remark 5. *In the classical case, i.e., L is independent of $\xi/|\xi|$ and* $\mathrm{Ker}\,(L) = \mathrm{Ker}\,(L^\sharp)$, *the nondegenerate conditions* $(\mathrm{NDC})_{low,1}$ *and* $(\mathrm{NDC})_{high,0}$ *hold, as commented in Remark 4 (2).*

The proof of Corollary 1 is omitted in this paper, as the derivation from the pointwise estimate in Theorem 3 is rather standard. We refer readers to [3,19] for the details.

3. Remark on the General Strategy and the Role of $L(\omega)^\sharp \geq 0$

Before going into the details of the proof for our main result, let us give some comments on the general strategy. Our proof for the semigroup estimate relies on the resolvent analysis studying the quantity called the pseudospectral bound. In the technical level, the argument is closely related to the reduction argument, systematically described by Kato [24] in the case of perturbations with one parameter. In essence, it is applied to investigate the asymptotic expansion of the eigenvalues or the resolvent for the operator $\mathcal{A}(r\omega) = irA(\omega) + L(\omega)$, in both the low frequency limit $r = |\xi| \to 0$ and the high frequency limit $r = |\xi| \to \infty$. One difficulty here is the additional parameter ω in the higher dimensional case $n \geq 2$, as the general theory is not available for perturbations with multi-parameters. This is the reason we have to introduce the condition of the no-splitting real eigenvalues, which enables us to avoid the unpleasant complexity coming from the possible bifurcation due to the continuous dependence on ω for $n \geq 2$.

In principle, under the suitable assumption on the no-splitting of the eigenvalues about ω, the reduction argument works for general couple $(A(\omega), L(\omega))$ without even symmetry of $A(\omega)$ or the nonnegativity $L(\omega)^\sharp \geq 0$. The problem here is that, however, if such structures of $A(\omega)$ and $L(\omega)$ are absent, the reduction argument becomes rather complicated in general, even under the assumption of the no-splitting eigenvalues about ω. This is indeed a serious problem for actual applications with concrete operators.

One important observation of this paper is that the symmetry of $A(\omega)$ and the nonnegativity of $L(\omega)^\sharp$ drastically simplify the reduction process, which would not be possible without these structures. To clarify this point, let us give a list of benefits brought by the conditions $A(\omega) = A(\omega)^\sharp$ and $L(\omega)^\sharp \geq 0$.

1. The operator $\mathcal{A}(r\omega) = irA(\omega) + L(\omega)$ becomes *m-accretive*, which enables us to obtain the semigroup bound directly from the pseudospectral bound, the resolvent estimate with resolvent parameters *only along the imaginary axis*, in virtue of the Gearhart–Püss type theorem by Wei [25] (see also [26]) (see Theorem A1). The pseudospectral bound was discussed by Gallagher, Gallay, and Nier [27], who studied the harmonic oscillator with some class of large skew-symmetric perturbations, which was also discussed, for example, by Li, Wei, and Zhang [28] and and Ibrahim, Maekawa, and Masmoudi [29] in order to study the semigroup estimate for the linearization around the stationary flows for the Navier–Stokes equations such as the Burgers vortex and the Kolmogorov flow.
2. In each reduction process for the uniform dissipativity, the leading operator M is either the skew-Hermitian (i.e., iM is Hermitian) or $M^\sharp \geq 0$. As a result, when $is \in i\mathbb{R}$, which is compatible with Statement 1 above, the orthogonal projection to $\mathrm{Ker}\,(isI + M)$ coincides with the spectral projection to $-M$. That is, any eigenvalue of $-M$ located on the imaginary axis must be semisimple, and furthermore, the associated eigenprojection is the orthogonal projection; see Lemma A1. This makes the reduction process much simpler, as the eigennilpotent (which would yield a serious complexity of the reduction formula) does not appear in the reduction process, and the orthogonal projection is easier to compute. This is also the reason we can describe the nondegenerate condition *only through the orthogonal projections* listed in Definition 1. It should be stressed that this remarkable feature is available only from the conditions $A(\omega) = A(\omega)^\sharp$ and $L(\omega)^\sharp \geq 0$.
3. We can derive the sufficient condition by making use of $\mathrm{Ker}\,(L(\omega)^\sharp)$, which gives a chance to stop the reduction process before making the full reduction. To be precise, as in the classical case of Sizuta and Kawashima [1], there are fruitful examples such

that the investigation of $\tilde{\mathcal{S}}_{low,1}$ and $\tilde{\mathcal{S}}_{high,0}$ is enough to achieve a uniform dissipation estimate, and the study of the full singular sets such as $\mathcal{S}_{loiw,1}$ and $\mathcal{S}_{high,0}$ is not always required. This is a great advantage in actual applications, which is why we introduce $\tilde{\mathcal{S}}_{low,1}$, $\tilde{\mathcal{S}}_{high,0}$, $\mathcal{S}^{(1)}_{high,1}$, and $\mathcal{S}^{(2)}_{high,1}$.

4. Analysis of Low Frequency

In this section, we study the case for $0 < |\xi| \leq 1$.

4.1. Resolvent Analysis

In this subsection, our interest is the quantitative estimate of the resolvent $(i\lambda I + irA(\omega) + L(\omega))|^{-1}_{X_{\omega,r}}$ with $\lambda \in \mathbb{R}$ and $0 < r \leq 1$; in particular, we aim to obtain the estimate with the concrete dependence on the parameter r uniformly in λ and ω. Let $\Psi(r,\omega)$ be the pseudospectral bound of the matrix $irA(\omega) + L(\omega)$ in $X_{\omega,r}$ defined by $\Psi(r,\omega) = 1/\Phi(r,\omega)$, where

$$\Phi(r,\omega) = \sup_{\lambda\in\mathbb{R}} \| \left(i\lambda I + irA(\omega) + L(\omega)\right)|^{-1}_{X_{\omega,r}} \|_{X_{\omega,r}\to X_{\omega,r}} . \tag{15}$$

The main result of this section is as follows.

Theorem 4. *Let $\alpha \in \{0,1\}$. Assume that* $(\mathrm{NDC})_{low,\alpha}$ *holds. Then, there exists $C > 0$ such that*

$$\sup_{\omega\in S^{n-1}} \Phi(r,\omega) \leq \frac{C}{r^{2\alpha}} , \qquad 0 < r \leq 1 . \tag{16}$$

Before going into the details of the proof, let us state a useful consequence of Theorem 4.

Corollary 2. *Assume that $X_{\omega,r} = X_\omega$ for all $r > 0$ and that* (SC) *and* (Inv) *hold, as well as that $\{(iL(\omega)|_{X_\omega}, X_\omega)\}_{\omega\in S^{n-1}}$ satisfies the no-splitting real eigenvalues. If there exists $s_0 \in \mathbb{R}$ such that* $\mathrm{Ker}\,(L(\omega)^\sharp|_{X_\omega}) \subset \mathrm{Ran}\,(\mathbb{P}_{s_0,\omega})$ *for all $\omega \in S^{n-1}$, then $\tilde{\mathcal{S}}_{low,1} = \emptyset$. As a consequence, we have* (16) *with $\alpha = 1$.*

In particular, if $\mathrm{rank}\,(\mathbb{D}_{\sharp,\omega}) \leq 1$ *for any $\omega \in S^{n-1}$, which is always valid when $m = 2$, then* (16) *with $\alpha = 1$ holds true.*

Remark 6. *Let us consider the case of the classical stability condition, where $X_\omega = X_{\omega,r} = \mathbb{C}^m$, L is independent of ω, and* $\mathrm{Ker}\,(L) = \mathrm{Ker}\,(L^\sharp)$. *In this case, we have* $\mathrm{Ker}\,(L^\sharp) = \mathrm{Ran}\,(\mathbb{P}_{s_0,\omega})$ *with $s_0 = 0$, and, hence, Corollary 2 is applied.*

Proof of Corollary 2. Suppose that $\mathrm{Ker}\,(L(\omega)^\sharp|_{X_\omega}) \subset \mathrm{Ran}\,(\mathbb{P}_{s_0,\omega})$. Let $u \in X_\omega$ be any vector in

$$\begin{aligned}
&\mathrm{Ker}\left(L(\omega)^\sharp\left(is_0 I + L(\omega)\right)|^{-1}_{\mathbb{P}^\perp_{s_0,\omega}X_\omega}\mathbb{P}^\perp_{s_0,\omega}A(\omega)|_{X_\omega}\right) \cap \mathrm{Ran}\,(\mathbb{P}_{s_1,s_0,\omega}) \\
&= \mathrm{Ker}\left(L(\omega)^\sharp\left(is_0 I + L(\omega)\right)|^{-1}_{\mathbb{P}^\perp_{s_0,\omega}X_\omega}\mathbb{P}^\perp_{s_0,\omega}A(\omega)|_{X_\omega}\right) \cap \mathrm{Ker}\left((s_1 I + \mathbb{P}_{s_0,\omega}A(\omega))|_{\mathbb{P}_{s_0,\omega}X_\omega}\right).
\end{aligned}$$

Then, since $\mathrm{Ran}\left((is_0 I + L(\omega))|^{-1}_{\mathbb{P}^\perp_{s_0,\omega}X_\omega}\mathbb{P}^\perp_{s_0,\omega}\right) \subset \mathrm{Ran}\,(\mathbb{P}^\perp_{s_0,\omega})$, we have from the assumption $\mathrm{Ker}\left(L(\omega)|^\sharp_{X_\omega}\right) \subset \mathrm{Ran}\,(\mathbb{P}_{s_0,\omega})$ and from $u \in \mathrm{Ker}\left(L(\omega)^\sharp(is_0 + L(\omega))|^{-1}_{\mathbb{P}^\perp_{s_0}X_\omega}\mathbb{P}^\perp_{s_0}A(\omega)|_{X_\omega}\right)$ that $(is_0 + L(\omega))|^{-1}_{\mathbb{P}^\perp_{s_0}X_\omega}\mathbb{P}^\perp_{s_0}A(\omega)u = 0$, i.e., $\mathbb{P}^\perp_{s_0,\omega}A(\omega)u = 0$. Hence, it follows from

$u \in \mathrm{Ker}\left((s_1 I + \mathbb{P}_{s_0,\omega} A(\omega))|_{\mathbb{P}_{s_0,\omega} X_\omega}\right)$ that $(s_1 I + A(\omega))u = 0$. On the other hand, since $u \in \mathrm{Ran}\,(\mathbb{P}_{s_0,\omega})$, we have $(is_0 I + L(\omega))u = 0$. Hence, we have

$$\begin{aligned} u &\in \mathrm{Ker}\left(i(s_1+s_0)I + iA(\omega) + L(\omega)\right) \cap X_\omega \\ &= \mathrm{Ker}\left((s_1+s_0)I + A(\omega) - iL(\omega)^\flat\right) \cap \mathrm{Ker}\,(L(\omega)^\sharp) \cap X_\omega\,. \end{aligned}$$

Here, we use Lemma A1 in the last line. Then, (SC) together with the assumption $X_{\omega,r} = X_\omega$ implies $u = 0$. Thus, we have $\tilde{\mathcal{S}}_{low,1} = \emptyset$, which gives (16) with $\alpha = 1$ by Theorem 4. We note that, if $\mathrm{rank}\,(\mathbb{D}_{\sharp,\omega}) = 1$ and $\mathrm{Ran}\,(\mathbb{P}_{s_0,\omega}) \neq \{0\}$, then $\mathrm{Ran}\,(\mathbb{P}_{s_0,\omega}) = \mathrm{Ker}\,(L(\omega)^\sharp|_{X_\omega})$ by Lemma A1, while, if either $\mathrm{rank}\,(\mathbb{D}_{\sharp,\omega}) = 0$ or $\mathrm{Ran}\,(\mathbb{P}_{s_0,\omega}) = \{0\}$, then $\mathrm{Ran}\,(\mathbb{P}_{s_0,\omega}) = \mathrm{Ker}\,(L(\omega)^\sharp|_{X_\omega}) = \{0\}$. Hence, the last statement of Corollary 2 holds. The proof is complete. □

Proof of Theorem 4. Set $M = 2\sup_{\omega\in S^{n-1}}(1 + \|A(\omega)\| + \|L(\omega)\|)$. It is easy to see from the Neumann series argument that

$$\sup_{0<r\le 1,\omega\in S^{n-1},|\lambda|\ge M} \|(i\lambda I + irA(\omega) + L(\omega))|^{-1}_{X_{\omega,r}}\|_{X_{\omega,r}\to X_{\omega,r}} \le C\,.$$

Thus, it suffices to consider the case $|\lambda| \le M$. The proof is based on the contradiction argument. Suppose that (16) does not hold. Then, there exist a sequence $\{r_N, \lambda_N, \omega_N, u_N\}$ with $r_N \in (0,1]$, $\omega_N \in S^{n-1}$, $\lambda_N \in \mathbb{R}$ with $|\lambda_N| \le M$, and $u_N \in X_{\omega_N,r_N}$ such that $|u_N| = 1$ and

$$\lim_{N\to\infty} r_N^{-2\alpha}(i\lambda_N I + ir_N A(\omega_N) + L(\omega_N))u_N = 0\,. \tag{17}$$

By taking a suitable subsequence if necessary, we may also assume

$$\lim_{N\to\infty}(r_N, \lambda_N, \omega_N, u_N) = (r_*, \lambda_*, \omega_*, u_*)$$

for some $r_* \in [0,1]$, $\lambda_* \in [-M, M]$, $\omega_* \in S^{n-1}$, $u_* \in X_{\omega_*,r_*}$ with $|u_*| = 1$. Then, the limit $u_* \in X_{\omega_*} \setminus \{0\}$ satisfies $(i\lambda_* I + ir_* A(\omega_*) + L(\omega_*))u_* = 0$. If $r_* > 0$, then u_* belongs to X_{ω_*,r_*} and this gives $u_* = 0$ due to (SC), which is a contradiction. Thus, it is enough to treat the case for $r_* = 0$. If $r_* = 0$, then $u_* \in \mathrm{Ker}\,((i\lambda_* I + L(\omega_*))|_{X_{\omega_*}}) = \mathrm{Ran}\,(\mathbb{P}_{\lambda_*,\omega_*})$, that is, λ_* must be an eigenvalue of $iL(\omega_*)|_{X_{\omega_*}}$ and $u_* \in X_{\omega_*,0} := \mathcal{P}_{\omega_*,0} X_{\omega_*}$. In the rest of this proof, we consider the two cases as follows.

Case $\alpha = 0$: Assume that $(\mathrm{NDC})_{low,0}$ holds. Then, u_* must be zero, which is a contradiction. The proof is complete in this case.

Case $\alpha = 1$: Assume that $(\mathrm{NDC})_{low,1}$ holds. Set $f_N = (i\lambda_N I + ir_N A(\omega_N) + L(\omega_N))u_N$. Then, we have $\langle L(\omega_N)^\sharp u_N, u_N\rangle = \Re\langle f_N, u_N\rangle$. Furthermore, since $L(\omega_N)^\sharp \ge 0$ and $L(\omega_N)^\sharp u_N = L(\omega_N)^\sharp \mathbb{D}^\perp_{\sharp,\omega_N} u_N$, we also have

$$\langle L(\omega_N)^\sharp u_N, u_N\rangle = |(L(\omega_N)^\sharp|_{\mathbb{D}^\perp_{\sharp,\omega_N} X_{\omega_N}})^{1/2}\mathbb{D}^\perp_{\sharp,\omega_N} u_N|^2. \tag{18}$$

Then, these lead to

$$\begin{aligned} |L(\omega_N)^\sharp u_N|^2 &= |L(\omega_N)^\sharp|_{\mathbb{D}^\perp_{\sharp,\omega_N} X_{\omega_N}} \mathbb{D}^\perp_{\sharp,\omega_N} u_N|^2 \\ &\le \|L(\omega_N)^\sharp|_{\mathbb{D}^\perp_{\sharp,\omega_N} X_{\omega_N}}\|\,|(L(\omega_N)^\sharp|_{\mathbb{D}^\perp_{\sharp,\omega_N} X_{\omega_N}})^{1/2}\mathbb{D}^\perp_{\sharp,\omega_N} u_N|^2 \\ &\le C|f_N|\,|u_N|\,. \end{aligned}$$

Therefore, employing $|u_N| = 1$ and $|f_N| = o(r_N^2)$, which comes from (17), we obtain

$$\lim_{N\to\infty} r_N^{-1}|L(\omega_N)^\sharp u_N| = 0\,. \tag{19}$$

Since $\{(iL(\omega)|_{X_\omega}, X_\omega)\}_{\omega\in S^{n-1}}$ has the no-splitting real eigenvalues and since $\lambda_* \in \mathbb{R}$ is the eigenvalue of $iL(\omega_*)$, there is a continuous curve $\{s_0(\omega)\}_{\omega\in S^{n-1}} \subset \mathbb{R}$ such that each $s_0(\omega)$ is an eigenvalue of $iL(\omega)|_{X_\omega}$ and $s_0(\omega_*) = \lambda_*$. Moreover, by the no-splitting property, we have

$$\inf_{\omega\in S^{n-1}} \inf_{\mu\in\sigma(iL(\omega)|_{X_\omega})\setminus\{s_0(\omega)\}} |\mu - s_0(\omega)| > 0$$

(see also Remark 3). Let us decompose u_N as $u_N = w_N + w_N^\perp$ with

$$w_N := \mathbb{P}_{0,N}u_N\,, \qquad w_N^\perp := \mathbb{P}_{0,N}^\perp u_N\,, \qquad \mathbb{P}_{0,N} := \mathbb{P}_{s_0(\omega_N),\omega_N}\,.$$

Then, it is easy to check that $w_N, w_N^\perp \in X_{\omega_N}$ by the invariance (Inv). Moreover, (ii) in Lemma A1 implies

$$\begin{aligned}\mathbb{P}_{0,N}(i\lambda_N I + L(\omega_N))u_N &= i(\lambda_N - s_0(\omega_N))w_N\,,\\ \mathbb{P}_{0,N}^\perp(i\lambda_N I + L(\omega_N))u_N &= (i\lambda_N I + L(\omega_N))|_{Y_N} w_N^\perp\,,\end{aligned}$$

where $Y_N := \mathbb{P}_{0,N}^\perp X_{\omega_N}$. Thus, f_N is also decomposed by

$$\begin{aligned}f_N &= \mathbb{P}_{0,N}f_N + \mathbb{P}_{0,N}^\perp f_N\\ &= (\lambda_N - s_0(\omega_N))w_N + r_N\mathbb{P}_{0,N}A(\omega_N)u_N + (i\lambda_N I + L(\omega_N))|_{Y_N}w_N^\perp + ir_N\mathbb{P}_{0,N}^\perp A(\omega_N)u_N.\end{aligned}$$

Then, since $f_N = o(r_N^2)$, this decomposition leads to

$$(\lambda_N - s_0(\omega_N))w_N + r_N\mathbb{P}_{0,N}A(\omega_N)u_N = o(r_N^2)\,, \tag{20}$$

$$(i\lambda_N I + L(\omega_N))|_{Y_N}w_N^\perp + ir_N\mathbb{P}_{0,N}^\perp A(\omega_N)u_N = o(r_N^2)\,. \tag{21}$$

Here, we note that

$$\big(i\lambda_N I + L(\omega_N) + ir_N\mathbb{P}^\perp_{s_0(\omega_N),\omega_N}A(\omega_N)\big)|_{Y_N} : Y_N \to Y_N$$

is invertible for large N by the Neumann series and the condition of the no-splitting real eigenvalues, and

$$\sup_N \|\big(i\lambda_N I + L(\omega_N) + ir_N\mathbb{P}_{0,N}^\perp A(\omega_N)\big)|_{Y_N}^{-1}\|_{Y_N\to Y_N} \le C\,.$$

Therefore, employing (21), we have

$$w_N^\perp = -ir_N\big(i\lambda_N I + L(\omega_N) + ir_N\mathbb{P}_{0,N}^\perp A(\omega_N)\big)|_{Y_N}^{-1}\mathbb{P}_{0,N}^\perp A(\omega_N)w_N + o(r_N^2)\,, \tag{22}$$

Hence, since (20) and (22) with $\lim_{N\to\infty} w_N = u_* \neq 0$ and $\lim_{N\to\infty}\lambda_N = \lambda_*$, we find that $|\lambda_N - s_0(\omega_N)| \le Cr_N$ and $|w_N^\perp| \le Cr_N$ are satisfied for large N, where C is independent of N. Then, we set

$$\tilde\lambda_N = \frac{\lambda_N - s_0(\omega_N)}{r_N}\,, \qquad \tilde w_N^\perp = \frac{w_N^\perp}{r_N} \in X_{\omega_N}\,,$$

which are bounded uniformly in N. Thus, by taking a subsequence if necessary, we may assume that $\lim_{N\to\infty}\tilde\lambda_N = \tilde\lambda_*$ and $\lim_{N\to\infty}\tilde w_N^\perp = \tilde w_*^\perp$. Since $\tilde w_N^\perp \in \mathrm{Ran}\,(\mathbb{P}_{0,N}^\perp)$ we have $\tilde w_*^\perp \in \mathrm{Ran}\,(\mathbb{P}^\perp_{\lambda_*,\omega_*})$, and we obtain

$$\tilde\lambda_* u_* + \mathbb{P}_{\lambda_*,\omega_*}A(\omega_*)u_* = 0\,, \qquad (i\lambda_* + L(\omega_*))|_{\mathbb{P}^\perp_{\lambda_*,\omega_*}X_{\omega_*}}\tilde w_*^\perp + i\mathbb{P}^\perp_{\lambda_*,\omega_*}A(\omega_*)u_* = 0\,,$$

that is,

$$u_* \in \mathrm{Ker}\left((\tilde{\lambda}_* I + \mathbb{P}_{\lambda_*,\omega_*} A(\omega_*))|_{\mathbb{P}_{\lambda_*,\omega_*} X_{\omega_*}}\right),$$

$$\tilde{w}_*^{\perp} = -i(i\lambda_* I + L(\omega_*))|^{-1}_{\mathbb{P}^{\perp}_{\lambda_*,\omega_*} X_{\omega_*}} \mathbb{P}^{\perp}_{\lambda_*,\omega_*} A(\omega_*) u_* .$$

Moreover, by using $L(\omega_*)|^{\sharp}_{X_{\omega_*}} \mathbb{P}_{\lambda_*,\omega_*} = O$ which follows from Lemma A1, we see

$$L(\omega_*)^{\sharp} \tilde{w}_*^{\perp} = \lim_{N\to\infty} L(\omega_N)^{\sharp} \frac{w_N^{\perp}}{r_N} = \lim_{N\to\infty} L(\omega_N)^{\sharp} \frac{u_N}{r_N} = 0 .$$

Here, we use (19) in the last equality. Thus, we get $u_* \in \mathrm{Ker}\,(L(\omega_*)^{\sharp}(i\lambda_* I + L(\omega_*))|^{-1}_{\mathbb{P}^{\perp}_{\lambda_*,\omega_*}} \mathbb{P}^{\perp}_{\lambda_*,\omega_*} A(\omega_*)|_{X_{\omega_*}})$. As a summary, we arrive at

$$u_* \neq 0 , \quad u_* \in \tilde{V}^{low,1}(\tilde{\lambda}_*, \lambda_*, \omega_*) .$$

In particular, $\tilde{V}^{low,1}(\tilde{\lambda}_*, \lambda_*, \omega_*)$ must be nontrivial, and thus we can use Conditions (i-a) and (i-b) in $(\mathrm{NDC})_{low,1}$. From (20) and (22) with $r_N \tilde{\lambda}_N = \lambda_N - s_0(\omega_N)$, we have

$$\begin{aligned} &\tilde{\lambda}_N w_N + \mathbb{P}_{0,N} A(\omega_N) w_N \\ &- i r_N \mathbb{P}_{0,N} A(\omega_N)\big(i\lambda_N I + L(\omega_N) + i r_N \mathbb{P}^{\perp}_{0,N} A(\omega_N)\big)|^{-1}_{Y_N} \mathbb{P}^{\perp}_{s_0(\omega_N),\omega_N} A(\omega_N) w_N = o(r_N) . \end{aligned}$$

Thus, from the Neumann series with $\lambda_N = s_0(\omega_N) + r_N \tilde{\lambda}_N$,

$$\tilde{\lambda}_N w_N + \mathbb{P}_{0,N} A(\omega_N) w_N - i r_N \mathbb{P}_{0,N} A(\omega_N)\big(i s_0(\omega_N) I + L(\omega_N)\big)|^{-1}_{Y_N} \mathbb{P}^{\perp}_{0,N} A(\omega_N) w_N = o(r_N) . \tag{23}$$

In virtue of Condition (i-a), there exists a continuous curve $s_1(\cdot) : S^{n-1} \to \mathbb{R}$ such that $s_1(\omega_*) = \tilde{\lambda}_*$ and each $s_1(\omega)$ is the eigenvalue of the Hermitian $\mathbb{P}_{s_0(\omega),\omega} A(\omega)|_{\mathbb{P}_{s_0(\omega),\omega} X_\omega}$. We recall that $\mathbb{P}_{\tau_1,\tau_0,\omega}$ is the orthogonal projection from $\mathbb{P}_{\tau_0,\omega} X_\omega$ to $\mathrm{Ker}\,((\tau_1 I + \mathbb{P}_{\tau_0,\omega} A(\omega))|_{\mathbb{P}_{\tau_0,\omega} X_\omega})$. Thus, each $\mathbb{P}_{s_1(\omega),s_0(\omega),\omega}$ is the eigenprojection to the eigenvalue $s_1(\omega)$ of $-\mathbb{P}_{s_0(\omega),\omega} A(\omega)|_{\mathbb{P}_{s_0(\omega),\omega} X_\omega}$. Then, by setting $\mathbb{P}_{1,0,N} = \mathbb{P}_{s_1(\omega_N),s_0(\omega_N),\omega_N}$ we have $\mathbb{P}_{1,0,N}\mathbb{P}_{0,N} = \mathbb{P}_{1,0,N}$, and we have from (23),

$$(\tilde{\lambda}_N - s_1(\omega_N))\mathbb{P}_{1,0,N} w_N - i r_N \mathbb{P}_{1,0,N} A(\omega_N)\big(i s_0(\omega_N) I + L(\omega_N)\big)|^{-1}_{Y_N} \mathbb{P}^{\perp}_{0,N} A(\omega_N) w_N = o(r_N) . \tag{24}$$

Note that both $s_1(\omega)$ and $\mathbb{P}_{s_1(\omega),s_0(\omega),\omega}$ are continuous in ω, in virtue of the no-splitting property. Since $\mathbb{P}_{s_1(\omega_*),s_0(\omega_*),\omega_*} u_* = u_* \neq 0$ and $\lim_{N\to\infty} w_N = u_*$ we have

$$\lim_{N\to\infty} \mathbb{P}_{1,0,N} w_N = u_* .$$

Thus, $|\mathbb{P}_{1,0,N} w_N|$ must be positive uniformly in $N \gg 1$, which implies from (24) that $\tilde{\lambda}'_N = (\tilde{\lambda}_N - s_1(\omega_N))/r_N$ is uniformly bounded in N, and, then, we may assume that $\tilde{\lambda}'_N$ converges to $\tilde{\lambda}'_*$ by taking a subsequence if necessary. Then, (24) with $s_1(\omega_*) = \tilde{\lambda}_*$ and $s_0(\omega_*) = \lambda_*$ imply

$$i\tilde{\lambda}'_* u_* + \mathbb{P}_{\tilde{\lambda}_*,\lambda_*,\omega_*} A(\omega_*)\big(i\lambda_* I + L(\omega_*)\big)|^{-1}_{\mathbb{P}^{\perp}_{\lambda_*,\omega_*} X_{\omega_*}} \mathbb{P}^{\perp}_{\lambda_*,\omega_*} A(\omega_*) u_* = 0 .$$

Thus, we have $u_* \in \mathrm{Ran}\,(\mathbb{P}_{\tilde{\lambda}'_*,\tilde{\lambda}_*,\lambda_*,\omega_*})$, and, then, $u_* = 0$ by Condition (i-b) of $(\mathrm{NDC})_{low,1}$, which is a contradiction. The proof is complete. □

4.2. Semigroup Estimate

The estimate of the semigroup is a consequence of the pseudospectral bound obtained in the previous subsection and the Gearhard-Prüss type theorem by Wei [25], stated in

Theorem A1. Let us recall that the semigroup considered here is $\{e^{-t\mathcal{A}(\xi)}\}_{t\geq 0}$ in $X_{\xi/|\xi|,|\xi|}$, where $\mathcal{A}(\xi)$ is defined by (14). The main result of this subsection is stated as follows.

Theorem 5. *There exist positive constants C and c such that the following statements hold. Let $\alpha \in \{0,1\}$ and assume that* $(\mathrm{NDC})_{low,\alpha}$ *holds. Then, for any $\xi \in \mathbb{R}^n$ with $0 < |\xi| \leq 1$,*

$$|e^{-t\mathcal{A}(\xi)}\boldsymbol{y}| \leq Ce^{-c|\xi|^{2\alpha}t}|\boldsymbol{y}|\,, \qquad \boldsymbol{y} \in X_{\xi/|\xi|,|\xi|}\,. \tag{25}$$

Proof. Theorem 4 implies the pseudospectral bound $\Psi(r,\omega) \geq r^{2\alpha}/C$, where the positive constant C is independent of $r \in (0,1]$ and $\omega \in S^{n-1}$. Then, the estimate (25) follows from Theorem A1. The proof is complete. □

4.3. Optimality

In this subsection, we show the optimality stated in Theorem 2 for the low frequency.

Theorem 6. *Assume that* (SC) *and* (Inv) *hold. Assume in addition that $\mathcal{A}(r\omega)\mathcal{P}^{\perp}_{\omega,r}X_\omega \subset \mathcal{P}^{\perp}_{\omega,r}X_\omega$ holds for any $r > 0$ and $\omega \in S^{n-1}$. Then, the following statement holds. Let $\alpha \in \{0,1\}$. If $\mathcal{S}_{low,\alpha} \neq \emptyset$, then $\{e^{-t\mathcal{A}(r\omega)}\}_{t\geq 0}$ does not have the uniform dissipative bound of order α at low frequency.*

Proof. We give the proof only for the case $\alpha = 1$; the case $\alpha = 0$ is proved in a similar manner and is much simpler. Assume that $\mathcal{S}_{low,1} \neq \emptyset$. Suppose that $\{e^{-t\mathcal{A}(\xi)}\}_{t\geq 0}$ has the uniform dissipative bound of order 1 at low frequency. Then, there exist positive constants C and c such that $\|e^{-t\mathcal{A}(r\omega)}\|_{X_{\omega,r}\to X_{\omega,r}} \leq Ce^{-cr^2t}$ for $t > 0$, $\omega \in S^{n-1}$, and $0 < r \leq 1$. Then, the Laplace transform for the resolvent,

$$(i\lambda I + \mathcal{A}(r\omega))|^{-1}_{X_{\omega,r}} = \int_0^\infty e^{-i\lambda t - t\mathcal{A}(r\omega)|_{X_{\omega,r}}}\,\mathrm{d}t\,,$$

implies

$$\begin{aligned}\sup_{\omega\in S^{n-1}} \Phi(r,\omega) &= \sup_{\lambda\in\mathbb{R},\omega\in S^{n-1}} \|(i\lambda I + \mathcal{A}(r\omega))|^{-1}_{X_{\omega,r}}\|_{X_{\omega,r}\to X_{\omega,r}} \\ &\leq \int_0^\infty Ce^{-cr^2t}\,\mathrm{d}t = \frac{C}{cr^2}\,, \quad 0 < r \leq 1\,.\end{aligned} \tag{26}$$

Hence, it suffices to reach the contradiction to the uniform resolvent estimate (26). Let $(s_2,s_1,s_0,\omega) \in \mathcal{S}_{low,1}$. Then, there exists $\boldsymbol{y}_0 \in V^{low,1}(s_2,s_1,s_0,\omega)$ with $|\boldsymbol{y}_0| = 1$. We set

$$\begin{aligned}\boldsymbol{y}_1 &= -i\big(i(s_0 + rs_1 + r^2s_2)I + L(\omega) + ir\mathbb{P}^{\perp}_{s_0,\omega}A(\omega)\big)|^{-1}_{\mathbb{P}^{\perp}_{s_0,\omega}X_\omega}\mathbb{P}^{\perp}_{s_0,\omega}A(\omega)\boldsymbol{y}_0\,,\\ \tilde{\boldsymbol{y}}_1 &= -\big((s_1 + rs_2)I + \mathbb{P}_{s_0,\omega}A(\omega)\big)|^{-1}_{\mathbb{P}^{\perp}_{s_1,s_0,\omega}X_\omega}\mathbb{P}^{\perp}_{s_1,s_0,\omega}\mathbb{P}_{s_0,\omega}A(\omega)\boldsymbol{y}_1\,,\\ \boldsymbol{y}_2 &= -i\big(i(s_0 + rs_1 + r^2s_2)I + L(\omega) + ir\mathbb{P}^{\perp}_{s_0,\omega}A(\omega)\big)|^{-1}_{\mathbb{P}^{\perp}_{s_0,\omega}X_\omega}\mathbb{P}^{\perp}_{s_0,\omega}A(\omega)\tilde{\boldsymbol{y}}_1\,.\end{aligned}$$

We note that the operator $\big(i(s_0 + rs_1 + r^2s_2)I + L(\omega) + ir\mathbb{P}^{\perp}_{s_0,\omega}A(\omega)\big)|^{-1}_{\mathbb{P}^{\perp}_{s_0,\omega}}$ and $\big((s_1 + rs_2)I + \mathbb{P}_{s_0,\omega}A(\omega)\big)|^{-1}_{\mathbb{P}^{\perp}_{s_1,s_0,\omega}}$ are well defined for small enough $r > 0$, and we have the uniform estimate for small enough $r > 0$ such that

$$|\boldsymbol{y}_1| + |\tilde{\boldsymbol{y}}_1| + |\boldsymbol{y}_2| \leq C_{s_1,s_0,\omega}\,, \tag{27}$$

where $C_{s_1,s_0,\omega}$ is a certain positive constant depends on s_1, s_0, and ω. Now, let us set $\boldsymbol{x}(r) = \boldsymbol{y}_0 + r(\boldsymbol{y}_1 + \tilde{\boldsymbol{y}}_1) + r^2\boldsymbol{y}_2$, which satisfies $1/2 \le |\boldsymbol{x}(r)| \le 2$ for any sufficiently small $r > 0$ in virtue of (27) and $|\boldsymbol{y}_0| = 1$. However, we have from $\tilde{\boldsymbol{y}}_1 \in \mathrm{Ran}\,(\mathbb{P}_{s_0,\omega})$,

$$\begin{aligned}
&\big(i(s_0 + rs_1 + r^2 s_2)I + irA(\omega) + L(\omega)\big)\boldsymbol{x}(r)\\
&= ir\big((s_1 + rs_2)I + A(\omega)\big)\boldsymbol{y}_0 - ir\mathbb{P}^{\perp}_{s_0,\omega}A(\omega)\boldsymbol{y}_0 + ir^2A(\omega)\boldsymbol{y}_1\\
&\quad - ir^2\mathbb{P}^{\perp}_{s_1,s_0,\omega}\mathbb{P}_{s_0,\omega}A(\omega)\boldsymbol{y}_1 + ir^2\mathbb{P}^{\perp}_{s_0,\omega}A(\omega)\tilde{\boldsymbol{y}}_1 - ir^2\mathbb{P}^{\perp}_{s_0,\omega}A(\omega)\boldsymbol{y}_1 - ir^2\mathbb{P}^{\perp}_{s_0,\omega}A(\omega)\tilde{\boldsymbol{y}}_1 + O(r^3)\\
&= ir\big(s_1 I + \mathbb{P}_{s_0,\omega}A(\omega)\big)\boldsymbol{y}_0 + ir^2\big(s_2\boldsymbol{y}_0 + \mathbb{P}_{s_1,s_0,\omega}A(\omega)\boldsymbol{y}_1\big) + O(r^3)\\
&= ir^2\big(s_2\boldsymbol{y}_0 + \mathbb{P}_{s_1,s_0,\omega}A(\omega)\boldsymbol{y}_1\big) + O(r^3)\,.
\end{aligned}$$

Since $\boldsymbol{y}_1 = -i\big(is_0 I + L(\omega)\big)|^{-1}_{\mathbb{P}^{\perp}_{s_0,\omega}X_\omega}\mathbb{P}^{\perp}_{s_0,\omega}A(\omega)\boldsymbol{y}_0 + O(r)$, we conclude from $\boldsymbol{y}_0 \in \mathrm{Ran}\,(\mathbb{P}_{s_2,s_1,s_0,\omega})$ that

$$\big(i(s_0 + rs_1 + r^2 s_2)I + irA(\omega) + L(\omega)\big)\boldsymbol{x}(r) = O(r^3)\,.$$

By acting the projection $\mathcal{P}_{\omega,r}$ on both sides above and by using the invariant condition $\mathcal{A}(r\omega)\mathcal{P}_{\omega,r}X_\omega \subset \mathcal{P}_{\omega,r}X_\omega$ and $\mathcal{A}(r\omega)\mathcal{P}^{\perp}_{\omega,r}X_\omega \subset \mathcal{P}^{\perp}_{\omega,r}X_\omega$, if the resolvent estimate (26) holds, then we must have $|\mathcal{P}_{\omega,r}\boldsymbol{x}(r)| \le O(r)$ for $0 < r \ll 1$, which contradicts with $\lim_{r\to 0}\mathcal{P}_{\omega,r}\boldsymbol{x}(r) = \boldsymbol{y}_0$ and $|\boldsymbol{y}_0| = 1$. The proof is complete. □

5. Analysis of High Frequency

In this section, we study the case for $|\xi| \ge 1$.

5.1. Resolvent Analysis

The key result of this section is as follows, which is the resolvent bound for the high frequency. Let us recall $\Phi(r,\omega)$ defined by (15). Then, the following theorem is obtained.

Theorem 7. *Let $\beta \in \{0,1\}$. Assume that* $(\mathrm{NDC})_{high,\beta}$ *holds. Then, there exists $C > 0$ such that*

$$\sup_{\lambda\in\mathbb{R},\omega\in S^{n-1}} \Phi(r,\omega) \le Cr^{2\beta}\,, \qquad r \ge 1\,.$$

Proof. The assertion is equivalent to

$$\sup_{\mu\in\mathbb{R},\omega\in S^{n-1}} \|(i\mu I + iA(\omega) + \tau L(\omega))|^{-1}_{X_{\omega,1/\tau}}\|_{X_{\omega,1/\tau}\to X_{\omega,1/\tau}} \le \frac{C}{\tau^{1+2\beta}}, \qquad \tau \in (0,1]. \tag{28}$$

We prove (28) by contradiction argument. It suffices to consider the case that $|\mu| \le M$, where $M := 2\sup_{\omega\in S^{n-1}}(1 + \|A(\omega)\| + \|L(\omega)\|)$, otherwise the uniform resolvent estimate is obtained by the Neumann series argument. Namely, we can apply the same argument proposed in Section 4.1. If the assertion (28) (but with $|\mu| \le M$) does not hold, then there exist a sequence $\{\tau_N, \mu_N, \omega_N, u_N\}$ with $\mu_N \in \mathbb{R}$, $|\mu_N| \le M$, $\tau_N \in (0,1]$, $\omega_N \in S^{n-1}$, and $u_N \in X_{\omega_N,1/\tau_N}$ such that $|u_N| = 1$ and

$$\lim_{N\to\infty} \tau_N^{-1-2\beta}|(i\mu_N + iA(\omega_N) + \tau_N L(\omega_N))u_N| = 0\,.$$

By taking a suitable subsequence if necessary, we may assume that there exist $\mu_* \in \mathbb{R}$ with $|\mu_*| \le 2M$, $\tau_* \in [0,1]$, $\omega_* \in S^{n-1}$, and $u_* \in X_{\omega_*,1/\tau_*}$ such that

$$\lim_{N\to\infty}(\mu_N, \tau_N, \omega_N, u_N) = (\mu_*, \tau_*, \omega_*, u_*)\,.$$

We have $|u_*| = 1$ and $(i\mu_* + iA(\omega_*) + \tau_* L(\omega_*))u_* = 0$, and the problem is reduced to the case $\tau_* = 0$; if $\tau_* > 0$, then it is easy to reach the contradiction to (SC). Hence, we

obtain $u_* \in \mathrm{Ker}\,((\mu_* I + A(\omega_*))|_{X_{\omega_*}}) \cap X_{\omega_*,\infty}$. Set $g_N = (i\mu_N + iA(\omega_N) + \tau_N L(\omega_N))u_N$, and we have $\tau_N \langle L(\omega_N)^\sharp u_N, u_N\rangle = \Re\langle g_N, u_N\rangle$. Here, we recall (18) and this gives

$$|L(\omega_N)^\sharp u_N|^2 = |L(\omega_N)^\sharp|_{\mathbb{D}^\perp_{\sharp,\omega_N} X_{\omega_N}} \mathbb{D}^\perp_{\sharp,\omega_N} u_N|^2 \le C\tau_N^{-1} |g_N|\,|u_N|\,.$$

Thus, employing $|u_N| = 1$ and $g_N = o(\tau_N^{1+2\beta})$, we obtain

$$\lim_{N\to\infty} \tau_N^{-\beta} |L(\omega_N)^\sharp u_N| = 0\,. \tag{29}$$

This gives $u_* \in \mathrm{Ker}\,(L(\omega_*)^\sharp|_{X_{\omega_*}})$. Since $\{(A(\omega')|_{X_{\omega'}}, X_{\omega'})\}_{\omega'\in S^{n-1}}$ satisfies the condition of the no-splitting real eigenvalues, we can take a continuous map $\omega \mapsto s_0(\omega) \in \mathbb{R}$ such that $s_0(\omega_*) = \mu_*$ and that each $s_0(\omega)$ is the eigenvalue of $A(\omega)|_{X_\omega}$.

Next, we decompose u_N as $u_N = w_N + w_N^\perp$, where $w_N := \mathbb{Q}_{0,N} u_N$, $w_N^\perp := \mathbb{Q}^\perp_{0,N} u_N$, and $\mathbb{Q}_{0,N} := \mathbb{Q}_{s_0(\omega_N),\omega_N}$. Then, we have $w_N, w_N^\perp \in X_{\omega_N}$ by the invariance (Inv). Furthermore, g_N is also decomposed by

$$\begin{aligned} g_N &= \mathbb{Q}_{0,N} g_N + \mathbb{Q}^\perp_{0,N} g_N \\ &= i(\mu_N - s_0(\omega_N)) w_N + \tau_N \mathbb{Q}_{0,N} L(\omega_N) u_N \\ &\quad + i(\mu_N + A(\omega_N))|_{\mathbb{Q}^\perp_{0,N} X_{\omega_N}} w_N^\perp + \tau_N \mathbb{Q}^\perp_{0,N} L(\omega_N) u_N\,. \end{aligned}$$

Then, since $g_N = o(\tau_N^{1+2\beta})$, this decomposition yields

$$i(\mu_N - s_0(\omega_N)) w_N + \tau_N \mathbb{Q}_{0,N} L(\omega_N) u_N = o(\tau_N^{1+2\beta}), \tag{30}$$

$$i(\mu_N + A(\omega_N))|_{\mathbb{Q}^\perp_{0,N} X_{\omega_N}} w_N^\perp + \tau_N \mathbb{Q}^\perp_{0,N} L(\omega_N) u_N = o(\tau_N^{1+2\beta})\,. \tag{31}$$

We notice that, for large N, the operator

$$\big(i\mu_N I + iA(\omega_N) + \tau_N \mathbb{Q}^\perp_{0,N} L(\omega_N)\big)|_{\mathbb{Q}^\perp_{0,N} X_{\omega_N}} : \mathbb{Q}^\perp_{0,N} X_{\omega_N} \to \mathbb{Q}^\perp_{0,N} X_{\omega_N}$$

is invertible on $\mathbb{Q}^\perp_{0,N} X_{\omega_N}$ with the uniform bound in N for its inverse. Thus, using (31), we get

$$w_N^\perp = -\tau_N \big(i\mu_N I + iA(\omega_N) + \tau_N \mathbb{Q}^\perp_{0,N} L(\omega_N)\big)|^{-1}_{\mathbb{Q}^\perp_{0,N} X_{\omega_N}} \mathbb{Q}^\perp_{0,N} L(\omega_N) w_N + o(\tau_N^{1+2\beta})\,. \tag{32}$$

Furthermore, since (30) and (32) with $\lim_{N\to\infty} w_N = u_*$ and $\lim_{N\to\infty} \mu_N = \mu_* = s_0(\omega_*)$, we obtain $|\mu_N - s_0(\omega_N)| \le C\tau_N$ and $|w_N^\perp| \le C\tau_N$, where C is independent of N. Then, we set

$$\tilde\mu_N = \frac{\mu_N - s_0(\omega_N)}{\tau_N}\,, \qquad \tilde w_N^\perp = \frac{w_N^\perp}{\tau_N} \in X_{\omega_N}\,, \tag{33}$$

which are bounded uniformly in N. Thus, by taking a subsequence if necessary, we may assume that $\lim_{N\to\infty} \tilde\mu_N = \tilde\mu_*$ and $\lim_{N\to\infty} \tilde w_N^\perp = \tilde w_*^\perp$. Then, we have $u_* \in \mathrm{Ker}(i\tilde\mu_* I + \mathbb{Q}_{\mu_*,\omega_*} L(\omega_*))$ and

$$\tilde w_*^\perp = i(\mu_* + A(\omega_*))|^{-1}_{\mathbb{Q}^\perp_{\mu_*,\omega_*} X_{\omega_*}} \mathbb{Q}^\perp_{\mu_*,\omega_*} L(\omega_*) u_*\,.$$

As a summary, we have $u_* \neq 0$ and $u_* \in V^{high,0}(\tilde\mu_*, \mu_*, \omega_*)$. In particular, $V^{high,0}(\tilde\mu_*, \mu_*, \omega_*)$ must be nontrivial.

Case $\beta = 0$: Assume that $(\mathrm{NDC})_{high,0}$ holds. Then, we reach the contradiction to $\mathcal{S}_{high,0} = \emptyset$ in $(\mathrm{NDC})_{high,0}$, and the proof is complete in this case.

Case $\beta = 1$: Assume that $(\mathrm{NDC})_{high,1}$ holds. Firstly we rewrite (30). By setting $K_N = -\big(s_0(\omega_N)I + A(\omega_N)\big)|^{-1}_{\mathbb{Q}^\perp_{0,N}X_{\omega_N}}\mathbb{Q}^\perp_{0,N}$, which is self-adjoint in $\mathbb{Q}^\perp_{0,N}X_{\omega_N}$, the Neumann series imply that

$$\begin{aligned}
&\big(i\mu_N I + iA(\omega_N) + \tau_N\mathbb{Q}^\perp_{0,N}L(\omega_N)\big)|^{-1}_{\mathbb{Q}^\perp_{0,N}X_{\omega_N}}\\
&= -i\big(s_0(\omega_N)I + A(\omega_N) + \tau_N(\tilde{\mu}_N - i\mathbb{Q}^\perp_{0,N}L(\omega_N))\big)|^{-1}_{\mathbb{Q}^\perp_{0,N}X_{\omega_N}}\\
&= iK_N + \tau_N\big(i\tilde{\mu}_N K_N^2 + K_N L(\omega_N)K_N\big) + O(\tau_N^2)\,.
\end{aligned} \tag{34}$$

Then, substituting (33) with (32) and (34) into (30), we obtain

$$\begin{aligned}
&i\tilde{\mu}_N w_N - i\tau_N\mathbb{Q}_{0,N}L(\omega_N)(K_N + \tau_N\tilde{\mu}_N K_N^2)L(\omega_N)w_N\\
&+ \mathbb{Q}_{0,N}L(\omega_N)w_N - \tau_N^2\mathbb{Q}_{0,N}L(\omega_N)K_N L(\omega_N)K_N L(\omega_N)w_N = o(\tau_N^2)\,.
\end{aligned} \tag{35}$$

Furthermore, thanks to the relation $L(\omega_N)w_N = -L(\omega_N)^* w_N + 2L(\omega_N)^\sharp w_N$ and $L(\omega_N)^\sharp w_N = L(\omega_N)^\sharp u_N - \tau_N L(\omega_N)^\sharp \tilde{w}_N^\perp = O(\tau_N)$ derived by (29), Equation (35) is rewritten as

$$i\tilde{\mu}_N w_N + i\tau_N A_N^{(1)} w_N + L_N^{(1)} w_N = o(\tau_N^2)\,, \tag{36}$$

where $A_N^{(1)} = A_N^{(1)}(\tau_N, \omega_N)$ and $L_N^{(1)} = L_N^{(1)}(\tau_N, \omega_N)$ are defined by

$$\begin{aligned}
A_N^{(1)}(\tau_N,\omega_N) &:= \mathbb{Q}_{0,N}L(\omega_N)(K_N + \tau_N\tilde{\mu}_N K_N^2)L(\omega_N)^*\,,\\
L_N^{(1)}(\tau_N,\omega_N) &:= \mathbb{Q}_{0,N}L(\omega_N) - 2i\tau_N\mathbb{Q}_{0,N}L(\omega_N)K_N L(\omega_N)^\sharp\\
&\quad + \tau_N^2\mathbb{Q}_{0,N}L(\omega_N)K_N L(\omega_N)K_N L(\omega_N)^*\,.
\end{aligned}$$

Note that $A_N^{(1)}$ is a Hermitian and $L_N^{(1)}$ is a nonnegative definite. Indeed, we compute

$$\begin{aligned}
\Re\langle L_N^{(1)}(\tau_N,\omega_N)w_N, w_N\rangle &= \langle L(\omega_N)^\sharp w_N, w_N\rangle + 2\Re\langle L(\omega_N)^\sharp w_N, i\tau_N K_N L(\omega_N)^* w_N\rangle\\
&\quad + \tau_N^2\langle L(\omega_N)^\sharp K_N L(\omega_N)^* w_N, K_N L(\omega_N)^* w_N\rangle\\
&= |w_N + i\tau_N K_N L(\omega_N)^* w_N|^2_{\sharp,\omega_N} \geq 0\,.
\end{aligned} \tag{37}$$

Here, $|\cdot|_{\sharp,\omega_N}$ is the weighted seminorm defined by $|f|^2_{\sharp,\omega_N} := \langle L(\omega_N)^\sharp f, f\rangle$. Thus, the operator $i\tilde{\mu}_N I|_{\mathbb{Q}_{0,N}X_{\omega_N}} + i\tau_N A_N^{(1)} + L_N^{(1)}$ has a similar structure as the one discussed in Theorem 4 for the analysis of the low frequency. The only difference is that the operators $A_N^{(1)}$ and $L_N^{(1)}$ depend on not only ω_N but also τ_N. The argument in the proof of Theorem 4 for the case $\alpha = 1$ but with (36) and (37) imply that

$$\lim_{N\to\infty}\tau_N^{-1}|w_N + i\tau_N K_N L(\omega_N)^* w_N|_{\sharp,\omega_N} = 0\,. \tag{38}$$

Furthermore, (38) also leads to

$$L(\omega_N)^\sharp w_N = -i\tau_N L(\omega_N)^\sharp K_N L(\omega_N)^* w_N + o(\tau_N)\,, \tag{39}$$

and these facts together with $L(\omega_N)^* w_N = -L(\omega_N)^\flat w_N + O(\tau_N)$ give

$$\lim_{N\to\infty}\tau_N^{-1}|w_N - i\tau_N K_N L(\omega_N)^\flat w_N|_{\sharp,\omega_N} = 0\,. \tag{40}$$

In the rest of the proof, we consider two cases and derive the contradiction in each case.

Case 1: Suppose that

$$L(\omega_N)^\sharp K_N L(\omega_N)^\flat \mathbb{Q}_{0,N} X_{\omega_N} \subset \mathbb{Q}_{0,N}^\perp X_{\omega_N} \tag{41}$$

for any N. Then, using (40) and

$$|w_N - i\tau_N K_N L(\omega_N)^\flat w_N|^2_{\sharp,\omega_N} = |w_N|^2_{\sharp,\omega_N} + \tau_N^2 |K_N L(\omega_N)^\flat w_N|^2_{\sharp,\omega_N}$$

given by the orthogonality, we obtain $\lim_{N\to\infty} |K_N L(\omega_N)^\flat w_N|_{\sharp,\omega_N} = 0$. Hence, we have

$$L(\omega_*)^\sharp K_\infty L(\omega_*)^\flat u_* = 0\,, \tag{42}$$

where $K_\infty := -\big(\mu_* I + A(\omega_*)\big)|^{-1}_{\mathbb{Q}^\perp_{\mu_*,\omega_*} X_{\omega_*}} \mathbb{Q}^\perp_{\mu_*,\omega_*}$. Hence, if (iii') of $(\mathrm{NDC})_{high,1}$ holds, then we reach the contradiction.

Case 2: Next, we consider the case (iii) of $(\mathrm{NDC})_{high,1}$ holds. If (41) is not necessary satisfied, then one cannot derive (42). Thus, we should analyze (36) in more detail. Substituting (39) into $L_N^{(1)}(\tau_N, \omega_N) w_N$, we have

$$L_N^{(1)}(\tau_N, \omega_N) w_N = \mathbb{Q}_{0,N} L(\omega_N) w_N - \tau_N^2 \mathbb{Q}_{0,N} L(\omega_N) K_N L(\omega_N)^* K_N L(\omega_N)^* w_N + o(\tau_N^2)\,,$$

and (36) is written as

$$i\tilde{\mu}_N w_N + i\tau_N A_N^{(1)} w_N + \mathbb{Q}_{0,N} L(\omega_N) w_N - \tau_N^2 \mathbb{Q}_{0,N} L(\omega_N) K_N L(\omega_N)^* K_N L(\omega_N)^* w_N = o(\tau_N^2)\,. \tag{43}$$

Then, we just follow the argument in the proof of Theorem 4 for the case $\alpha = 1$; below, only a sketch of the proof is given. Firstly, we decompose w_N as $w_N = y_N + y_N^\perp$, where $y_N := \mathbb{Q}_{1,N} w_N$, $y_N^\perp := \mathbb{Q}_{1,N}^\perp w_N$, and $\mathbb{Q}_{1,N} := \mathbb{Q}_{s_1(\omega_N), s_0(\omega_N), \omega_N}$. Here, $s_1(\cdot) : S^{n-1} \to \mathbb{R}$ is a continuous map associated with the no-splitting real eigenvalues and thus satisfying $s_1(\omega_*) = \tilde{\mu}_*$. Similar to above, y_N and $y_N^\perp$ satisfy

$$i(\tilde{\mu}_N - s_1(\omega_N)) y_N + i\tau_N \mathbb{Q}_{1,N} A_N^{(1)} w_N - \tau_N^2 \mathbb{Q}_{1,N} L(\omega_N) K_N L(\omega_N)^* K_N L(\omega_N)^* w_N = o(\tau_N^2)\,,$$

$$y_N^\perp = -i\tau_N \big(i\tilde{\mu}_N I + \mathbb{Q}_{0,N} L(\omega_N)\big)|^{-1}_{\mathbb{Q}^\perp_{1,N} X_{\omega_N}} \mathbb{Q}^\perp_{1,N} A_N^{(1)} y_N + O(\tau_N^2)\,,$$

which comes from (43). Then, we set

$$\sigma_N = \frac{\tilde{\mu}_N - s_1(\omega_N)}{\tau_N}\,, \qquad \tilde{y}_N^\perp = \frac{y_N^\perp}{\tau_N} \in X_{\omega_N}\,,$$

and this gives from $A_N^{(1)} = \mathbb{Q}_{0,N} L(\omega_N)(K_N + \tau_N \tilde{\mu}_N K_N^2) L(\omega_N)^*$,

$$\begin{aligned} &i\big(\sigma_N I + \mathbb{Q}_{1,N} L(\omega_N) K_N L(\omega_N)^*\big) y_N + i\tau_N \tilde{\mu}_N \mathbb{Q}_{1,N} L(\omega_N) K_N^2 L(\omega_N)^* y_N \\ &+ \tau_N \mathbb{Q}_{1,N} L(\omega_N) K_N L(\omega_N)^* A_N^{(2)} y_N = o(\tau_N) \end{aligned} \tag{44}$$

and

$$\tilde{y}_N^\perp = -i\big(i\tilde{\mu}_N I + \mathbb{Q}_{0,N} L(\omega_N)\big)|^{-1}_{\mathbb{Q}^\perp_{1,N} X_{\omega_N}} \mathbb{Q}^\perp_{1,N} A_N^{(1)} y_N + O(\tau_N)\,,$$

where $A_N^{(2)} := \big(i\tilde{\mu}_N I + \mathbb{Q}_{0,N} L(\omega_N)\big)|^{-1}_{\mathbb{Q}^\perp_{1,N} X_{\omega_N}} \mathbb{Q}^\perp_{1,N} A_N^{(1)} - K_N L(\omega_N)^*$. Thus, letting $N \to \infty$ and setting $\lim_{N\to\infty} \sigma_N = \sigma_*$, $\lim_{N\to\infty} \tilde{y}_N^\perp = \tilde{y}_*^\perp$, and

$$A_\infty^{(1)} := \mathbb{Q}_{\mu_*,\omega_*} L(\omega_*) K_\infty L(\omega_*)^*|_{\mathbb{Q}_{\mu_*,\omega_*} X_{\omega_*}}\,,$$

we obtain $u_* \in \mathrm{Ker}(\sigma_* I + \mathbb{Q}_{\tilde{\mu}_*,\mu_*,\omega_*} A_\infty^{(1)})$ (i.e. $u_* \in \mathrm{Ran}\,(\mathbb{Q}_{\sigma_*,\tilde{\mu}_*,\mu_*,\omega_*})$) and

$$\tilde{y}_*^\perp = -i\big(i\tilde{\mu}_* I + \mathbb{Q}_{\mu_*,\omega_*} L(\omega_*)\big)|^{-1}_{\mathbb{Q}^\perp_{\tilde{\mu}_*,\mu_*,\omega_*} X_{\omega_*}} \mathbb{Q}^\perp_{\tilde{\mu}_*,\mu_*,\omega_*} A_\infty^{(1)} u_* \,. \tag{45}$$

Since (38) and $\mathrm{Ran}\,(\mathbb{Q}_{1,0,N}) \subset \mathrm{Ker}(L(\omega_N)^\sharp)$, we get

$$L(\omega_*)^\sharp \tilde{y}_*^\perp + iL(\omega_*)^\sharp K_\infty L(\omega_*)^* u_* = 0\,.$$

Thus, this fact and (45) give

$$A_\infty^{(2)} u_* \in \mathrm{Ker}\big(L(\omega_*)^\sharp\big)\,,$$

where

$$A_\infty^{(2)} := \big(i\tilde{\mu}_* I + \mathbb{Q}_{\mu_*,\omega_*} L(\omega_*)\big)|^{-1}_{\mathbb{Q}^\perp_{\tilde{\mu}_*,\mu_*,\omega_*} X_{\omega_*}} \mathbb{Q}^\perp_{\tilde{\mu}_*,\mu_*,\omega_*} A_\infty^{(1)} - K_\infty L(\omega_*)^*|_{\mathbb{Q}_{\mu_*,\omega_*} X_{\omega_*}}\,.$$

Finally, we also decompose y_N as $y_N = z_N + z_N^\perp$, where $z_N := \mathbb{Q}_{2,N} y_N$, $z_N^\perp := \mathbb{Q}^\perp_{2,N} y_N$, and $\mathbb{Q}_{2,N} := \mathbb{Q}_{s_2(\omega_N),s_1(\omega_N),s_0(\omega_N),\omega_N}$. Here, $s_2(\cdot) : S^{n-1} \to \mathbb{R}$ is a continuous map associated with the no-splitting real eigenvalues and thus satisfying $s_2(\omega_*) = \sigma_*$. Then, (44) leads to

$$\begin{aligned} i(\sigma_N - s_2(\omega_N))z_N + i\tau_N\tilde{\mu}_N \mathbb{Q}_{2,N} L(\omega_N) K_N^2 L(\omega_N)^* y_N \\ + \tau_N \mathbb{Q}_{2,N} L(\omega_N) K_N L(\omega_N)^* A_N^{(2)} y_N = o(\tau_N)\,, \end{aligned}$$

and we arrive at, by recalling $K_\infty = -\big(\mu_* I + A(\omega_*)\big)|^{-1}_{\mathbb{Q}^\perp_{\mu_*,\omega_*} X_{\omega_*}} \mathbb{Q}^\perp_{\mu_*,\omega_*}$,

$$i\tilde{\sigma}_* u_* + i\tilde{\mu}_* \mathbb{Q}_{\sigma_*,\tilde{\mu}_*,\mu_*,\omega_*} L(\omega_*) K_\infty^2 L(\omega_*)^* u_* + \mathbb{Q}_{\sigma_*,\tilde{\mu}_*,\mu_*,\omega_*} L(\omega_*) K_\infty L(\omega_*)^* A_\infty^{(2)} u_* = 0\,, \tag{46}$$

where $\tilde{\sigma}_N := (\sigma_N - s_2(\omega_N))/\tau_N$ and $\lim_{N\to\infty} \tilde{\sigma}_N = \tilde{\sigma}_*$. Consequently, we achieve $u_* \in V^{high,1}(\tilde{\sigma}_*, \sigma_*, \tilde{\mu}_*, \mu_*, \omega_*)$ and it must be zero by the condition $(\mathrm{NDC})_{high,1}$, which is a contradiction. The proof is complete. □

Remark 7. *The matrix in (46) has a similar structure to the original one "ir$A(\omega) + L(\omega)$". Indeed, it is straightforward to see that $L(\omega)K(s_0,\omega)^2 L(\omega)^*$ with $K(s_0,\omega) = -\big(s_0 I + A(\omega)\big)|^{-1}_{\mathbb{Q}^\perp_{s_0,\omega} X_\omega} \mathbb{Q}^\perp_{s_0,\omega}$ is Hermitian. On the other hand, we set an operator*

$$\tilde{A}^{(1)}(s_1,s_0,\omega) := \big(is_1 I + \mathbb{Q}_{s_0,\omega} L(\omega)\big)|^{-1}_{\mathbb{Q}^\perp_{s_1,s_0,\omega} X_\omega} \mathbb{Q}^\perp_{s_1,s_0,\omega} A^{(1)}(s_0,\omega) : \mathbb{Q}_{s_0,\omega} X_\omega \to \mathbb{Q}_{s_0,\omega} X_\omega$$

with $A^{(1)}(s_0,\omega) = \mathbb{Q}_{s_0,\omega} L(\omega) K(s_0,\omega) L(\omega)^*|_{\mathbb{Q}_{s_0,\omega} X_\omega}$, and define

$$\begin{aligned} &L^{(2)}(s_1,s_0,\omega) \\ &:= -A^{(1)}(s_0,\omega)\tilde{A}^{(1)}(s_1,s_0,\omega) + \mathbb{Q}_{s_0,\omega} L(\omega) K(s_0,\omega) L(\omega)^* K(s_0,\omega) L(\omega)^*|_{\mathbb{Q}_{s_0,\omega} X_\omega}\,. \end{aligned}$$

Then, we find $-L^{(2)}(\tilde{\mu}_*,\mu_*,\omega_*) = \mathbb{Q}_{\mu_*,\omega_*} L(\omega_*) K_\infty L(\omega_*)^* A_\infty^{(2)}$, which appears in (46). We study the property of $L^{(2)}(s_1,s_0,\omega)$. The Hermitian part of $L^{(2)}(s_1,s_0,\omega)$ is given by

$$\begin{aligned} &L^{(2)}(s_1,s_0,\omega)^\sharp \\ &= \Big(\tilde{A}^{(1)}(s_1,s_0,\omega)^* - \mathbb{Q}_{s_0,\omega} L(\omega) K(s_0,\omega)\Big) L(\omega)^\sharp \Big(\tilde{A}^{(1)}(s_1,s_0,\omega) - K(s_0,\omega) L(\omega)^*|_{\mathbb{Q}_{s_0,\omega} X_\omega}\Big), \end{aligned} \tag{47}$$

which implies that $L^{(2)}(s_1,s_0,\omega)^\sharp$ *is nonnegative definite on* $\mathbb{Q}_{s_0,\omega}X_\omega$*, while the skew-Hermitian part of* $L^{(2)}(s_1,s_0,\omega)$ *is given by*

$$\begin{aligned}L^{(2)}(s_1,s_0,\omega)^\flat = &-\tilde{A}^{(1)}(s_1,s_0,\omega)^*(is_1I+L(\omega)^\flat)\tilde{A}^{(1)}(s_1,s_0,\omega)\\ &+\tilde{A}^{(1)}(s_1,s_0,\omega)^*\big(A^{(1)}(s_0,\omega)-L(\omega)^\flat K(s_0,\omega)L(\omega)^*|_{\mathbb{Q}_{s_0,\omega}X_\omega}\big)\\ &-\big(A^{(1)}(s_0,\omega)+\mathbb{Q}_{s_0,\omega}L(\omega)K(s_0,\omega)L(\omega)^\flat\big)\tilde{A}^{(1)}(s_1,s_0,\omega)\\ &-\mathbb{Q}_{s_0,\omega}L(\omega)K(s_0,\omega)L(\omega)^\flat K(s_0,\omega)L(\omega)^*|_{\mathbb{Q}_{s_0,\omega}X_\omega}\,.\end{aligned}$$

To check these identities, we start from (47) *and then observe that*

$$\begin{aligned}&L^{(2)}(s_1,s_0,\omega)^\sharp\\ &=A^{(1)}(s_0,\omega)\tilde{A}^{(1)}(s_1,s_0,\omega)+\tilde{A}^{(1)}(s_1,s_0,\omega)^*(is_1I+L(\omega)^\flat)\tilde{A}^{(1)}(s_1,s_0,\omega)\\ &\quad-A^{(1)}(s_0,\omega)\tilde{A}^{(1)}(s_1,s_0,\omega)-\mathbb{Q}_{s_0,\omega}L(\omega)K(s_0,\omega)L(\omega)^\flat\tilde{A}^{(1)}(s_1,s_0,\omega)\\ &\quad-\tilde{A}^{(1)}(s_1,s_0,\omega)^*A^{(1)}(s_0,\omega)+\tilde{A}^{(1)}(s_1,s_0,\omega)^*L(\omega)^\flat K(s_0,\omega)L(\omega)^*|_{\mathbb{Q}_{s_0,\omega}X_\omega}\\ &\quad+\mathbb{Q}_{s_0,\omega}L(\omega)K(s_0,\omega)L(\omega)^*K(s_0,\omega)L(\omega)^*|_{\mathbb{Q}_{s_0,\omega}X_\omega}\\ &\quad+\mathbb{Q}_{s_0,\omega}L(\omega)K(s_0,\omega)L(\omega)^\flat K(s_0,\omega)L(\omega)^*|_{\mathbb{Q}_{s_0,\omega}X_\omega}\\ &=-A^{(1)}(s_0,\omega)\tilde{A}^{(1)}(s_1,s_0,\omega)+\mathbb{Q}_{s_0,\omega}L(\omega)K(s_0,\omega)L(\omega)^*K(s_0,\omega)L(\omega)^*|_{\mathbb{Q}_{s_0,\omega}X_\omega}-L^{(2),\flat}_{s_1,s_0,\omega}\,,\end{aligned}$$

where we use that $L(\omega)^\sharp=L(\omega)^*+L(\omega)^\flat=L(\omega)-L(\omega)^\flat$*. It is not difficult to check that* $L^{(2)}(s_1,s_0,\omega)^\sharp$ *is Hermitian and* $L^{(2)}(s_1,s_0,\omega)^\flat$ *is skew-Hermitian on* $\mathbb{Q}_{s_0,\omega}X_\omega$*. Moreover, if* $u\in\mathrm{Ran}\,(\mathbb{Q}_{s_3,s_2,s_1,\omega})$*, then* $u\in\mathrm{Ker}\,(L^{(2)}(s_1,s_0,\omega)^\sharp)$*, i.e.,*

$$L(\omega)^\sharp\Big(\tilde{A}^{(1)}(s_1,s_0,\omega)-K(s_0,\omega)L(\omega)^*|_{\mathbb{Q}_{s_0,\omega}X_\omega}\Big)u=0 \tag{48}$$

is satisfied.

5.2. Semigroup Estimate

As in Theorem 5, by applying the result of Wei [25], we obtain from Theorem 7 the following semigroup estimate for the high frequency.

Theorem 8. *There exist positive constants* C *and* c *such that the following statements hold. Let* $\beta\in\{0,1\}$ *and assume that* $(\mathrm{NDC})_{high,\beta}$ *holds. Then, for any* $\xi\in\mathbb{R}^n$ *with* $|\xi|\geq 1$*,*

$$|e^{-t\mathcal{A}(\xi)}\boldsymbol{y}|\leq Ce^{-c|\xi|^{-2\beta}t}|\boldsymbol{y}|\,,\qquad \boldsymbol{y}\in X_{\xi/|\xi|,|\xi|}\,. \tag{49}$$

Proof. Theorem 7 implies the pseudospectral bound $\Psi(r,\omega)\geq r^{-2\beta}/C$, where the positive constant C is independent of $r\geq 1$ and $\omega\in S^{n-1}$. Then, the estimate (49) follows from Theorem A1. The proof is complete. □

5.3. Optimality

The optimality for the high frequency is stated as follows.

Theorem 9. *Assume that* (SC) *and* (Inv) *hold. Assume in addition that* $\mathcal{A}(r\omega)\mathcal{P}^\perp_{\omega,r}X_\omega\subset\mathcal{P}^\perp_{\omega,r}X_\omega$ *holds for any* $r>0$ *and* $\omega\in S^{n-1}$*. Then, the following statement holds. Let* $\beta\in\{0,1\}$*. If* $\mathcal{S}_{high,\beta}\neq\emptyset$*, then* $\{e^{-t\mathcal{A}(r\omega)}\}_{t\geq 0}$ *does not have the uniform dissipative bound of order* β *at high frequency.*

Proof. The proof is similar to the one in Section 4.3. Assume that $\mathcal{S}_{high,0}\neq\emptyset$. Suppose that $\{e^{-t\mathcal{A}(\xi)}\}_{t\geq 0}$ has the uniform dissipative bound of order 0 at high frequency. Then, there

exist positive constants C and c such that $\|e^{-t\mathcal{A}(r\omega)}\|_{X_{\omega,r}\to X_{\omega,r}} \le Ce^{-ct}$ for $t>0$, $\omega\in S^{n-1}$, and $r\ge 1$. Then, the Laplace transform for the resolvent implies

$$\sup_{\omega\in S^{n-1}}\Phi(r,\omega)=\sup_{\lambda\in\mathbb{R},\omega\in S^{n-1}}\|(i\lambda I+\mathcal{A}(r\omega))|^{-1}_{X_{\omega,r}}\|_{X_{\omega,r}\to X_{\omega,r}}\le\int_0^\infty Ce^{-ct}\,\mathrm{d}t=\frac{C}{c}\,,\qquad r\ge 1,$$

which is equivalent with

$$\sup_{s\in\mathbb{R},\omega\in S^{n-1}}\|\big(isI+iA(\omega)+\tau L(\omega)\big)|^{-1}_{X_{\omega,1/\tau}}\|_{X_{\omega,1/\tau}\to X_{\omega,1/\tau}}\le\frac{C}{c\tau}\,,\qquad 0<\tau\le 1\,,\tag{50}$$

Hence, it suffices to reach the contradiction to the uniform resolvent estimate (50). Let $(s_1,s_0,\omega)\in\mathcal{S}_{high,0}$. Then, there exists $\boldsymbol{y}_0\in V^{high,0}(s_1,s_0,\omega)$ with $|\boldsymbol{y}_0|=1$. We set

$$\boldsymbol{y}_1=-\big(i(s_0+\tau s_1)I+iA(\omega)+\tau\mathbb{Q}^\perp_{s_0,\omega}L(\omega)\big)|^{-1}_{\mathbb{Q}^\perp_{s_0,\omega}X_\omega}\mathbb{Q}^\perp_{s_0,\omega}L(\omega)\boldsymbol{y}_0\,.$$

Note that the operator $(i(s_0+\tau s_1)I+A(\omega)+\mathbb{Q}^\perp_{s_0,\omega}L(\omega))|^{-1}_{\mathbb{Q}^\perp_{s_0,\omega}X_\omega}$ is well defined for small enough $\tau>0$, and we have the uniform estimate for small enough $\tau>0$ such that

$$|\boldsymbol{y}_1|\le C_{s_0,\omega}\,.\tag{51}$$

Now, let us set $\boldsymbol{x}(\tau)=\boldsymbol{y}_0+\tau\boldsymbol{y}_1$, which satisfies $1/2\le|\boldsymbol{x}(\tau)|\le 2$ for any sufficiently small $\tau>0$ in virtue of (51) and $|\boldsymbol{y}_0|=1$. However, we have

$$\big(i(s_0+\tau s_1)I+iA(\omega)+\tau L(\omega)\big)\boldsymbol{x}(\tau)=i\tau s_1\boldsymbol{y}_0+\tau\mathbb{Q}_{s_0,\omega}L(\omega)\boldsymbol{y}_0+O(\tau^2)=O(\tau^2)\,.$$

Here, we use $\boldsymbol{y}_0\in\mathrm{Ran}\,(\mathbb{Q}_{s_1,s_0,\omega})$ in the last line. By acting the projection $\mathcal{P}_{\omega,1/\tau}$ on both sides above and by using the invariant condition $\mathcal{A}(r\omega)\mathcal{P}_{\omega,r}X_\omega\subset\mathcal{P}_{\omega,r}X_\omega$ and $\mathcal{A}(r\omega)\mathcal{P}^\perp_{\omega,r}X_\omega\subset\mathcal{P}^\perp_{\omega,r}X_\omega$, if the resolvent estimate (50) holds, then we must have $|\mathcal{P}_{\omega,1/\tau}\boldsymbol{x}(\tau)|\le O(\tau)$ for $0<\tau\ll 1$, which contradicts with $\lim_{\tau\to 0}\mathcal{P}_{\omega,1/\tau}\boldsymbol{x}(\tau)=\boldsymbol{y}_0$ and $|\boldsymbol{y}_0|=1$.

Next, we assume that $\mathcal{S}_{high,1}\neq\emptyset$. In this case, it suffices to reach the contradiction to the estimate

$$\sup_{s\in\mathbb{R},\omega\in S^{n-1}}\|(isI+iA(\omega)+\tau L(\omega))|^{-1}_{X_{\omega,1/\tau}}\|_{X_{\omega,1/\tau}\to X_{\omega,1/\tau}}\le\frac{C}{c\tau^3}\,,\qquad 0<\tau\le 1\,,\tag{52}$$

Let us take $\boldsymbol{y}_0\in V^{high,1}(s_3,s_2,s_1,s_0,\omega)$ with $|\boldsymbol{y}_0|=1$. Set

$$\begin{aligned}
\boldsymbol{y}_1&=-\big(i(s_0+\tau s_1+\tau^2s_2+\tau^3s_3)I+iA(\omega)+\tau\mathbb{Q}^\perp_{s_0,\omega}L(\omega)\big)|^{-1}_{\mathbb{Q}^\perp_{s_0,\omega}}\mathbb{Q}^\perp_{s_0,\omega}L(\omega)\boldsymbol{y}_0\,,\\
\tilde{\boldsymbol{y}}_1&=-\big(i(s_1+\tau s_2+\tau^2s_3)I+\mathbb{Q}_{s_0,\omega}L(\omega)\big)|^{-1}_{\mathbb{Q}^\perp_{s_1,s_0,\omega}}\mathbb{Q}^\perp_{s_1,s_0,\omega}\mathbb{Q}_{s_0,\omega}L(\omega)\boldsymbol{y}_1\,,\\
\boldsymbol{y}_2&=-\big(i(s_0+\tau s_1+\tau^2s_2+\tau^3s_3)I+iA(\omega)+\tau\mathbb{Q}^\perp_{s_0,\omega}L(\omega)\big)|^{-1}_{\mathbb{Q}^\perp_{s_0,\omega}}\mathbb{Q}^\perp_{s_0,\omega}L(\omega)\tilde{\boldsymbol{y}}_1\,,\\
\tilde{\boldsymbol{y}}_2&=-\big(i(s_1+\tau s_2+\tau^2s_3)I+\mathbb{Q}_{s_0,\omega}L(\omega)\big)|^{-1}_{\mathbb{Q}^\perp_{s_1,s_0,\omega}}\mathbb{Q}^\perp_{s_1,s_0,\omega}\mathbb{Q}_{s_0,\omega}L(\omega)\boldsymbol{y}_2\,,\\
\boldsymbol{y}_3&=-\big(i(s_0+\tau s_1+\tau^2s_2+\tau^3s_3)I+iA(\omega)\big)|^{-1}_{\mathbb{Q}^\perp_{s_0,\omega}}\mathbb{Q}^\perp_{s_0,\omega}L(\omega)\tilde{\boldsymbol{y}}_2\,,
\end{aligned}$$

and $\boldsymbol{x}(\tau)=\boldsymbol{y}_0+\tau(\boldsymbol{y}_1+\tilde{\boldsymbol{y}}_1)+\tau^2(\boldsymbol{y}_2+\tilde{\boldsymbol{y}}_2)+\tau^3\boldsymbol{y}_3$. Then, we find that

$$\begin{aligned}
&\big(i(s_0+\tau s_1+\tau^2s_2+\tau^3s_3)I+iA(\omega)+L(\omega)\big)\boldsymbol{x}(\tau)\\
&=\tau^2(is_2\boldsymbol{y}_0+\mathbb{Q}_{s_1,s_0,\omega}L(\omega)\boldsymbol{y}_1)+\tau^3(is_3\boldsymbol{y}_0+\mathbb{Q}_{s_1,s_0,\omega}L(\omega)\boldsymbol{y}_2)+O(\tau^4)\,,
\end{aligned}$$

while

$$\boldsymbol{y}_1 = -iK(s_0,\omega)L(\omega)\boldsymbol{y}_0 - i\tau K(s_0,\omega)(s_1 I - i\mathbb{Q}^{\perp}_{s_0,\omega}L(\omega))K(s_0,\omega)L(\omega)\boldsymbol{y}_0 + O(\tau^2)$$

and

$$\begin{aligned}\boldsymbol{y}_2 &= -iK(s_0,\omega)L(\omega)\tilde{\boldsymbol{y}}_1 + O(\tau)\\ &= iK(s_0,\omega)L(\omega)\big(is_1 I + \mathbb{Q}_{s_0,\omega}L(\omega)\big)|^{-1}_{\mathbb{Q}^{\perp}_{s_1,s_0,\omega}}\mathbb{Q}^{\perp}_{s_1,s_0,\omega}\mathbb{Q}_{s_0,\omega}L(\omega)\boldsymbol{y}_1 + O(\tau)\\ &= K(s_0,\omega)L(\omega)\big(is_1 I + \mathbb{Q}_{s_0,\omega}L(\omega)\big)|^{-1}_{\mathbb{Q}^{\perp}_{s_1,s_0,\omega}}\mathbb{Q}^{\perp}_{s_1,s_0,\omega}\mathbb{Q}_{s_0,\omega}L(\omega)K(s_0,\omega)L(\omega)\boldsymbol{y}_0 + O(\tau)\,.\end{aligned}$$

We recall here that $L(\omega)f = -L(\omega)^*f + 2L(\omega)^\sharp f$ in general. Since $\boldsymbol{y}_0 \in \mathrm{Ker}\,(L(\omega)^\sharp)$, we see

$$\begin{aligned}\boldsymbol{y}_1 &= iK(s_0,\omega)L(\omega)^*\boldsymbol{y}_0 + i\tau K(s_0,\omega)(s_1 I - i\mathbb{Q}^{\perp}_{s_0,\omega}L(\omega))K(s_0,\omega)L(\omega)^*\boldsymbol{y}_0 + O(\tau^2)\\ &= iK(s_0,\omega)L(\omega)^*\boldsymbol{y}_0 + i\tau s_1 K(s_0,\omega)^2L(\omega)^*\boldsymbol{y}_0\\ &\quad - \tau K(s_0,\omega)L(\omega))^*K(s_0,\omega)L(\omega)^*\boldsymbol{y}_0 + 2\tau K(s_0,\omega)L(\omega))^\sharp K(s_0,\omega)L(\omega)^*\boldsymbol{y}_0 + O(\tau^2)\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{y}_2 &= -K(s_0,\omega)L(\omega)\big(is_1 I + \mathbb{Q}_{s_0,\omega}L(\omega)\big)|^{-1}_{\mathbb{Q}^{\perp}_{s_1,s_0,\omega}}\mathbb{Q}^{\perp}_{s_1,s_0,\omega}\mathbb{Q}_{s_0,\omega}L(\omega)K(s_0,\omega)L(\omega)^*\boldsymbol{y}_0 + O(\tau)\\ &= -K(s_0,\omega)L(\omega)\big(is_1 I + \mathbb{Q}_{s_0,\omega}L(\omega)\big)|^{-1}_{\mathbb{Q}^{\perp}_{s_1,s_0,\omega}}\mathbb{Q}^{\perp}_{s_1,s_0,\omega}A^{(1)}(s_0,\omega)\boldsymbol{y}_0 + O(\tau)\\ &= K(s_0,\omega)L(\omega)^*\big(is_1 I + \mathbb{Q}_{s_0,\omega}L(\omega)\big)|^{-1}_{\mathbb{Q}^{\perp}_{s_1,s_0,\omega}}\mathbb{Q}^{\perp}_{s_1,s_0,\omega}A^{(1)}(s_0,\omega)\boldsymbol{y}_0\\ &\quad - 2K(s_0,\omega)L(\omega)^\sharp\big(is_1 I + \mathbb{Q}_{s_0,\omega}L(\omega)\big)|^{-1}_{\mathbb{Q}^{\perp}_{s_1,s_0,\omega}}\mathbb{Q}^{\perp}_{s_1,s_0,\omega}A^{(1)}(s_0,\omega)\boldsymbol{y}_0 + O(\tau)\\ &= K(s_0,\omega)L(\omega)^*\big(is_1 I + \mathbb{Q}_{s_0,\omega}L(\omega)\big)|^{-1}_{\mathbb{Q}^{\perp}_{s_1,s_0,\omega}}\mathbb{Q}^{\perp}_{s_1,s_0,\omega}A^{(1)}(s_0,\omega)\boldsymbol{y}_0\\ &\quad - 2K(s_0,\omega)L(\omega)^\sharp K(s_0,\omega)L(\omega)^*\boldsymbol{y}_0 + O(\tau)\,.\end{aligned}$$

Here, we use (48) on the last line. Then, $\tau^2(is_2\boldsymbol{y}_0 + \mathbb{Q}_{s_1,s_0,\omega}L(\omega)\boldsymbol{y}_1) + \tau^3(is_3\boldsymbol{y}_0 + \mathbb{Q}_{s_1,s_0,\omega}L(\omega)\boldsymbol{y}_2) = O(\tau^4)$, and, therefore,

$$\big(i(s_0 + \tau s_1 + \tau^2 s_2 + \tau^3 s_3)I + iA(\omega) + L(\omega)\big)\boldsymbol{x}(\tau) = O(\tau^4)\,,$$

which leads to a contradiction with (52), by discussing as in the case $\beta = 0$. The proof is complete. □

6. Proof of Main Theorems

The results of the previous sections imply the theorems stated in Section 2. Indeed, Theorem 1 follows from Theorems 4, 6, 7, and 9; Theorem 2 follows from Theorems 6 and 9; and Theorem 3 follows from Theorems 4 and 7. The proof is complete.

7. Application

In this section, we apply our main theorems to some models.

7.1. Classical Case

We recall the known results obtained by Shizuta and Kawashima [1] and Umeda, Kawashima, and Shizuta [2].

Proposition 1 (Classical case). *Let* $X_{\xi/|\xi|,|\xi|} = \mathbb{C}^m$. *Assume that* $L(\omega) = L$, $\mathrm{Ker}\,(L) = \mathrm{Ker}\,(L^\sharp)$, *and* (SC). *Then, for any* $\hat{f} \in \mathbb{C}^m$ *and* $\xi \in \mathbb{R}^n \setminus \{0\}$, *the solution operator to the system* (2) *satisfies*

$$|e^{-t\mathcal{A}(\xi)}\hat{f}| \le Ce^{-c\frac{|\xi|^2}{1+|\xi|^2}t}|\hat{f}|, \qquad t > 0. \tag{53}$$

Since $(\mathrm{NDC})_{low,1}$ and $(\mathrm{NDC})_{high,0}$ hold in this classical case, the assertion follows by Theorem 3. Furthermore, Theorems 1 and 2 lead to the optimality for the estimate (53).

7.2. Dissipative Timoshenko System

We consider the linear dissipative Timoshenko system described as

$$\begin{cases} \partial_t^2\phi - \partial_x(\partial_x\phi - \psi) = 0, \\ \partial_t^2\psi - a^2\partial_x^2\psi - (\partial_x\phi - \psi) + \gamma\partial_t\psi = 0. \end{cases} \tag{54}$$

Here, a and γ are positive constants and $\phi = \phi(t,x)$ and $\psi = \psi(t,x)$ are unknown scalar functions of $t > 0$ and $x \in \mathbb{R}$. The Timoshenko system is a model system describing the vibration of the beam called the Timoshenko beam, and ϕ and ψ denote the transversal displacement and the rotation angle of the beam, respectively.

We introduce the vector function $u = (\partial_x\phi - \psi, \partial_t\phi, a\partial_x\psi, \partial_t\psi)^\top$. Then, the Timoshenko system (54) is written in the form of the first equation in (1) with coefficient matrices

$$A_1 = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & a & 0 \end{pmatrix}, \qquad L^\sharp = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}, \qquad L^\flat = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{55}$$

Here, we have $A(\omega) = A\omega$ for $\omega \in S^{1-1} = \{\pm 1\}$ in (3). Then, Ide, Haramoto, and Kawashima [12] considered the system (2) with (55) and derived the pointwise estimate of solutions in Fourier space by the energy method. In this section, we employ our main theorems and derive the same pointwise estimate derived by [12]. We remark that the optimality of the obtained pointwise estimate is guaranteed by Theorem 1.

Proposition 2 (Linear dissipative Timoshenko system). *For any* $\hat{f} \in \mathbb{C}^4$ *and* $\xi \in \mathbb{R}\setminus\{0\}$, *the solution operator to the system* (2) *with* (55) *satisfies*

$$|e^{-t\mathcal{A}(\xi)}\hat{f}| \le \begin{cases} Ce^{-\frac{c|\xi|^2}{1+|\xi|^4}t}|\hat{f}| & (a \ne 1), \\ Ce^{-\frac{c|\xi|^2}{1+|\xi|^2}t}|\hat{f}| & (a = 1). \end{cases}$$

Proof. (Low frequency part) The eigenvalues of $iL^\flat$ are $0, \pm 1$ and the dimensions of each eigenspace are 2 (for the eigenvalue 0) and 1 (for the eigenvalue ± 1), respectively. Moreover, we have $\mathrm{Ran}\,(\mathbb{P}_{\mu,\omega}) = \{0\}$ for any $\mu \ne 0$, and

$$\mathbb{D}_{\sharp,\omega}\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ 0 \end{pmatrix}, \qquad \mathbb{P}_{0,\omega}\boldsymbol{y} = \begin{pmatrix} 0 \\ y_2 \\ y_3 \\ 0 \end{pmatrix}.$$

where $\boldsymbol{y} = (y_1, y_2, y_3, y_4)^\top \in \mathbb{C}^4$. Here, let us recall that $\mathbb{P}_{\mu,\omega}$ is the orthogonal projection to $\mathrm{Ker}\,(i\mu I + L) = \mathrm{Ker}\,(\mu I - iL^\flat) \cap \mathrm{Ker}\,(L^\sharp)$ with $\omega \in S^{n-1} = \{\pm 1\}$ (see Lemma A1). Hence, we have $\mathcal{S}_{low,0} = \{(0,\omega) \mid \omega = \pm 1\} \ne \emptyset$. This means that $(\mathrm{NDC})_{low,0}$ is not satisfied.

It is not difficult to find that $\mathbb{P}_{0,\omega}A(\omega)|_{\mathbb{P}_{0,\omega}\mathbb{C}^4} = O|_{\mathbb{P}_{0,\omega}\mathbb{C}^4}$ for all $\omega \in \{\pm 1\}$. Hence, the associated eigenprojection of the eigenvalue 0 to $\mathbb{P}_{0,\omega}A(\omega)|_{\mathbb{P}_{0,\omega}\mathbb{C}^4}$ is the identity map $I|_{\mathbb{P}_{0,\omega}\mathbb{C}^4}$. The direct computation shows that, for

$$\tilde{V}^{low,1}(\mu,0,\omega) = \mathrm{Ker}\left(L^\sharp L|^{-1}_{\mathbb{P}^\perp_{0,\omega}\mathbb{C}^4}\mathbb{P}^\perp_{0,\omega}A(\omega)\right) \cap \mathrm{Ker}\left((\mu I + \mathbb{P}_{0,\omega}A(\omega))|_{\mathbb{P}_{0,\omega}\mathbb{C}^4}\right),$$

we have

$$\tilde{V}^{low,1}(\mu,0,\omega) = \begin{cases} \left\{ \begin{pmatrix} 0 \\ 0 \\ y_3 \\ 0 \end{pmatrix} \,|\, y_3 \in \mathbb{C} \right\} & \text{for} \quad \mu = 0\,, \\ \{0\} & \text{for} \quad \mu \neq 0\,. \end{cases}$$

Hence, we obtain $\tilde{\mathcal{S}}_{low,1} = \{(0,0,\pm 1)\} \neq \emptyset$, and, thus, it suffices to consider the set $V^{low,1}(s_2,0,0,\pm 1)$ in checking (i-b) of $(\mathrm{NDC})_{low,1}$. For $\boldsymbol{y} = (0,0,y_3,0)^\top$ and $s_2 \in \mathbb{R}$, we have

$$is_2\boldsymbol{y} + \mathbb{P}_0A(\omega)L|^{-1}_{\mathbb{P}^\perp_0\mathbb{C}^4}\mathbb{P}^\perp_0A(\omega)\boldsymbol{y} = \begin{pmatrix} 0 \\ -ay_3 \\ is_2y_3 \\ 0 \end{pmatrix}.$$

Thus, this gives

$$\mathrm{Ker}\left(\left(is_2I + \mathbb{P}_0A(\omega)L|^{-1}_{\mathbb{P}^\perp_0\mathbb{C}^4}\mathbb{P}^\perp_0A(\omega)\right)|_{\mathbb{P}_{0,0,\omega}\mathbb{C}^4}\right) \cap \tilde{V}^{low,1}(0,0,\omega) = \{0\}$$

for all $s_2 \in \mathbb{R}$ and $\omega \in \{\pm 1\}$. This implies $\mathcal{S}_{low,1} = \emptyset$ and therefore Condition (i-b) of $(\mathrm{NDC})_{low,1}$ is satisfied. Hence, the proof is complete for the low frequency part.

(High frequency part) Let us recall that $a > 0$. The eigenvalues of $-A(\omega)$ are ± 1 and $\pm a$, where the dimensions of each eigenspace are 1 (if $a \neq 1$) and 2 (if $a = 1$).
Case $a \neq 1$: Set

$$\boldsymbol{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_a = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{e}_{-a} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}. \tag{56}$$

Then, we have $\mathbb{Q}_{\mu,\omega} = \langle \cdot, \boldsymbol{e}_\mu \rangle \boldsymbol{e}_\mu$ for $\mu \in \{\pm 1, \pm a\}$ and $\mathbb{Q}_{\mu,\omega} = O$ otherwise. Therefore, we also obtain

$$\tilde{V}^{high,0}(\pm 1,\omega) = \{c\boldsymbol{e}_{\pm 1} \,|\, c \in \mathbb{R}\}\,, \qquad \tilde{V}^{high,0}(s_0,\omega) = \{0\} \quad \text{for} \quad s_0 \neq \pm 1\,.$$

Namely, we arrive at $\tilde{\mathcal{S}}_{high,0} = \{(\pm 1,\omega) \,|\, \omega = \pm 1\}$. Next, we observe that

$$\mathbb{Q}_{\pm 1,\omega}L\boldsymbol{e}_{\pm 1} = \mathbb{Q}_{\pm 1,\omega}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = 0\,,$$

which gives $\mathbb{Q}_{\pm1,\omega}L|_{\mathbb{Q}_{\pm1,\omega}\mathbb{C}^4} = O|_{\mathbb{Q}_{\pm1,\omega}\mathbb{C}^4}$. Therefore,

$$V^{high,0}(0,\pm1,\omega) = \tilde{V}^{high,0}(\pm1,\omega), \qquad \mathbb{Q}_{0,\pm1,\omega} = I|_{\mathbb{Q}_{\pm1,\omega}\mathbb{C}^4},$$

$$V^{high,0}(s_1,\pm1,\omega) = \{0\} \quad (s_1 \neq 0),$$

which implies $\mathcal{S}_{high,0} = \{(0,\pm1,\omega) \,|\, \omega = \pm1\}$. Let us check Condition (ii) in $(\mathrm{NDC})_{high,1}$ for $\mathbb{Q}_{\pm1,\omega}$ and $K(\pm1,\omega) = -((\pm I + A(\omega))|_{\mathbb{Q}^{\perp}_{\pm,\omega}\mathbb{C}^4})^{-1}\mathbb{Q}^{\perp}_{\pm1,\omega}$. The direct calculation shows that, for any $\boldsymbol{y} \in \mathbb{Q}_{\pm1,\omega}\mathbb{C}^4$, the vector $K(\pm1,\omega)L^{\flat}\boldsymbol{y}$ is of the form $(0,0,x_3,x_4)^{\top}$, and, thus, $L^{\sharp}K(\pm1,\omega)L^{\flat}\boldsymbol{y}$ is of the form $(0,0,0,\gamma x_4)^{\top}$, Hence, the definition of $\mathbb{Q}_{\pm1,\omega}$ implies $\mathbb{Q}_{\pm1,\omega}L^{\sharp}K(\pm1,\omega)L^{\flat}\boldsymbol{y} = 0$, as desired. Finally, we show $\mathcal{S}^{(1)}_{high,1} = \emptyset$. It suffices to consider the set

$$V^{high,1,(1)}(0,\pm1,\omega) = \mathrm{Ker}\left(L^{\sharp}(\pm I + A(\omega))|^{-1}_{\mathbb{Q}^{\perp}_{\pm1,\omega}\mathbb{C}^4}\mathbb{Q}^{\perp}_{\pm1,\omega}L^*\right) \cap V^{high,0}(0,\pm1,\omega).$$

To this end, suppose that $c\boldsymbol{e}_{\pm1}$ satisfies

$$L^{\sharp}(\pm I + A(\omega))|^{-1}_{\mathbb{Q}^{\perp}_{\pm1,\omega}\mathbb{C}^4}\mathbb{Q}^{\perp}_{\pm1,\omega}L^*c\boldsymbol{e}_{\pm1} = 0.$$

The direct computation shows that

$$(\pm I + A(\omega))|^{-1}_{\mathbb{Q}^{\perp}_{\pm1,\omega}\mathbb{C}^4}\mathbb{Q}^{\perp}_{\pm1,\omega}L^*c\boldsymbol{e}_{\pm1} = \frac{c\tilde{c}}{1-a^2}\begin{pmatrix}0\\0\\a\\\pm1\end{pmatrix}$$

for some nonzero real number $\tilde{c}$. Hence, the condition

$$L^{\sharp}\frac{c\tilde{c}}{1-a^2}\begin{pmatrix}0\\0\\a\\\pm1\end{pmatrix} = 0$$

yields $c = 0$, which implies $V^{high,1,(1)}(0,\pm1,\omega) = \{0\}$. Thus, we prove $\mathcal{S}^{(1)}_{high,1} = \emptyset$, and the condition $(\mathrm{NDC})_{high,1}$ is proved for the case $a \neq 1$.

Case $a = 1$: In this case, we have $\mathbb{Q}_{\pm1,\omega} = \langle\cdot,\boldsymbol{e}_{\pm1}\rangle\boldsymbol{e}_{\pm1} + \langle\cdot,\boldsymbol{e}_{\pm a}\rangle\boldsymbol{e}_{\pm a}$, where $\boldsymbol{e}_{\pm a}$ is defined as (56). As in the case $a \neq 1$, we have

$$\tilde{V}^{high,0}(\pm1,\omega) = \{c\boldsymbol{e}_{\pm1} \,|\, c \in \mathbb{R}\}, \qquad \tilde{V}^{high,0}(s_0,\omega) = \{0\}\ (s_0 \neq \pm1),$$

and, thus, $\tilde{\mathcal{S}}_{high,0} = \{(\pm1,\omega) \,|\, \omega = \pm1\}$. Suppose that $c\boldsymbol{e}_{\pm1}$ satisfies $(is_1 + \mathbb{Q}_{\pm1,\omega}L)c\boldsymbol{e}_{\pm1} = 0$ for some $s_1 \in \mathbb{R}$. Since

$$\mathbb{Q}_{\pm1,\omega}L\boldsymbol{e}_{\pm1} = \mathbb{Q}_{\pm1,\omega}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\-1\end{pmatrix} = \mp\frac{1}{2}\boldsymbol{e}_{\pm a},$$

we must have $ics_1\boldsymbol{e}_{\pm1} \mp c\boldsymbol{e}_{\pm a}/2 = 0$, which is possible only when $c = 0$. Thus, we conclude that $\mathcal{S}_{high,0} = \emptyset$ when $a = 1$, and the condition $(\mathrm{NDC})_{high,0}$ is proved for the case $a = 1$. The proof is complete. □

7.3. Compressible Euler–Maxwell System

As an application of our theorems, we deal with the compressible Euler–Maxwell system

$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = 0, \\ (\rho v)_t + \operatorname{div}(\rho v \otimes v) + \nabla p(\rho) = -\rho(E + v \times B) - \rho v, \\ E_t - \operatorname{rot} B = \rho v, \\ B_t + \operatorname{rot} E = 0, \end{cases} \tag{57}$$

$$\operatorname{div} E = \rho_\infty - \rho, \qquad \operatorname{div} B = 0. \tag{58}$$

Here, the density $\rho > 0$, the velocity $v \in \mathbb{R}^3$, the electric field $E \in \mathbb{R}^3$, and the magnetic induction $B \in \mathbb{R}^3$ are unknown functions of $t > 0$ and $x \in \mathbb{R}^3$. Assume that the pressure $p(\rho)$ is a given smooth function of ρ satisfying $p'(\rho) > 0$ for $\rho > 0$, and ρ_∞ is a positive constant.

From the analysis in [14,15], we know that the system (57) can be written in the form of a symmetric hyperbolic system. The reader is also referred to the works of Ruggeri and Strumia [30] and Boillat [31] for the general result about the symmetrization and the convex entropy of balance laws. Let us introduce that $w = (\rho, v, E, B)^\top$, $w_\infty = (\rho_\infty, 0, 0, B_\infty)^\top$, which are regarded as column vectors in $\mathbb{R}^{10}$, where $B_\infty \in \mathbb{R}^3$ is an arbitrarily fixed constant. Then, the Euler–Maxwell system (57) is rewritten as

$$\tilde{A}_0(w)\partial_t w + \sum_{j=1}^{3} \tilde{A}_j(w)\partial_{x_j} w + \tilde{L}(w)w = 0, \tag{59}$$

where the coefficient matrices are given explicitly as

$$\tilde{A}_0(w) = \begin{pmatrix} p'(\rho)/\rho & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & \rho I_3 & O_3 & O_3 \\ \mathbf{0}^\top & O_3 & I_3 & O_3 \\ \mathbf{0}^\top & O_3 & O_3 & I_3 \end{pmatrix}, \qquad \tilde{L}(w) = \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & \rho(I - \Omega_B) & \rho I_3 & O_3 \\ \mathbf{0}^\top & -\rho I_3 & O_3 & O_3 \\ \mathbf{0}^\top & O_3 & O_3 & O_3 \end{pmatrix},$$

$$\sum_{j=1}^{3} \tilde{A}_j(w)\xi_j = \begin{pmatrix} (p'(\rho)/\rho)(v \cdot \xi) & p'(\rho)\xi & \mathbf{0} & \mathbf{0} \\ p'(\rho)\xi^\top & \rho(v \cdot \xi) I_3 & O_3 & O_3 \\ \mathbf{0}^\top & O_3 & O_3 & -\Omega_\xi \\ \mathbf{0}^\top & O_3 & \Omega_\xi & O_3 \end{pmatrix}.$$

Here, $\mathbf{0} = (0, 0, 0)$, $\xi = (\xi_1, \xi_2, \xi_3)$, $B = (B_1, B_2, B_3) \in \mathbb{R}^3$, I_3 denotes the 3×3 identity matrix, O_3 denotes the 3×3 zero matrix, and Ω_ξ is the skew-symmetric matrix defined by

$$\Omega_\xi = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}$$

for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$.

To consider the linearization of (59) around the equilibrium state w_∞, we regard $w - w_\infty$ by w again. Then, the linearization of the system (59) can be written as

$$\tilde{A}_0\partial_t w + \sum_{j=1}^{3} \tilde{A}_j\partial_{x_j} w + \tilde{L}w = 0, \tag{60}$$

where $\tilde{A}_0 = \tilde{A}_0(w_\infty)$, $\tilde{A}_j = \tilde{A}_j(w_\infty)$ and $\tilde{L} = \tilde{L}(w_\infty)$. More precisely, the coefficient matrices are written as

$$\tilde{A}_0 = \begin{pmatrix} a_\infty & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & \rho_\infty I_3 & O_3 & O_3 \\ \mathbf{0}^\top & O_3 & I_3 & O_3 \\ \mathbf{0}^\top & O_3 & O_3 & I_3 \end{pmatrix}, \qquad \sum_{j=1}^{3} \tilde{A}_j \xi_j = \begin{pmatrix} 0 & b_\infty \xi & \mathbf{0} & \mathbf{0} \\ b_\infty \xi^\top & O_3 & O_3 & O_3 \\ \mathbf{0}^\top & O_3 & O_3 & -\Omega_\xi \\ \mathbf{0}^\top & O_3 & \Omega_\xi & O_3 \end{pmatrix},$$

$$\tilde{L}^\sharp = \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & \rho_\infty I_3 & O_3 & O_3 \\ \mathbf{0}^\top & O_3 & O_3 & O_3 \\ \mathbf{0}^\top & O_3 & O_3 & O_3 \end{pmatrix}, \qquad \tilde{L}^\flat = \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & -\rho_\infty \Omega_{B_\infty} & \rho_\infty I_3 & O_3 \\ \mathbf{0}^\top & -\rho_\infty I_3 & O_3 & O_3 \\ \mathbf{0}^\top & O_3 & O_3 & O_3 \end{pmatrix},$$

where $a_\infty = p'(\rho_\infty)/\rho_\infty$ and $b_\infty = p'(\rho_\infty)$ are positive constants. Furthermore, because $\tilde{A}_0$ is a positive definite, we introduce the new function $u = \tilde{A}_0^{-1/2} w$ and (60) is rewritten as the first equation in (1) with

$$\begin{aligned} &\sum_{j=1}^{3} A_j \xi_j = \begin{pmatrix} 0 & \sqrt{b_\infty}\xi & \mathbf{0} & \mathbf{0} \\ \sqrt{b_\infty}\xi^\top & O_3 & O_3 & O_3 \\ \mathbf{0}^\top & O_3 & O_3 & -\Omega_\xi \\ \mathbf{0}^\top & O_3 & \Omega_\xi & O_3 \end{pmatrix}, \\ &L^\sharp = \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & I_3 & O_3 & O_3 \\ \mathbf{0}^\top & O_3 & O_3 & O_3 \\ \mathbf{0}^\top & O_3 & O_3 & O_3 \end{pmatrix}, \qquad L^\flat = \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & -\Omega_{B_\infty} & \sqrt{\rho_\infty} I_3 & O_3 \\ \mathbf{0}^\top & -\sqrt{\rho_\infty} I_3 & O_3 & O_3 \\ \mathbf{0}^\top & O_3 & O_3 & O_3 \end{pmatrix}. \end{aligned} \tag{61}$$

Next, we consider the constraint condition which comes from (58). Since (58), the solution to the linearized system (60) is considered under

$$\sum_{j=1}^{3} Q_j \partial_{x_j} w + \tilde{R} w = 0,$$

where

$$\sum_{j=1}^{3} Q_j \xi_j = \begin{pmatrix} 0 & \mathbf{0} & \xi & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} & \xi \end{pmatrix}, \qquad \tilde{R} = \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Thus, this gives

$$\sum_{j=1}^{3} Q_j \partial_{x_j} u + R u = 0, \tag{62}$$

with

$$R = \begin{pmatrix} 1/\sqrt{a_\infty} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Inspired by the condition (62), we introduce the closed subspace of $\mathbb{C}^{10}$, for $\omega \in S^2$, that

$$\begin{aligned} X_\omega &= \{ \boldsymbol{y} = (y_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \boldsymbol{y}_4) \in \mathbb{C}^{10} \,|\, \boldsymbol{y}_4 \cdot \omega = 0,\ y_1 \in \mathbb{C},\ \boldsymbol{y}_2, \boldsymbol{y}_3, \boldsymbol{y}_4 \in \mathbb{C}^3 \}, \\ X_{\omega,r} &= \{ \boldsymbol{y} \in X_\omega \,|\, y_1/\sqrt{a_\infty} - ir\boldsymbol{y}_3 \cdot \omega = 0 \}. \end{aligned}$$

The limit spaces are given as

$$X_{\omega,0} = \{ \boldsymbol{y} \in X_\omega | y_1 = 0 \}, \qquad X_{\omega,\infty} = \{ \boldsymbol{y} \in X_\omega | \boldsymbol{y}_3 \cdot \omega = 0 \}.$$

In this situation, we consider the system (2) under the constraint

$$\hat{u}(t,\xi) \in X_{\xi/|\xi|,|\xi|}\,, \qquad t \geq 0\,.$$

It is easy to see that the invariant condition (Inv) and the general stability condition (SC) hold true. Therefore, we can apply our theorems to the linearized Euler–Maxwell system and derive the pointwise estimate of solutions and optimality of its estimate as follows.

Proposition 3 (Linearized compressible Euler–Maxwell system)**.** *For any* $\hat{f} \in X_{\xi/|\xi|,|\xi|}$ *and* $\xi \in \mathbb{R}^3\backslash\{0\}$*, the solution operator to the system* (2) *with* (61) *satisfies*

$$|e^{-t\mathcal{A}(\xi)}\hat{f}| \leq Ce^{-\frac{c|\xi|^2}{1+|\xi|^4}t}|\hat{f}|\,.$$

Proof. (Low frequency part) The eigenvalues of $iL^\flat$ are $0, \pm\mu_*$ with some $\mu_* > 0$. Moreover, we have $\mathrm{Ran}(\mathbb{P}_{\mu,\omega}) = \{0\}$ for any $\mu \neq 0$, and

$$\mathbb{D}_{\sharp,\omega}\boldsymbol{y} = \begin{pmatrix} y_1 \\ \mathbf{0}^\top \\ \boldsymbol{y}_3^\top \\ \boldsymbol{y}_4^\top \end{pmatrix}, \qquad \mathbb{P}_{0,\omega}\boldsymbol{y} = \begin{pmatrix} y_1 \\ \mathbf{0}^\top \\ \mathbf{0}^\top \\ \boldsymbol{y}_4^\top \end{pmatrix}. \tag{63}$$

Therefore, we obtain $V^{low,0}(0,\omega) \neq \{0\}$ and $V^{low,0}(\mu,\omega) = \{0\}$ for $\mu \neq 0$. Furthermore, (63) gives $\mathbb{P}_{0,\omega}A(\omega)|_{\mathbb{P}_{0,\omega}X_\omega} = O$. Thus, the associated eigenprojection of the eigenvalue 0 is the identity map, i.e.,

$$\mathrm{Ran}\,(\mathbb{P}_{s_1,0,\omega}) = \begin{cases} \{0\} & (s_1 \neq 0)\,, \\ \mathbb{P}_{0,\omega}X_\omega & (s_1 = 0)\,. \end{cases}$$

Hence, $\tilde{V}^{low,1}(\mu,0,\omega) = \{0\}$ for $\mu \neq 0$, and the direct computation shows

$$\begin{aligned} \tilde{V}^{low,1}(0,0,\omega) &= \mathrm{Ran}\,(\mathbb{P}_{0,0,\omega}) \cap \mathrm{Ker}\left(L(\omega)^\sharp\big(L(\omega)\big)|^{-1}_{\mathbb{P}^\perp_{0,\omega}X_\omega}\mathbb{P}^\perp_{0,\omega}A(\omega)|_{X_\omega}\right) \cap X_{\omega,0} \\ &= \left\{ \begin{pmatrix} 0 \\ \mathbf{0}^\top \\ \mathbf{0}^\top \\ \boldsymbol{y}_4^\top \end{pmatrix} \in \mathbb{C}^{10} \,|\, \omega^\top \times \boldsymbol{y}_4^\top = 0 \right\} = \{0\}\,. \end{aligned}$$

Here, we use $\omega \in S^2$ and $\boldsymbol{y}_4 \cdot \omega = 0$ for $\boldsymbol{y} \in X_{\omega,0}$. Hence, $\tilde{\mathcal{S}}_{low,1} = \emptyset$, and Condition (i) of $(\mathrm{NDC})_{low,1}$ holds. The proof is complete for the low frequency part.

(High frequency part) The eigenvalues of $-A(\omega)|_{X_\omega}$ are $0, \pm 1, \pm\sqrt{b_\infty}$. By taking $\boldsymbol{h}_1, \boldsymbol{h}_2 = \omega^\top \times \boldsymbol{h}_1 \in \mathbb{R}^3$ (column vectors) as the orthonormal basis of the plane $\{\boldsymbol{x}^\top \in \mathbb{R}^3 \,|\, \boldsymbol{x} \cdot \omega = 0\}$, we have for $\sqrt{b_\infty} \neq 1$,

$$\mathbb{Q}_{0,\omega}\boldsymbol{y} = (\boldsymbol{y}_2 \cdot \boldsymbol{h}_1)\begin{pmatrix} 0 \\ \boldsymbol{h}_1 \\ \boldsymbol{0}^\top \\ \boldsymbol{0}^\top \end{pmatrix} + (\boldsymbol{y}_2 \cdot \boldsymbol{h}_2)\begin{pmatrix} 0 \\ \boldsymbol{h}_2 \\ \boldsymbol{0}^\top \\ \boldsymbol{0}^\top \end{pmatrix} + (\boldsymbol{y}_3 \cdot \omega)\begin{pmatrix} 0 \\ \boldsymbol{0}^\top \\ \omega^\top \\ \boldsymbol{0}^\top \end{pmatrix},$$

$$\mathbb{Q}_{\sqrt{b_\infty},\omega}\boldsymbol{y} = \frac{y_1 - (\boldsymbol{y}_2 \cdot \omega)}{2}\begin{pmatrix} 1 \\ -\omega^\top \\ \boldsymbol{0}^\top \\ \boldsymbol{0}^\top \end{pmatrix}, \qquad \mathbb{Q}_{-\sqrt{b_\infty},\omega}\boldsymbol{y} = \frac{y_1 + (\boldsymbol{y}_2 \cdot \omega)}{2}\begin{pmatrix} 1 \\ \omega^\top \\ \boldsymbol{0}^\top \\ \boldsymbol{0}^\top \end{pmatrix},$$

$$\mathbb{Q}_{1,\omega}\boldsymbol{y} = \langle \boldsymbol{y}, \boldsymbol{g}_{1,1}\rangle \boldsymbol{g}_{1,1} + \langle \boldsymbol{y}, \boldsymbol{g}_{1,2}\rangle \boldsymbol{g}_{1,2}\,, \qquad \mathbb{Q}_{-1,\omega}\boldsymbol{y} = \langle \boldsymbol{y}, \boldsymbol{g}_{-1,1}\rangle \boldsymbol{g}_{-1,1} + \langle \boldsymbol{y}, \boldsymbol{g}_{-1,2}\rangle \boldsymbol{g}_{-1,2}\,.$$

Here,

$$\boldsymbol{g}_{1,j} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \boldsymbol{0}^\top \\ \omega^\top \times \boldsymbol{h}_j \\ \boldsymbol{h}_j \end{pmatrix}, \qquad \boldsymbol{g}_{-1,j} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \boldsymbol{0}^\top \\ -\omega^\top \times \boldsymbol{h}_j \\ \boldsymbol{h}_j \end{pmatrix}.$$

When $\sqrt{b_\infty} = 1$, the orthogonal eigenprojection to the eigenvalue ± 1 is $\tilde{\mathbb{Q}}_{\pm 1,\omega} := \mathbb{Q}_{\pm 1,\omega} + \mathbb{Q}_{\pm\sqrt{b_\infty},\omega}$ by keeping the same notation as above. Then, we can check that for $s_1 \in \mathbb{R}$,

$$\begin{aligned}\mathrm{Ker}\left((is_1 I + \mathbb{Q}_{0,\omega}L)|_{\mathbb{Q}_{0,\omega}X_\omega}\right) \cap \mathrm{Ker}\left(L^\sharp|_{X_\omega}\right) &= \mathrm{Ker}\left((s_1 I - \mathbb{Q}_{0,\omega} i L^\flat)|_{\mathbb{Q}_{0,\omega}X_\omega}\right) \cap \mathrm{Ker}\left(L^\sharp|_{X_\omega}\right) \\ &= \begin{cases} \left\{ c\begin{pmatrix} 0 \\ \boldsymbol{0}^\top \\ \omega^\top \\ \boldsymbol{0}^\top \end{pmatrix} \,|\, c \in \mathbb{C}\right\} & (s_1 = 0)\,, \\ \{0\} & (s_1 \neq 0)\,. \end{cases}\end{aligned}$$

Hence, $\mathrm{Ker}\left((is_1 I + \mathbb{Q}_{0,\omega}L)|_{\mathbb{Q}_{0,\omega}X_\omega}\right) \cap \mathrm{Ker}\left(L^\sharp|_{X_\omega}\right) \cap X_{\omega,\infty} = \{0\}$ for any $s_1 \in \mathbb{R}$. It is also easy to see that $\mathrm{Ker}\left((is_1 I + \mathbb{Q}_{\pm a_1,\omega}L)|_{\mathbb{Q}_{\pm a_1,\omega}X_\omega}\right) \cap \mathrm{Ker}\left(L^\sharp|_{X_\omega}\right) \cap X_{\omega,\infty} = \{0\}$ for any $s_1 \in \mathbb{R}$. Next, we observe that $\mathbb{Q}_{\pm 1,\omega}L|_{\mathbb{Q}_{\pm 1,\omega}X_\omega} = O$ and $\mathbb{Q}_{\pm 1,\omega}X_\omega \subset X_{\omega,\infty}$, and, therefore, for $s_1 \in \mathbb{R}$ and $\sqrt{b_\infty} \neq 1$,

$$\begin{aligned} V^{high,0}(s_1, \pm 1, \omega) &= \mathrm{Ker}\left((is_1 I + \mathbb{Q}_{\pm 1,\omega}L)|_{\mathbb{Q}_{\pm 1,\omega}X_\omega}\right) \cap \mathrm{Ker}\left(L^\sharp|_{X_\omega}\right) \cap X_{\omega,\infty} \\ &= \mathrm{Ker}\left((s_1 - \mathbb{Q}_{\pm 1,\omega} i L^\flat)|_{\mathbb{Q}_{\pm 1,\omega}X_\omega}\right) \cap \mathrm{Ker}\left(L^\sharp|_{X_\omega}\right) \\ &= \begin{cases} \mathbb{Q}_{\pm 1,\omega}X_\omega & (s_1 = 0)\,, \\ \{0\} & (s_1 \neq 0)\,. \end{cases}\end{aligned}$$

A similar result is valid when $\sqrt{b_\infty} = 1$: in this case, we have

$$\begin{aligned} V^{high,0}(s_1, \pm 1, \omega) &= \mathrm{Ker}\left((is_1 I + \tilde{\mathbb{Q}}_{\pm 1,\omega}L)|_{\tilde{\mathbb{Q}}_{\pm 1,\omega}X_\omega}\right) \cap \mathrm{Ker}\left(L^\sharp|_{X_\omega}\right) \cap X_{\omega,\infty} \\ &= \begin{cases} \mathbb{Q}_{\pm 1,\omega}X_\omega & (s_1 = 0)\,, \\ \{0\} & (s_1 \neq 0)\,. \end{cases}\end{aligned}$$

Notice that, even in the case $\sqrt{b_\infty} = 1$, the projection in the right-hand side is $\mathbb{Q}_{\pm 1,\omega}$ defined as above, rather than $\tilde{\mathbb{Q}}_{\pm 1,\omega}$, due to the presence of $\mathrm{Ker}\,(L^\sharp|_{X_\omega})$ in the left-hand side. This implies $\mathcal{S}_{high,0} = \{(0,\pm 1,\omega) | \omega \in S^{n-1}\}$ for any $\sqrt{b_\infty} > 0$. Next, we find

$$\mathbb{Q}_{\pm 1,\omega'} L(\omega')^\sharp K(\pm 1,\omega') L(\omega')^\flat|_{\mathbb{Q}_{\pm 1,\omega'} X_{\omega'}} = O\,,$$

where $K(\pm 1,\omega') = -\big((\pm I + A(\omega'))|_{\mathbb{Q}^\perp_{\pm 1,\omega'} X_{\omega'}}\big)^{-1} \mathbb{Q}^\perp_{\pm 1,\omega'}$. Indeed, this follows from the stronger cancellation property

$$\mathbb{Q}_{\pm 1,\omega'} L(\omega')^\sharp \mathbb{Q}^\perp_{\pm 1,\omega'} = O\,, \tag{64}$$

which is straightforward to check from the definition of the projections (including the case $a_1 = 1$, where $\mathbb{Q}_{\pm 1,\omega'}$ in (64) is replaced by $\tilde{\mathbb{Q}}_{\pm 1,\omega} = \mathbb{Q}_{\pm 1,\omega'} + \mathbb{Q}_{\pm a_1,\omega'}$). Suppose that $\boldsymbol{y} \in V^{high,0}(0,\pm 1,\omega) = \mathbb{Q}_{\pm 1,\omega} X_\omega$ satisfies $L(\omega)^\sharp (\pm I + A(\omega))|^{-1}_{\mathbb{Q}^\perp_{\pm 1,\omega} X_\omega} \mathbb{Q}^\perp_{\pm 1,\omega} L(\omega)^* \boldsymbol{y} = 0$. Set

$$\boldsymbol{x} = (\pm I + A(\omega))|^{-1}_{\mathbb{Q}^\perp_{\pm 1,\omega} X_\omega} \mathbb{Q}^\perp_{\pm 1,\omega} L(\omega)^* \boldsymbol{y} \in \mathbb{Q}^\perp_{\pm 1,\omega} X_\omega\,.$$

Then, (64) implies $\mathbb{Q}^\perp_{\pm 1,\omega} L^\sharp \boldsymbol{x} = 0$, and one can check that for $\sqrt{b_\infty} \neq 1$,

$$\mathrm{Ker}\,(\mathbb{Q}^\perp_{\pm 1,\omega} L^\sharp|_{\mathbb{Q}^\perp_{\pm 1,\omega} X_\omega}) = \{c \begin{pmatrix} 0 \\ \mathbf{0}^\top \\ \omega^\top \\ \mathbf{0}^\top \end{pmatrix} \mid c \in \mathbb{C}\} \oplus \mathbb{Q}_{\mp 1,\omega} X_\omega\,.$$

When $a_1 = 1$, the equality is valid as well by replacing only the left-hand side by

$$\mathrm{Ker}\,(\tilde{\mathbb{Q}}^\perp_{\pm 1,\omega} L^\sharp|_{\tilde{\mathbb{Q}}^\perp_{\pm 1,\omega} X_\omega})\,.$$

On the other hand, when $\boldsymbol{y} = c_1 \boldsymbol{g}_{\pm 1,1} + c_2 \boldsymbol{g}_{\pm 1,2}$, the vector $\boldsymbol{x}$ is of the form

$$cc_1 \begin{pmatrix} 0 \\ \omega^\top \times \boldsymbol{h}_1 \\ \mathbf{0}^\top \\ \mathbf{0}^\top \end{pmatrix} + c'c_2 \begin{pmatrix} 0 \\ \omega^\top \times \boldsymbol{h}_2 \\ \mathbf{0}^\top \\ \mathbf{0}^\top \end{pmatrix}$$

with some nonzero constants c, c', and, thus, in order for $\boldsymbol{x}$ to belong to $\mathrm{Ker}\,(\mathbb{Q}^\perp_{\pm 1,\omega} L^\sharp|_{\mathbb{Q}^\perp_{\pm 1,\omega} X_\omega})$ we must have $c_1 = c_2 = 0$ since $\boldsymbol{h}_2 = \omega^\top \times \boldsymbol{h}_1$. Thus, we have $V^{high,1,(1)}(0,\pm 1,\omega) = \{0\}$, that is, $\mathcal{S}^{(1)}_{high,1} = \emptyset$. Hence, the condition $(\mathrm{NDC})_{high,1}$ holds. The proof is complete for the high frequency part. □

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Appendix A. Gearhart-Prüss Type Theorem

Let X be a Hilbert space and let $A : D(A) \to X$ be a densely defined closed operator in X with the domain $D(A) \subset X$. The operator A is called m-accretive if the left open half-plane is contained in the resolvent set $\rho(A)$ with $\|(\lambda I + A)^{-1}\|_{X\to X} \le 1/\Re\lambda$ for $\lambda \in \mathbb{C}$ with $\Re\lambda > 0$. We denote by $\Psi(A)$ the pseudospectral bound of A:

$$\Psi(A) = \Big(\sup_{\lambda\in\mathbb{R}} \|(i\lambda I + A)^{-1}\|_{X\to X}\Big)^{-1}.$$

Theorem A1 (Wei [25]). *Let A be an m-accretive operator in a Hilbert space X. Then*

$$\|e^{-tA}\|_{X\to X} \le e^{-t\Psi(A)+\pi/2}, \qquad t > 0\,.$$

Appendix B. Basic Lemma for Matrix with Nonnegative Definite

Lemma A1. *Let X be a subspace of $\mathbb{C}^m$ and let $\mathcal{P}_X : \mathbb{C}^m \to X$ be the orthogonal projection. Assume that the $m \times m$ matrix M satisfies $M^\sharp \ge 0$ on $\mathbb{C}^m$. Let $\mu \in \mathbb{R}$.*
(i) *It follows that*

$$\mathrm{Ker}\,\Big((i\mu I + \mathcal{P}_X M)|_X\Big) = \mathrm{Ker}\,\Big((\mu I - i\mathcal{P}_X M^\flat)|_X\Big) \cap \mathrm{Ker}\,(M^\sharp)\,.$$

(ii) *Let $\mathbb{P}_\mu : X \to \mathrm{Ker}\,\Big((i\mu I + \mathcal{P}_X M)|_X\Big)$ be the orthogonal projection from X to $\mathrm{Ker}\,\Big((i\mu I + \mathcal{P}_X M)|_X\Big)$. Then, $\mathbb{P}_\mu(i\mu I + \mathcal{P}_X M)|_X = O$. As a consequence, the restriction $(i\mu I + \mathcal{P}_X M)|_{\mathbb{P}_\mu^\perp X} : \mathbb{P}_\mu^\perp X \to \mathbb{P}_\mu^\perp X$, where $\mathbb{P}_\mu^\perp = I|_X - \mathbb{P}_\mu$, is well-defined and invertible.*

Proof. (i) Let $u \in \mathrm{Ker}\,((i\mu I + \mathcal{P}_X M)|_X)$. Then, $\langle (i\mu + \mathcal{P}_X M)u, u\rangle = 0$. Taking the real part and using $\mathcal{P}_X u = u$, we have $\langle M^\sharp u, u\rangle = 0$, and, thus, $\langle M^\sharp|_{\mathbb{D}^\perp_{\sharp,M}} \mathbb{D}^\perp_{\sharp,M} u, \mathbb{D}^\perp_{\sharp,M} u\rangle = 0$, where $\mathbb{D}_{\sharp,M} : \mathbb{C}^m \to \mathrm{Ker}\,(M^\sharp)$ is the orthogonal projection and $\mathbb{D}^\perp_{\sharp,M} = I - \mathbb{D}_{\sharp,M}$. Since $M^\sharp \ge 0$ and $M^\sharp|_{\mathbb{D}^\perp_{\sharp,M}\mathbb{C}^m} : \mathbb{D}^\perp_{\sharp,M}\mathbb{C}^m \to \mathbb{D}^\perp_{\sharp,M}\mathbb{C}^m$ is invertible, we have $M^\sharp|_{\mathbb{D}^\perp_{\sharp,M}\mathbb{C}^m} > 0$ in $\mathbb{D}^\perp_{\sharp,M}\mathbb{C}^m$. Thus, $\mathbb{D}^\perp_{\sharp,M} u = 0$, which yields $M^\sharp u = M^\sharp(\mathbb{D}^\perp_{\sharp,M} u + \mathbb{D}_{\sharp,M} u) = 0$. Then, we have $(\mu - i\mathcal{P}_X M^\flat)u = -i(i\mu + \mathcal{P}_X M^\flat)u = -i(i\mu + \mathcal{P}_X M^\flat + \mathcal{P}_X M^\sharp)u = 0$. Hence, $\mathrm{Ker}\,((i\mu I + \mathcal{P}_X M)|_X) \subset \mathrm{Ker}\,((\mu I - i\mathcal{P}_X M^\flat)|_X) \cap \mathrm{Ker}\,(M^\sharp)$. The converse inclusion is trivial.
(ii) For any $u, v \in X$ we have

$$\begin{aligned}\langle \mathbb{P}_\mu(i\mu I + M)u, v\rangle &= \langle (i\mu I + M)u, \mathbb{P}_\mu v\rangle \\ &= i\langle (\mu I - iM^\flat)u, \mathbb{P}_\mu v\rangle + \langle M^\sharp u, \mathbb{P}_\mu v\rangle \\ &= i\langle u, (\mu I - iM^\flat)\mathbb{P}_\mu v\rangle + \langle u, M^\sharp \mathbb{P}_\mu v\rangle = 0\,.\end{aligned}$$

Here, we use (i). The proof is complete. □

Remark A1. *From Lemma A1 we have the following important result: if $M^\sharp \ge 0$ then each eigenvalue of M located in $i\mathbb{R}$ (i.e., purely imaginary eigenvalue) must be semisimple, and furthermore, the associated eigenprojection is orthogonal.*

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