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Neighbor Sum Distinguishing Total Choosability of IC-Planar Graphs without Theta Graphs $\Theta_{2,1,2}$

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Abstract: A theta graph $\Theta_{2,1,2}$ is a graph obtained by joining two vertices by three internally disjoint paths of lengths 2, 1, and 2. A neighbor sum distinguishing (NSD) total coloring ϕ of G is a proper total coloring of G such that $\sum_{z\in E_G(u)\cup\{u\}}\phi(z) \neq \sum_{z\in E_G(v)\cup\{v\}}\phi(z)$ for each edge $uv \in E(G)$, where $E_G(u)$ denotes the set of edges incident with a vertex u. In 2015, Pilśniak and Woźniak introduced this coloring and conjectured that every graph with maximum degree Δ admits an NSD total $(\Delta+3)$-coloring. In this paper, we show that the listing version of this conjecture holds for any IC-planar graph with maximum degree $\Delta \geq 9$ but without theta graphs $\Theta_{2,1,2}$ by applying the Combinatorial Nullstellensatz, which improves the result of Song et al.

Keywords: IC-planar graphs; neighbor sum distinguishing total choosibility; Combinatorial Nullstellensatz

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1. Introduction

The graphs mentioned in this paper are finite, undirected, and simple. For undefined terminology and notations, here we follow [1]. Let $G = (V(G), E(G))$ be a simple graph. For a vertex $u \in V(G)$, let $E_G(u)$ denote the set of edges incident with u, and we use $d_G(u)$ and $N_G(u)$ to represent the degree and the neighborhood of u, respectively. Let $\Delta(G)$ (or Δ) and $\delta(G)$ (or δ) denote the maximum degree and the minimum degree of G, respectively. A t-cycle (t^+-cycle, t^--cycle) is a cycle of length t (at least t, at most t). In particular, a 3-cycle with vertex set $\{v_1, v_2, v_3\}$ is called a $(d_G(v_1), d_G(v_2), d_G(v_3))$-cycle. A theta graph Θ_{t_1,t_2,t_3} is a graph obtained by joining two vertices by three internally disjoint paths of lengths t_1, t_2 and t_3.

Let k be a positive integer and $T(G) = V(G) \cup E(G)$. A mapping $\phi : T(G) \to \{1, \cdots, k\}$ is called a *proper k-total coloring* of G if $\phi(z_1) \neq \phi(z_2)$ for any two adjacent or incident elements z_1, z_2 in $T(G)$. A proper k-total coloring ϕ of G is *neighbor sum distinguishing* (for short, *NSD*) if for each edge $uv \in E(G)$,

$$\sum_{z\in E_G(u)\cup\{u\}}\phi(z) \neq \sum_{z\in E_G(v)\cup\{v\}}\phi(z).$$

The *NSD total chromatic number* of G, denoted by $\chi^t_\Sigma(G)$, is the smallest integer k such that G has an NSD k-total coloring. Pilśniak and Woźniak [2] posed an important conjecture in the following.

Conjecture 1 ([2]). *For a graph G, $\chi^t_\Sigma(G) \leq \Delta(G) + 3$.*

Pilśniak and Woźniak [2] showed that Conjecture 1 holds for complete graphs, bipartite graphs, cubic graphs, and 2-degenerate graphs with maximum degree $\Delta \leq 3$. Yao et al. [3] confirmed this conjecture for any 2-degenerate graph. Li et al. [4] verified the conjecture for K_4-minor free graphs. Yang et al. [5] proved that any planar graph with maximum degree $\Delta = 10$ satisfies this conjecture.

An *IC*-planar graph, introduced by Alberson [6] in 2008, is a graph that can be embedded in a plane such that each edge is crossed at most one other edge and two pairs of crossing edges share no common end vertex, i.e., two distinct crossings are independent. There are also many results about the NSD total coloring of IC-planar as follows.

Theorem 1 ([7–9]). *Conjecture 1 holds for the following families of IC-planar graphs.*
(1) Any IC-planar graph with maximum degree $\Delta \geq 13$.
(2) Every IC-planar graph with maximum degree $\Delta \geq 7$ *but without* 3*-cycles.*
(3) Any IC-planar graph with maximum degree $\Delta \geq 10$ *but without theta graphs* $\Theta_{2,1,2}$.

A *k-list total assignment* of G is a mapping L that assigns to each member $z \in T(G)$ a set $L(z)$ of k real numbers. For a list total assignment L of G, a mapping ϕ is called an NSD *total L-coloring* of G if ϕ is an NSD total coloring of G and $\phi(z) \in L(z)$ for each $z \in T(G)$. The smallest integer k such that G has an NSD total L-coloring for any k-list total assignment L is called the NSD *total choice number* of G, denoted by $\mathrm{ch}^t_\Sigma(G)$. Clearly, $\chi^t_\Sigma(G) \leq \mathrm{ch}^t_\Sigma(G)$.

There are also many results about the list version of Conjecture 1 in the following.

Conjecture 2 ([2]). *For a graph* G, $\mathrm{ch}^t_\Sigma(G) \leq \Delta(G) + 3$.

Obviously, Conjecture 2 implied Conjecture 1. Qu et al. [10] proved that this conjecture holds for any planar graph G with maximum degree $\Delta(G) \geq 13$. Wang et al. [11] confirmed Conjecture 2 for every planar graph G with maximum degree $\Delta(G) \geq 8$ but without theta graphs $\Theta_{2,1,2}$. Song et al. [12] discussed the NSD total L-coloring of any IC-planar graph and obtained the following theorem.

Theorem 2 ([12]). *Let* G *be an IC-planar graph. Then,* $\mathrm{ch}^t_\Sigma(G) \leq \max\{\Delta(G) + 3, 17\}$.

In this paper, we reduce the condition $\Delta \geq 10$ of (3) in Theorem 1 to $\Delta \geq 9$ and obtain the list version result as follows.

Theorem 3. *Let* G *be an IC-planar graph without theta graphs* $\Theta_{2,1,2}$. *Then,*

$$\mathrm{ch}^t_\Sigma(G) \leq \max\{\Delta(G) + 3, 12\}.$$

2. Preliminaries

Let G be a simple graph. An ℓ-vertex (ℓ^+-vertex, ℓ^--vertex) u of G is a vertex with $d_G(u) = \ell$ ($d_G(u) \geq \ell$, $d_G(u) \leq \ell$). We use $n^\ell_G(u)$ ($n^{\ell^+}_G(u)$, $n^{\ell^-}_G(u)$) to denote the number of ℓ-vertices (ℓ^+-vertices, ℓ^--vertices) adjacent to u.

In 1999, Alon developed a general algebraic technique that is called Combinatorial Nullstellensatz. It has numerous applications in additive number theory, combinatorics, and graph coloring problems. In the paper, we also use Combinatorial Nullstellensatz in the following to discuss the local structure of minimal counterexample to Theorem 3.

Lemma 1 ([13]). *Let* $\mathbb{F}$ *be an arbitrary field and* $P \in \mathbb{F}[x_1, \ldots, x_n]$ *with degree* $deg(P) = \sum_{k=1}^n i_k$, *where each* i_k *is a natural integer. If the coefficient* $c_P(x_1^{i_1} \cdots x_n^{i_n})$ *of the monomial* $x_1^{i_1} \ldots x_n^{i_n}$ *in* P *is nonzero, and if* $S_1, \ldots, S_n$ *are subsets of* $\mathbb{F}$ *with* $|S_k| > i_k$, *then there are* $s_1 \in S_1, \ldots, s_n \in S_n$ *such that* $P(s_1, \ldots, s_n) \neq 0$.

Let $t \geq 2$ be a positive integer and $S_1, \cdots, S_t$ be t finite sets of real numbers. Define

$$S = S_1 \oplus S_2 \oplus \cdots \oplus S_t = \{ \sum_{i=1}^{t} s_i \mid s_i \in S_i \text{ and } s_i \neq s_j \text{ for } 1 \leq i < j \leq t \}.$$

Lemma 2 ([10]). *Let $t \geq 2$ be a positive integer and $S_1, \cdots, S_t$ be t finite sets of real numbers, where $|S_i| = n_i$ and $n_1 \geq \cdots \geq n_t$. Define $n'_1, \cdots, n'_t$ by*

$$n'_1 = n_1 \text{ and } n'_i = \min\{n'_{i-1} - 1, n_i\}, \text{ for } 2 \leq i \leq t.$$

If $n'_i > 0$ for $1 \leq i \leq t$, then

$$|S| \geq \sum_{i=1}^{t} n'_i - \frac{1}{2}t(t+1) + 1.$$

3. Proof of Theorem 3

Let G be a counterexample to Theorem 3 with $|E(G)|$ being minimal and $k = \max\{\Delta(G) + 3, 12\}$. For any k-list total assignment L, every subgraph G' of G has an NSD total L-coloring ϕ' by the minimality of G. In the following, we will extend the NSD total L-coloring ϕ' of G' to an NSD total L-coloring ϕ of G to obtain a contradiction. As G' is a subgraph of G, we have $T(G) \cap T(G') = T(G')$. Not stated otherwise, $\phi(z) = \phi'(z)$ for any $z \in T(G')$. Let $m(u) = \sum_{z \in E_G(u) \cup \{u\}} \phi(z)$. For the coloring ϕ', the definition of $m'(u)$ is the same as $m(u)$. Let X be a subset of $T(G)$ and $\theta : X \to \mathbb{R}$ be a mapping. For each $z \in T(G)$, let $S(z) = L(z) \setminus \{\theta(x) \mid x \in X \text{ and } x \text{ is adjacent or incident to } z \text{ in } G\}$.

Assume that u is a 3^--vertex of G. For any k-list total assignment L of G, let a map $\phi' : T(G) \setminus \{u\} \to \mathbb{R}$ satisfy the following three conditions:

(i) $\phi'(z) \in L(z)$ for every $z \in T(G) \setminus \{u\}$;
(ii) $\phi'(z_1) \neq \phi'(z_2)$ for any two adjacent or incident members $z_1, z_2 \in T(G) \setminus \{u\}$;
(iii) $\sum_{z \in E_G(v_1) \cup \{v_1\}} \phi'(z) \neq \sum_{z \in E_G(v_2) \cup \{v_2\}} \phi'(z)$ for any two adjacent vertices $v_1, v_2 \in V(G) \setminus \{u\}$.

As $|S(u)| \geq k - 2d_G(u) > d_G(u) + 1$ follows from the assumption that $k \geq 12$ and $d_G(u) \leq 3$, there is a color in $S(u)$ to color u such that the resulting coloring ϕ obtained from ϕ' satisfies $m(u) \neq m'(z)$ for each $z \in N_G(u)$ and $\phi(u) \neq \phi'(z)$ for each $z \in N_G(u) \cup E_G(u)$. Therefore, ϕ satisfies the definition of NSD total L-coloring. Therefore, ϕ' can be extend to an NSD total L-coloring ϕ of G. Thus, for simplicity, we will omit the colors of all 3^--vertices in the following.

Theorem 3 follows from Theorem 2 if $\Delta(G) > 13$. Thus, the following Claim 1 is immediate.

Claim 1. $\Delta(G) \leq 13$.

Claim 2. *There is no edge uv in G with $d_G(u) \leq 5$ and $d_G(v) \leq 4$.*

Proof. By contradiction, suppose that there is an edge uv in G with $d_G(u) \leq 5$ and $d_G(v) \leq 4$. Without loss of generality, set $d_G(u) = 5$ and $d_G(v) = 4$. Let $G' = G - uv$. Then, G' has an NSD total L-coloring ϕ'. In order to extend the coloring ϕ' to an NSD total L-coloring ϕ of G, we erase the colors on u and v. Then,

$$\begin{aligned} |S(u)| &\geq k - 2(d_G(u) - 1) \geq 4, \\ |S(uv)| &\geq k - (d_G(u) - 1) - (d_G(v) - 1) \geq 5, \text{ and} \\ |S(v)| &\geq k - 2(d_G(v) - 1) \geq 6. \end{aligned}$$

Assign a variable x_1 to u, a variable y_1 to uv and a variable x_2 to v, respectively. Let

$$\begin{aligned} P(x_1, y_1, x_2) \quad = \quad & (x_1 - y_1)(x_1 - x_2)(x_2 - y_1)\big(m(u) - m(v)\big) \\ & \cdot \prod_{z \in N_G(u) \setminus \{v\}} \big(m(u) - m'(z)\big) \prod_{z \in N_G(v) \setminus \{u\}} \big(m(v) - m'(z)\big), \end{aligned}$$

where $m(u) = x_1 + y_1 + m'(u) - \phi'(u)$ and $m(v) = x_2 + y_1 + m'(v) - \phi'(v)$. By Appendix A, we know that $c_P(x_1^3 y_1^4 x_2^4) = 5$. By Lemma 1, there are $s_1 \in S(u), s_2 \in S(uv)$ and $s_3 \in S(v)$ such that $P(s_1, s_2, s_3) \neq 0$. By the definitions of P and NSD total L-coloring, we can extend ϕ' to an NSD total L-coloring ϕ of G by recoloring u and v with colors s_1, s_3 and coloring uv with color s_2. It is a contradiction. □

Claim 3. *There is no edge uv in G with $d_G(u) \leq 6$ and $d_G(v) \leq 2$.*

Proof. Suppose to be contrary that there is an edge uv in G with $d_G(u) \leq 6$ and $d_G(v) \leq 2$. Without loss of generality, set $d_G(u) = 6$. Let $G' = G - uv$. Then, G' has an NSD total L-coloring ϕ'. In order to extend the coloring ϕ' to an NSD total L-coloring ϕ of G, we erase the colors on u and v. Note that v is a 2^--vertex. The color of v can be omitted when we extend ϕ' to an NSD total L-coloring ϕ of G. Then,

$$|S(u)| \geq k - 2(d_G(u) - 1) \geq 2, \text{ and}$$
$$|S(uv)| \geq k - (d_G(u) - 1) - (d_G(v) - 1) \geq 6.$$

By Lemma 2, we have that

$$|S(u) \oplus S(uv)| \geq 2 + 6 - \frac{1}{2} \times 2 \times 3 + 1 > d_G(u) - 1.$$

Thus, there is a color in $S(u)$ to color u and a color in $S(uv)$ to color uv such that the resulting color ϕ obtain from ϕ' satisfies $m(u) \neq m'(z)$ for each $z \in N_G(u) \setminus \{v\}$. Therefore, we can extend ϕ' to an NSD total L-coloring ϕ of G. It is a contradiction. □

The proofs of the following Claims 4 and 5 are similar to the proof of Claim 2. To avoid duplication, we omit the proofs.

Claim 4. *Let u be a 6-vertex in G. Then, $n_G^3(u) \leq 1$; furthermore, $n_G^{5^-}(u) = 1$ when $n_G^3(u) = 1$.*

Claim 5. *There is no $(5^-, 5^-, 5^-)$-cycle in G.*

The proof of the following Claim 6 is similar to the proof of Claim 3. To avoid duplication, we omit the proof of Claim 6.

Claim 6. *Let u be an ℓ-vertex of G. Then, each of the following results must hold.*
(1) If $\ell = 7$, then $n_G^{3^-}(u) \leq 1$.
(2) If $\ell = 8$, then $n_G^3(u) \leq 3$.
(3) If $8 \leq \ell \leq 9$, then $n_G^{2^-}(u) \leq \lceil \frac{\ell}{4} \rceil$; furthermore, $n_G^{3^-}(u) \leq \lceil \frac{\ell}{4} \rceil$ when $n_G^{2^-}(u) \geq 1$.
(4) If $10 \leq \ell \leq 13$, then $n_G^{2^-}(u) \leq \lfloor \frac{3\ell}{7} \rfloor$; furthermore, $n_G^{3^-}(u) \leq \lfloor \frac{3\ell}{7} \rfloor$ when $n_G^{2^-}(u) \geq 1$.

Let $V_i(G)$ $\left(V_{i^-}(G), V_{i^+}(G)\right)$ be the set of vertices in G of degree i (at most i, at least i) and let

$$H = G - V_{2^-}(G).$$

Then, $d_H(v) = d_G(v) - n_G^{2^-}(v)$ for each $v \in V(H)$.

Claim 7. *For the graph H, each of the following results must hold.*
(1) $\delta(H) \geq 3$.
(2) $d_H(v) = d_G(v)$ when $3 \leq d_G(v) \leq 6$.
(3) $d_H(v) \geq 6$ when $d_G(v) \geq 7$.
(4) $n_H^3(v) \leq 1$ when $6 \leq d_H(v) \leq 7$, $n_H^3(v) = 0$ when $d_H(v) = 6$ and $d_G(v) \geq 7$, $n_H^{5^-}(v) = 1$ when $d_H(v) = 6$ and $n_H^3(v) = 1$, and $n_H^3(v) \leq 3$ when $d_H(v) = 8$.
(5) $n_H^{4^-}(v) = 0$ when $d_H(v) \leq 5$.

Proof. Let v be a vertex of H. Then, $3 \leq d_G(v) \leq \Delta(G) \leq 13$ by the definition of H and Claim 1.

(1)–(3) By Claims 3 and 6, we have that

$$\begin{cases} n_G^{2^-}(v) = 0 & \text{if } d_G(v) \leq 6; \\ n_G^{2^-}(v) \leq d_G(v) - 6 & \text{if } d_G(v) \in \{7,8,9,10\}; \\ n_G^{2^-}(v) \leq d_G(v) - 7 & \text{if } d_G(v) \in \{11,12\}; \\ n_G^{2^-}(v) \leq d_G(v) - 8 & \text{if } d_G(v) = 13. \end{cases} \tag{1}$$

Thus, $d_H(v) = d_G(v)$ when $3 \leq d_G(v) \leq 6$ and $d_H(v) \geq 6$ when $d_G(v) \geq 7$ by Equation (1). Therefore, $\delta(H) \geq 3$.

(4) By Claims 4 and 6, we have that

$$\begin{cases} n_G^3(v) \leq 1 & \text{if } d_G(v) = 6; \\ n_G^{3^-}(v) \leq 1 & \text{if } d_G(v) = 7; \\ n_G^3(v) = 0 & \text{if } n_G^{2^-}(v) = d_G(v) - 6 \text{ and } d_G(v) \in \{7,8,9,10\}; \\ n_G^3(v) = 0 & \text{if } n_G^{2^-}(v) = d_G(v) - 7 \text{ and } d_G(v) \in \{11,12\}; \\ n_G^3(v) = 0 & \text{if } n_G^{2^-}(v) = d_G(v) - 8 \text{ and } d_G(v) = 13. \end{cases} \tag{2}$$

and

$$\begin{cases} n_G^3(v) \leq 3 & \text{if } d_G(v) = 8; \\ n_G^3(v) \leq 2 & \text{if } n_G^{2^-}(v) = d_G(v) - 8 \text{ and } d_G(v) \in \{9,10\}; \\ n_G^3(v) \leq 1 & \text{if } n_G^{2^-}(v) = d_G(v) - 8 \text{ and } d_G(v) \in \{11,12\}; \\ n_G^3(v) = 0 & \text{if } n_G^{2^-}(v) = d_G(v) - 8 \text{ and } d_G(v) = 13. \end{cases} \tag{3}$$

Note that $d_H(v) = 3$ if and only if $d_G(v) = 3$ follows from (2) and (3). Thus, $n_H^3(v) \leq 1$ when $6 \leq d_H(v) \leq 7$ and $n_H^3(v) = 0$ when $d_H(v) = 6$ and $d_G(v) \geq 7$ by Equation (2). As $n_H^3(v) = 0$ when $d_H(v) = 6$ and $d_G(v) \geq 7$, $d_H(v) = d_G(v) = 6$ when $d_H(v) = 6$ and $n_H^3(v) = 1$. Therefore, $n_H^{5^-}(v) = 1$ when $d_H(v) = 6$ and $n_H^3(v) = 1$ by Claim 4. By Equation (3), it is easy to know that $n_H^3(v) \leq 3$ when $d_H(v) = 8$.

(5) Note that $d_H(v) = d_G(v)$ when $d_H(v) \leq 5$ by (1)–(3). Thus, $n_H^{4^-}(v) = 0$ when $d_H(v) \leq 5$ by Claim 2. □

Claim 8. *Each 3-cycle in H is either a $(3^+, 6^+, 6^+)$-cycle or a $(5, 5, 6^+)$-cycle.*

Proof. Let v be a vertex of H. Then, $d_H(v) = d_G(v)$ when $d_G(v) \leq 5$ and $d_H(v) \geq 6$ when $d_G(v) \geq 6$ by Claim 7. Thus, there is no $(5^-, 5^-, 5^-)$-cycle in H as G contains no $(5^-, 5^-, 5^-)$-cycle by Claim 5. By Claim 7, we know that each 4^--vertex is not adjacent to any 5^--vertex in H. Therefore, each 3-cycle in H is either a $(3^+, 6^+, 6^+)$-cycle or a $(5, 5, 6^+)$-cycle. □

For a planar graph, we call a face a *t-face* (resp., a *t^+-face*, a *t^--face*, an *(ℓ_1, ℓ_2, ℓ_3)-face*) if its boundary is a t-cycle (resp., a t^+-cycle, a t^--cycle, an (ℓ_1, ℓ_2, ℓ_3)-cycle). A face is said to be *incident with* the vertices and edges in its boundary.

From now on, we assume that the IC-planar graph G has been embedded on a plane such that every edge is crossed by at most one other edge and the number of crossings is as small as possible. We turn all crossings of G into new 4-vertices on the plane and obtain a planar graph $G^\times$, which is called the *associated planar graph* of G. A vertex in $G^\times$ is called a *false vertex* if it is not a vertex of G and *real vertex* otherwise. We call a face f in $G^\times$ a *false face* if it is incident with one false vertex and a *real face* otherwise.

Let $H^\times$ be the associated planar graph of H. For each real vertex $v \in V(H^\times)$, we use $f(v)$ and $f'(v)$ to denote the number of real 3-faces and false 3-faces incident with v, respectively.

Claim 9. *Let v be a real vertex of $H^\times$. Then, each of the following results must hold.*

(1) $f(v) \leq \lfloor \frac{d_{H^\times}(v)}{2} \rfloor$.

(2) $f(v) \leq \lfloor \frac{d_{H^\times}(v)}{2} \rfloor - 1$ *if v is adjacent to one false 4-vertex and* $d_{H^\times}(v) \equiv 0 \ (mod\ 2)$.

(3) $f(v) \leq \lfloor \frac{d_{H^\times}(v)}{2} \rfloor - 1$ *if* $f'(v) = 2$ *and* $d_{H^\times}(v) \equiv 1 \ (mod\ 2)$.

(4) $f(v) \leq \lfloor \frac{d_{H^\times}(v)}{2} \rfloor - 2$ *if* $f'(v) = 2$ *and* $d_{H^\times}(v) \equiv 0 \ (mod\ 2)$.

Proof. (1) As G (and thus H) is an IC-planar graph without theta graphs $\Theta_{2,1,2}$, any two real 3-faces have no common edge in H. Thus, a real vertex v of $H^\times$ is incident with at most $\lfloor \frac{d_{H^\times}(v)}{2} \rfloor$ real 3-faces as each 3-face contains two edges incident with v. Furthermore, statement (1) holds.

(2) By contradiction, suppose that $f(v) = \lfloor \frac{d_{H^\times}(v)}{2} \rfloor$. If $d_{H^\times}(v) \equiv 0 \ (mod\ 2)$, then $\lfloor \frac{d_{H^\times}(v)}{2} \rfloor = \frac{d_{H^\times}(v)}{2}$. Thus, each edge incident with v belongs to a real 3-face incident with v since each 3-face contains two edges. Moreover, v is not adjacent to any false 4-vertex, a contradition. Therefore, $f(v) \leq \lfloor \frac{d_{H^\times}(v)}{2} \rfloor - 1$.

(3) and (4) Let v be incident with two false 3-faces. Then, the two false 3-faces contain three edges incident with v since G (and thus H) is an IC-planar graph without theta graphs $\Theta_{2,1,2}$. Thus, the two false 3-faces cause that $f(v)$ is reduced by 1 when $d_{H^\times}(v) \equiv 1 \ (mod\ 2)$ and 2 when $d_{H^\times}(v) \equiv 0 \ (mod\ 2)$. Therefore, (3) and (4) hold. □

The discharging method, first developed in the study of the coloring of planar graphs about 100 years ago, is an important proof technique in graph theory. The method has been applied in many types of problems, especially in various graph coloring problems.

The general process of discharging is that members (usually vertices or vertices and faces) of a graph are assigned charges by certain "charging rules", then the graph is discharged by certain "discharging rules", during which some members get charges, and some members lose charges, while the sum of the charges keeps unchanged.

In the following, we will apply the discharging method on the associated planar graph $H^\times$ to show that $H^\times$ (and thus H) does not exist. Therefore, G does not exist.

Let $F(H^\times)$ denote the set of faces in $H^\times$. For each $z \in V(H^\times) \cup F(H^\times)$, assign the initial charge $\omega(z) = d_{H^\times}(z) - 4$. By Euler's formula, we have

$$\sum_{z \in V(H^\times) \cup F(H^\times)} \omega(z) = -8.$$

Next, we make some discharging rules to redistribute charges among vertices and faces and keep the total charges unchanged. For simplicity, a real ℓ-vertex is still called an ℓ-vertex in the following discussion.

The discharging rules as follows:

(R1) Each 3-vertex receives $\frac{1}{3}$ from each neighbor.

(R2) Let z be a false 4-vertex and x be one neighbor of z in $H^\times$.

(R2.1) When $d_{H^\times}(x) = 5$, z receives $\frac{1}{2}$ from x.

(R2.2) When $\left(d_{H^\times}(x), n^3_{H^\times}(x)\right) = (6, 0)$, z receives 1 from x.

(R2.3) When $\left(d_{H^\times}(x), n^3_{H^\times}(x)\right) = (6, 1)$, z receives $\frac{7}{6}$ from x if $f'(x) = 2$ and $\frac{2}{3}$ from x otherwise.

(R2.4) When $d_{H^\times}(x) = 7$, z receives $\frac{5}{3}$ from x if $f'(x) = 2$ and $\frac{7}{6}$ from x otherwise.

(R2.5) When $d_{H^\times}(x) \geq 8$, z receives $\frac{5}{3}$ from x if $f'(x) = 2$ and $\frac{4}{3}$ from x otherwise.

(R3) Each false 3-face receives 1 from the false 4-vertex incident with it.

(R4) Each real 3-face receives $\frac{1}{4}$ from each 5-vertex incident with it and $\frac{1}{2}$ from 6^+-vertex incident with it.

In the following, we give a specific example about the charge change of some false 4-vertex by the discharging rules. Let u be a false 4-vertex in $H^\times$ and $N_{H^\times}(u) = \{v_1, v_2, v_3, v_4\}$ with $d_{H^\times}(v_1) = 3, d_{H^\times}(v_2) = 6, d_{H^\times}(v_3) = 7$ and $d_{H^\times}(v_4) \geq 8$. We take the configuration

of $H^{\times}[\{u\} \cup N_{H^{\times}}(u)]$ (see Figure 1) as a specific example to illustrate how the charge of the false 4-vertex u changes by the discharging rules.

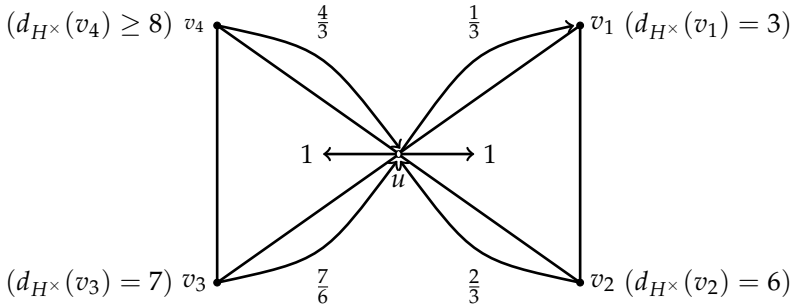


Figure 1. A specific example about the charge change of the false 4-vertex u.

Next, we discuss the new charge of each $z \in V(H^{\times}) \cup F(H^{\times})$ after the discharging process. Let $\omega'(z)$ denote the new charge for each $z \in V(H^{\times}) \cup F(H^{\times})$. Then,

$$\sum_{z \in V(H^{\times}) \cup F(H^{\times})} \omega'(z) = \sum_{z \in V(H^{\times}) \cup F(H^{\times})} \omega(z) = -8 < 0.$$

In the following, we show $\omega'(z) \geq 0$ for each $z \in V(H^{\times}) \cup F(H^{\times})$ to obtain a contradiction.

First, we prove $\omega'(z) \geq 0$ for each (real or false) face $z \in F(H^{\times})$. Pick arbitrarily a face z from $F(H^{\times})$. If z is a false 3-face, then it must be incident with a false 4-vertex. Therefore, $\omega'(z) = 3 - 4 + 1 = 0$ by (R3). If z is a real 3-face, then it is easy to verify $\omega'(z) \geq 0$ by (R4) since each real 3-face is either a $(3^+, 6^+, 6^+)$-face or a $(5, 5, 6^+)$-face by Claim 8. If z is a (real or false) 4^+-face, then $\omega'(z) = \omega(z) = d_{H^{\times}}(z) - 4 \geq 0$ as no rule is applied to it. Thus, $\omega'(z) \geq 0$ for each $z \in F(H^{\times})$.

Note that $V(H^{\times})$ consists of all real vertices in $V(H)$ and all false 4-vertices.

Second, we show $\omega'(z) \geq 0$ for each real vertex $z \in V(H)$. Choose arbitrarily a vertex z from $V(H)$. Since $\delta(H) \geq 3$, $\delta(H^{\times}) \geq 3$. Note that each real vertex z gives no charge to any false 3-face and $d_{H^{\times}}(z) = d_H(z)$. If z is a 3-vertex, then $\omega'(z) = 3 - 4 + 3 \cdot \frac{1}{3} = 0$ by (R1). If z is a 4-vertex, then $\omega'(z) = \omega(z) = 4 - 4 = 0$ as each 4-vertex gives nothing away. In the following, we discuss $z \in V_{5^+}(H)$.

Note that as H is an IC-planar graph, each real vertex z is adjacent to at most a false 4-vertex and $0 \leq f'(z) \leq 2$.

Let $d_{H^{\times}}(z) = 5$. Then, $f(z) \leq 2$ by Claim 9. Thus, $\omega'(z) \geq d_{H^{\times}}(z) - 4 - (2 \cdot \frac{1}{4} + \frac{1}{2}) = 0$ by (R2) and (R4) as z is not adjacent to any 3-vertex by Claim 7.

Let $d_{H^{\times}}(z) = 6$. Then, $n^3_{H^{\times}}(z) \leq n^3_H(z) \leq 1$ by Claim 7. If z is not adjacent to any false 4-vertex, then $f(z) \leq 3$ by Claim 9. Thus, $\omega'(z) \geq d_{H^{\times}}(z) - 4 - (3 \cdot \frac{1}{2} + \frac{1}{3}) = \frac{1}{6}$ by (R1) and (R4). If z is adjacent to one false 4-vertex and $f'(z) \leq 1$, then $f(z) \leq 2$ by Claim 9. When z is not adjacent to any 3-vertex in $H^{\times}$, $\omega'(z) \geq d_{H^{\times}}(z) - 4 - (2 \cdot \frac{1}{2} + 1) = 0$ by (R2) and (R4). When z is adjacent to a 3-vertex in $H^{\times}$, $\omega'(z) \geq d_{H^{\times}}(z) - 4 - (2 \cdot \frac{1}{2} + \frac{2}{3} + \frac{1}{3}) = 0$ by (R1), (R2), and (R4). If z is adjacent to one false 4-vertex and $f'(z) = 2$, then $f(z) \leq 1$ by Claim 9. When z is not adjacent to any 3-vertex in $H^{\times}$, $\omega'(z) \geq d_{H^{\times}}(z) - 4 - (\frac{1}{2} + 1) = \frac{1}{2}$ by (R2) and (R4). When z is adjacent to a 3-vertex in $H^{\times}$, $\omega'(z) \geq d_{H^{\times}}(z) - 4 - (\frac{1}{2} + \frac{7}{6} + \frac{1}{3}) = 0$ by (R1), (R2), and (R4).

Let $d_{H^{\times}}(z) = 7$. Then, $n^3_{H^{\times}}(z) \leq n^3_H(z) \leq 1$ by Claim 7. If $f'(z) \leq 1$, then $f(z) \leq 3$ by Claim 9. Thus, $\omega'(z) \geq d_{H^{\times}}(z) - 4 - (3 \cdot \frac{1}{2} + \frac{7}{6} + \frac{1}{3}) = 0$ by (R1), (R2), and (R4). If $f'(z) = 2$, then $f(z) \leq 2$ by Claim 9. Thus, $\omega'(z) \geq d_{H^{\times}}(z) - 4 - (2 \cdot \frac{1}{2} + \frac{5}{3} + \frac{1}{3}) = 0$ by (R1), (R2), and (R4).

Let $d_{H^{\times}}(z) = 8$. Then, $n^3_{H^{\times}}(z) \leq n^3_H(z) \leq 3$ by Claim 7. If z is not adjacent to any false 4-vertex, then $f(z) \leq 4$ by Claim 9. Thus, $\omega'(z) \geq d_{H^{\times}}(z) - 4 - (3 \cdot \frac{1}{3} + 4 \cdot \frac{1}{2}) = 1$ by (R1) and (R4). If z is adjacent to one false 4-vertex and $f'(z) \leq 1$, then $f(z) \leq 3$

by Claim 9. Thus, $\omega'(z) \geq d_{H^\times}(z) - 4 - (\frac{4}{3} + 3 \cdot \frac{1}{3} + 3 \cdot \frac{1}{2}) = \frac{1}{6}$ by (R1), (R2), and (R4). If z is adjacent to one false 4-vertex and $f'(z) = 2$, then $f(z) \leq 2$ by Claim 9. Thus, $\omega'(z) \geq d_{H^\times}(z) - 4 - (\frac{5}{3} + 3 \cdot \frac{1}{3} + 2 \cdot \frac{1}{2}) = \frac{1}{3}$ by (R1), (R2), and (R4).

Let $d_{H^\times}(z) = \ell$ ($9 \leq \ell \leq 13$). Note that each (real or false) 3-face is incident with at most one 3-vertex as two 3-vertices are not adjacent in H (and thus $H^\times$) by Claim 7. If $f'(z) \leq 1$, then $f(z) \leq \lfloor \frac{\ell}{2} \rfloor$ and $n^3_{H^\times}(z) + f(z) \leq \ell$ by Claim 9. Thus, $\omega'(z) \geq d_{H^\times}(z) - 4 - (\frac{4}{3} + \max\{\frac{1}{3} \cdot n^3_{H^\times}(z) + \frac{1}{2} \cdot f(z) \mid f(z) \leq \lfloor \frac{\ell}{2} \rfloor$ and $n^3_{H^\times}(z) + f(z) \leq \ell\})$ by (R1), (R2), and (R4). Therefore, $\omega'(z) \geq 0$ when $\ell = 9$ and $\omega'(z) \geq \frac{1}{2}$ when $10 \leq \ell \leq 13$. If $f'(z) = 2$, then $f(z) \leq \lfloor \frac{\ell}{2} \rfloor - 1$ and $n^3_{H^\times}(z) + f(z) \leq \ell - 1$ by Claim 9. Thus, $\omega'(z) \geq d_{H^\times}(z) - 4 - (\frac{5}{3} + \max\{\frac{1}{3} \cdot n^3_{H^\times}(z) + \frac{1}{2} \cdot f(z) \mid f(z) \leq \lfloor \frac{\ell}{2} \rfloor - 1$ and $n^3_{H^\times}(z) + f(z) \leq \ell - 1\}) \geq \frac{1}{12}$ by (R1), (R2), and (R4).

Therefore, $\omega'(z) \geq 0$ for each real vertex $z \in V(H)$.

Finally, we prove $\omega'(z) \geq 0$ for each false 4-vertex $z \in V(H^\times) \setminus V(H)$. Pick arbitrarily a vertex z from $V(H^\times) \setminus V(H)$. Let $N_{H^\times}(z) = \{v_1, v_2, v_3, v_4\}$. Then, up to isomorphism, the induced subgraph $H^\times[\{z\} \cup N_{H^\times}(z)]$ is one of four configurations in Figure 2.



Figure 2. Four different configurations of $H^\times[\{z\} \cup N_{H^\times}(z)]$.

Note that z is incident with at most two false 3-faces as H is an IC-planar graph without theta graphs $\Theta_{2,1,2}$. By Claim 7, z is adjacent to at most two ℓ-vertices with $3 \leq \ell \leq 4$ in $H^\times$.

⟨1⟩ Suppose that the configuration of $H^\times[\{z\} \cup N_{H^\times}(z)]$ is F_1 in Figure 2. As z is not incident with any false 3-face and adjacent to at most two 3-vertices, it is easy to verify $\omega'(z) \geq 0$ by (R1) and (R2).

⟨2⟩ Suppose that the configuration of $H^\times[\{z\} \cup N_{H^\times}(z)]$ is F_2 in Figure 2. Then, z is incident with one false 3-face.

⟨2.1⟩ Assume that z is not adjacent to any 3-vertex. Then, it is adjacent to at least two 5^+-vertices by Claim 7. Thus, $\omega'(z) \geq 4 + 2 \cdot \frac{1}{2} - 4 - 1 = 0$ by (R2) and (R3).

⟨2.2⟩ Assume that z is adjacent to exactly one 3-vertex. Then, it is adjacent to at most one 4-vertex by Claim 7. If z is not adjacent to any 4-vertex, then it is adjacent to two 5^+-vertices and one 6^+-vertex by Claim 7. Thus, $\omega'(z) \geq 4 - 4 + 2 \cdot \frac{1}{2} + \frac{2}{3} - \frac{1}{3} - 1 = \frac{1}{3}$ by (R1)∼(R3). If z is adjacent to exactly one 4-vertex, then it is adjacent to two 6^+-vertices by Claim 7. Thus, $\omega'(z) \geq 4 - 4 + 2 \cdot \frac{2}{3} - \frac{1}{3} - 1 = 0$ by (R1)∼(R3).

⟨2.3⟩ Assume that z is adjacent to two 3-vertices. Then, z is adjacent to two 6^+-vertices. If z is adjacent to two 6-vertices, then $\omega'(z) \geq 4 - 4 + 2 \cdot 1 - 2 \cdot \frac{1}{3} - 1 = \frac{1}{3}$ by (R1)∼(R3) as each 6-vertex is not adjacent to any 3-vertex in $H^\times$ by Claim 7. If it is adjacent to one 6^+-vertex and one 7^+-vertex, then $\omega'(z) \geq 4 - 4 + \frac{2}{3} + \frac{7}{6} - 2 \cdot \frac{1}{3} - 1 = \frac{1}{6}$ by (R1)∼(R3).

⟨3⟩ Suppose that the configuration of $H^\times[\{z\} \cup N_{H^\times}(z)]$ is F_3 in Figure 2. Then, z is incident with two false 3-faces.

⟨3.1⟩ Assume that z is not adjacent to any 3-vertex. Then z is adjacent to at most two 4-vertices by Claim 7.

⟨3.1.1⟩ Let z be not adjacent to any 4-vertex. Then, it is adjacent to four 5^+-vertices by Claim 7. Thus, $\omega'(z) \geq 4 + 4 \cdot \frac{1}{2} - 4 - 2 = 0$ by (R2) and (R3).

⟨3.1.2⟩ Let z be adjacent to exactly one 4-vertex. Then, it is adjacent to one 5^+-vertices and two 6^+-vertices by Claim 7. Without loss of generosity, suppose that $d_{H^\times}(v_1) = 4$. Then, $d_{H^\times}(v_i) \geq 6$ $(i = 2, 3)$. If $d_{H^\times}(v_2) = 6$, then v_2 is not adjacent to any 3-vertex in H (and thus $H^\times$) by Claim 7 as it is adjacent to one 4-vertex in H. Thus, $\omega'(z) \geq 4 + \frac{1}{2} + \frac{2}{3} + 1 - 4 - 2 = \frac{1}{6}$ by (R2) and (R3). If $d_{H^\times}(v_2) \geq 7$, then $\omega'(z) \geq 4 + \frac{1}{2} + \frac{2}{3} + \frac{7}{6} - 4 - 2 = \frac{1}{3}$ by (R2) and (R3).

⟨3.1.3⟩ Let z be adjacent to two 4-vertices. As two 4-vertices are not adjacent in H (and thus $H^\times$) by Claim 7, without loss of generosity, suppose that $d_{H^\times}(v_i) = 4$ $(i = 1, 4)$. Then, $d_{H^\times}(v_i) \geq 6$ $(i = 2, 3)$ by Claim 7. Note that if $d_{H^\times}(v_i) = 6$ $(i = 2, 3)$, then v_i is not adjacent to any 3-vertex in H (and thus $H^\times$) by Claim 7 as it is adjacent to one 4-vertex in H. If $d_{H^\times}(v_2) = d_{H^\times}(v_3) = 6$, then $\omega'(z) \geq 4 + 2 \cdot 1 - 4 - 2 = 0$ by (R2) and (R3). If $d_{H^\times}(v_2) = 6$ and $d_{H^\times}(v_3) \geq 7$, then $\omega'(z) \geq 4 + 1 + \frac{7}{6} - 4 - 2 = \frac{1}{6}$ by (R2) and (R3). If $d_{H^\times}(v_2) \geq 7$ and $d_{H^\times}(v_3) = 6$, then $\omega'(z) \geq 4 + \frac{7}{6} + 1 - 4 - 2 = \frac{1}{6}$ by (R2) and (R3). If $d_{H^\times}(v_2) \geq 7$ and $d_{H^\times}(v_3) \geq 7$, then $\omega'(z) \geq 4 + 2 \cdot \frac{7}{6} - 4 - 2 = \frac{1}{3}$ by (R2) and (R3).

⟨3.2⟩ Assume that z is adjacent to exactly one 3-vertex. Without loss of generosity, suppose that $d_{H^\times}(v_1) = 3$. Then $d_{H^\times}(v_i) \geq 6$ $(i = 2, 3)$ by Claim 7.

⟨3.2.1⟩ Let $d_{H^\times}(v_3) = 6$. As v_3 is adjacent to a 3-vertex v_1 in H, it is not adjacent to any other 5^--vertex except v_1 in H by Claim 7. Thus, $d_{H^\times}(v_4) \geq 6$ and $n^3_{H^\times}(v_3) = 0$. Therefore, $\omega'(z) \geq 4 - 4 + 2 \cdot \frac{2}{3} + 1 - \frac{1}{3} - 2 = 0$ by (R1)∼(R3).

⟨3.2.2⟩ Let $d_{H^\times}(v_3) \geq 7$. If $d_{H^\times}(v_2) = 6$, then it is not adjacent to any other 5^--vertex except v_1 in H as v_2 is adjacent to a 3-vertex v_1 in H by Claim 7. Thus, $d_{H^\times}(v_4) \geq 6$. Therefore, $\omega'(z) \geq 4 - 4 + 2 \cdot \frac{2}{3} + \frac{7}{6} - \frac{1}{3} - 2 = \frac{1}{6}$ by (R1)∼(R3). If $d_{H^\times}(v_2) \geq 7$, then $\omega'(z) \geq 4 - 4 + 2 \cdot \frac{7}{6} - \frac{1}{3} - 2 = 0$ by (R1)∼(R3).

⟨3.3⟩ Assume that z is adjacent to two 3-vertices. Then, z is adjacent to two 8^+-vertices by Claim 7. Thus, $\omega'(u) \geq 4 - 4 + 2 \cdot \frac{4}{3} - 2 \cdot \frac{1}{3} - 2 = 0$ by (R1)∼(R3).

⟨4⟩ Suppose that the configuration of $H^\times[\{z\} \cup N_{H^\times}(z)]$ is F_4 in Figure 2. Then, z is incident with two false 3-faces. Note that v_2 is also incident with two false 3-faces.

⟨4.1⟩ Assume that z is not adjacent to any 3-vertex. Then, z is adjacent to at most two 4-vertices by Claim 7.

⟨4.1.1⟩ Let z be not adjacent to any 4-vertex. Then, it is adjacent to four 5^+-vertices by Claim 7. Thus, $\omega'(z) \geq 4 + 4 \cdot \frac{1}{2} - 4 - 2 = 0$ by (R2) and (R3).

⟨4.1.2⟩ Let z be adjacent to exactly one 4-vertex.

⟨4.1.2*a*⟩ Let $d_{H^\times}(v_1) = 4$. Then, $d_{H^\times}(v_i) \geq 6$ $(i = 2, 3)$ by Claim 7. If $d_{H^\times}(v_2) = 6$, then v_2 is not adjacent to any 3-vertex in H (and thus $H^\times$) by Claim 7 as it is adjacent to one 4-vertex v_1 in H. Note that $d_{H^\times}(v_4) \geq 5$. Thus, $\omega'(z) \geq 4 + \frac{1}{2} + \frac{2}{3} + 1 - 4 - 2 = \frac{1}{6}$ by (R2) and (R3). If $d_{H^\times}(v_2) \geq 7$, $\omega'(z) \geq 4 + \frac{1}{2} + \frac{2}{3} + \frac{7}{6} - 4 - 2 = \frac{1}{3}$ by (R2) and (R3).

By symmetry, we know that $\omega'(z) \geq 0$ when $d_{H^\times}(v_3) = 4$.

⟨4.1.2*b*⟩ Let $d_{H^\times}(v_2) = 4$. Then, $d_{H^\times}(v_i) \geq 6$ $(i = 2, 3, 4)$ by Claim 7. Thus, $\omega'(z) \geq 4 + 3 \cdot \frac{2}{3} - 4 - 2 = 0$ by (R2) and (R3).

⟨4.1.2*c*⟩ Let $d_{H^\times}(v_4) = 4$. Then $d_{H^\times}(v_2) \geq 6$ by Claim 7. Note that $d_{H^\times}(v_i) \geq 5$ $(i = 1, 3)$. If $d_{H^\times}(v_2) = 6$, then v_2 is not adjacent to any 3-vertex in H (and thus $H^\times$) by Claim 7 as it is adjacent to one 4-vertex v_4 in H. Thus, $\omega'(z) \geq 4 + 2 \cdot \frac{1}{2} + 1 - 4 - 2 = 0$ by (R2) and (R3). If $d_{H^\times}(v_2) \geq 7$, $\omega'(z) \geq 4 + 2 \cdot \frac{1}{2} + \frac{7}{6} - 4 - 2 = \frac{1}{6}$ by (R2) and (R3).

⟨4.1.3⟩ Let z be adjacent to two 4-vertices. As two 4-vertices are not adjacent in H by Claim 7, $d_{H^\times}(v_4) = 4$. Then, $d_{H^\times}(v_2) \geq 6$ by Claim 7. Without loss of generosity, suppose that $d_{H^\times}(v_1) = 4$. Then, $d_{H^\times}(v_3) \geq 6$ by Claim 7. Note that if $d_{H^\times}(v_i) = 6$ $(i = 2, 3)$, then v_i is not adjacent to any 3-vertex in H (and thus $H^\times$) by Claim 7 as it is adjacent to one 4-vertex in H. If $d_{H^\times}(v_2) = d_{H^\times}(v_3) = 6$, then $\omega'(z) \geq 4 + 2 \cdot 1 - 4 - 2 = 0$ by (R2) and (R3). If $d_{H^\times}(v_2) = 6$ and $d_{H^\times}(v_3) \geq 7$, then $\omega'(z) \geq 4 + 1 + \frac{7}{6} - 4 - 2 = \frac{1}{6}$ by (R2) and (R3). If $d_{H^\times}(v_2) \geq 7$ and $d_{H^\times}(v_3) = 6$, then $\omega'(z) \geq 4 + \frac{7}{6} + 1 - 4 - 2 = \frac{1}{6}$ by (R2) and (R3). If $d_{H^\times}(v_2) \geq 7$ and $d_{H^\times}(v_3) \geq 7$, then $\omega'(z) \geq 4 + 2 \cdot \frac{7}{6} - 4 - 2 = \frac{1}{3}$ by (R2) and (R3).

⟨4.2⟩ Assume that z is adjacent to exactly one 3-vertex.

⟨4.2.1⟩ Let $d_{H^\times}(v_1) = 3$. Then, $d_{H^\times}(v_i) \geq 6$ $(i = 2, 3)$ by Claim 7.

⟨4.2.1*a*⟩ Suppose that $d_{H^\times}(v_2) = 6$. As v_2 is adjacent to a 3-vertex v_1 in H, it is not adjacent to any other 5^--vertex except v_1 in H by Claim 7. Thus, $d_{H^\times}(v_4) = d_H(v_4) \geq 6$. As $d_{H^\times}(v_3) \geq 6$ and $f'(v_2) = 2$, $\omega'(z) \geq 4 - 4 + \frac{7}{6} + 2 \cdot \frac{2}{3} - \frac{1}{3} - 2 = \frac{1}{6}$ by (R1)∼(R3).

⟨4.2.1*b*⟩ Suppose that $d_{H^\times}(v_2) \geq 7$. As $d_{H^\times}(v_3) \geq 6$ and $f'(v_2) = 2$, Thus, $\omega'(z) \geq 4 - 4 + \frac{2}{3} + \frac{5}{3} - \frac{1}{3} - 2 = 0$ by (R1)∼(R3)

By symmetry, we know that $\omega'(z) \geq 0$ when $d_{H^\times}(v_3) = 3$.

⟨4.2.2⟩ Let $d_{H^\times}(v_2) = 3$. Then, $d_{H^\times}(v_i) \geq 6$ $(i = 1, 3, 4)$ by Claim 7. If $d_{H^\times}(v_4) = 6$, then v_4 is not adjacent to any other 3-vertex in H (and thus $H^\times$) by Claim 7 as it is adjacent to a 3-vertex v_2 in H. Thus, $\omega'(z) \geq 4 - 4 + 2 \cdot \frac{2}{3} + 1 - \frac{1}{3} - 2 = 0$ by (R1)∼(R3). If $d_{H^\times}(v_4) \geq 7$, then $\omega'(z) \geq 4 - 4 + 2 \cdot \frac{2}{3} + \frac{7}{6} - \frac{1}{3} - 2 = \frac{1}{6}$ by (R1)∼(R3).

⟨4.2.3⟩ Let $d_{H^\times}(v_4) = 3$. Then, $d_{H^\times}(v_2) \geq 6$ by Claim 7.

⟨4.2.3*a*⟩ Suppose that $d_{H^\times}(v_2) = 6$. Since v_2 is adjacent to a 3-vertex v_4 in H, it is not adjacent to any other 5^--vertex in H by Claim 7. Thus, $d_{H^\times}(v_i) = d_H(v_i) \geq 6$ $(i = 1, 3)$ by Claim 7. Thus, $\omega'(z) \geq 4 - 4 + 2 \cdot \frac{2}{3} + 1 - \frac{1}{3} - 2 = 0$ by (R1)∼(R3).

⟨4.2.3*b*⟩ Suppose that $d_{H^\times}(v_2) \geq 7$. Note that $d_{H^\times}(v_i) \geq 4$ $(i = 1, 3)$. If $d_{H^\times}(v_1) = 4$, then $d_{H^\times}(v_3) \geq 6$ by Claim 7. Thus, $\omega'(z) \geq 4 - 4 + \frac{5}{3} + \frac{2}{3} - \frac{1}{3} - 2 = 0$ by (R1)∼(R3) since $f'(v_2) = 2$. If $d_{H^\times}(v_1) \geq 5$, then $d_{H^\times}(v_3) \geq 5$ by Claim 7. Thus, $\omega'(z) \geq 4 - 4 + \frac{5}{3} + 2 \cdot \frac{1}{2} - \frac{1}{3} - 2 = \frac{1}{3}$ by (R1)∼(R3) as $f'(v_2) = 2$.

⟨4.3⟩ Assume that z is adjacent to two 3-vertices. Then, $d_{H^\times}(v_4) = 3$ as two 3-vertices are not adjacent in H (and thus $H^\times$) by Claim 7. Thus, $d_{H^\times}(v_2) \geq 8$. For v_1 and v_3, without loss of generosity, suppose that $d_{H^\times}(v_1) = 3$. Then, $d_{H^\times}(v_3) \geq 6$. If $d_{H^\times}(v_3) = 6$, then v_3 is not adjacent to any 3-vertex in $H^\times$ as it is adjacent to one 3-vertex v_1 in H. Thus, $\omega'(z) \geq 4 - 4 + \frac{5}{3} + 1 - 2 \cdot \frac{1}{3} - 2 = 0$ by (R1)∼(R3) as $f'(v_2) = 2$. If $d_{H^\times}(v_3) \geq 7$, $\omega'(z) \geq 4 - 4 + \frac{5}{3} + \frac{7}{6} - 2 \cdot \frac{1}{3} - 2 = \frac{1}{6}$ by (R1)∼(R3) as $f'(v_2) = 2$.

Therefore, $\omega'(z) \geq 0$ for each false 4-vertex $z \in V(H^\times) \setminus V(H)$.

In summary, $\omega'(z) \geq 0$ for each $z \in V(H^\times) \cup F(H^\times)$, which contradicts $\sum_{z \in V(H^\times) \cup F(H^\times)} \omega'(z) = -8 < 0$. The proof of Theorem 3 is completed.

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Appendix A

%% Note that $P(x_1, y_1, x_2)$ in Claim 2 and P in the following have the same set of the monomials with highest degree and nonzero coefficient in their expansions.
%% The NoteBook of Mathematica to compute the coefficient.
% INPUT

% Claim 2
P = $(x_1 - y_1)(x_1 - x_2)(x_2 - y_1)(x_1 - x_2)(x_1 + y_1)^4(x_2 + y_1)^3$
Cp = Coefficient[P, $x_1^3 y_1^4 x_2^4$]
% To calculate the coefficient of $x_1^3 y_1^4 x_2^4$
% OUTPUT
Cp = 5

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