Article

Splitting of Framelets and Framelet Packets

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Abstract: Due to resilience to background noise, stability of sparse reconstruction, and ability to capture local time-frequency information, the frame theory is becoming a dynamic forefront topic in data science. In this study, we overcome the disadvantages in the construction of traditional framelet packets derived by frame multiresolution analysis and square iterative matrices. We propose two novel approaches: One is to directly split known framelets again and again; the other approach is based on a generalized scaling function whose shifts are not a frame of some space. In these two approaches, the iterative matrices used are not square and the number of rows in the iterative matrix can be any integer number.

Keywords: framelet packet; splitting trick; generalized scaling function

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1. Introduction

The notion of wavelet packets was first introduced by Coifman, Meyer, and Wick-erhauser [1]. A wavelet packet is a library from which various orthonormal bases can be picked. Wavelet packets can extract better time-frequency features than wavelets, so wavelet packets play an important role in the application of wavelets. Later, for the high-dimensional case, the tensor product wavelet packets were constructed by Chui and Li [2], whereas the non-tensor product wavelet packets were constructed by Shen [3].

As a generalization of wavelet packets, framelet packets were first constructed from frame multiresolution analyses (FMRA) [4–6]. Since generally the scaling function of FMRA is discontinuous in frequency domain, the derived framelet packets cannot possess nice time-domain localization. Moreover, the iterative matrix in the construction of framelet packets is square, and only when the matrix is unitary, the iterative process can be operated up to infinitely many times which can lead to framelet packet with finer and finer frequency bands.

In order to solve the above problems, we propose two approaches. One approach is to abandon multiresolution structure used in traditional construction of framelet packet. Instead, we will directly split framelets by various iterative matrices. The other approach is to remove the use of scaling function in FMRA, i.e., it starts from a generalized scaling function whose shifts are not a frame of some space. In these two approaches, all the iterative matrices are not square and the number of rows in iterative matrix can be any integer number. The framelet packets constructed by us possess fine properties, including short supported, high approximation orders, symmetry, and smoothness.

2. Preliminaries

Denote the space of square-integrable functions on $\mathbb{R}^d$ by $L^2(\mathbb{R}^d)$ and the space of $2\pi\mathbb{Z}^d$ -periodic bounded function by $L^\infty(T^d)$. Denote the inner product by $(\cdot, \cdot)$ and the norm by $\|\cdot\|$. We define the Fourier transform $\hat{f}$ of $f \in L^1(\mathbb{R}^d)$ by

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t) e^{-it \cdot \omega} dt, \quad \omega \in \mathbb{R}^d.$$

We denote the set of vertexes of the cube $[0, 1]^d$ by $\{0, 1\}^d$. The notation δ_{ij} is the Kronecker delta symbol, i.e., $\delta_{ij} = 0$ ($i \neq j$) and $\delta_{ii} = 1$.

Let $\{h_n\}_1^\infty$ be a sequence in $L^2(\mathbb{R}^d)$. If there exists a $B > 0$ such that

$$\sum_{n=1}^{\infty} |(f, h_n)|^2 \leq B \|f\|^2 \quad \forall f \in L^2(\mathbb{R}^d),$$

then $\{h_n\}_1^\infty$ is called a Bessel sequence for $L^2(\mathbb{R}^d)$. If there exist $A, B > 0$ such that

$$A \|f\|^2 \leq \sum_{n=1}^{\infty} |(f, h_n)|^2 \leq B \|f\|^2 \quad \forall f \in L^2(\mathbb{R}^d),$$

then $\{h_n\}_1^\infty$ is called a frame for $L^2(\mathbb{R}^d)$ with bounds A and B . If $A = B$, then it is called a tight frame.

Let $\{\psi_\mu\}_1^l \subset L^2(\mathbb{R}^d)$ and

$$\psi_{\mu,j,k} := 2^{\frac{j \cdot d}{2}} \psi_\mu(2^j \cdot -k), \quad \mu = 1, \dots, l; \quad j \in \mathbb{Z}; \quad k \in \mathbb{Z}^d$$

If the affine system $\{\psi_{\mu,j,k}\}$ is a frame for $L^2(\mathbb{R}^d)$, then the set $\{\psi_\mu\}_1^l$ is called a framelet.

3. Splitting of Framelets

Recently, a lot of framelets with short supported, high approximation orders, symmetry, and smoothness were constructed [7–13]. In this section, we split these nice framelets by non-square iterative matrices to generate new framelets with better time-frequency localization.

Suppose that $\{\psi_\mu\}_0^l \subset L^2(\mathbb{R}^d)$ and the affine system $\{\psi_{\mu,j,k}\}_{\mu=0,\dots,l; j \in \mathbb{Z}; k \in \mathbb{Z}^d}$ is a frame for $L^2(\mathbb{R}^d)$ with bounds A and B . Let $\{H_\mu\}_0^s \subset L^\infty(T^d)$. With the help of these filters $H_0, \dots, H_s$, the original functions $\{\psi_\mu\}_0^l$ are split into $(s+1)(l+1)$ functions $\{\psi_{\mu_1, \mu_2}\}_{\mu_1=0,1,\dots,s; \mu_2=0,\dots,l}$:

$$\hat{\psi}_{\mu_1, \mu_2}(2 \cdot) = H_{\mu_1} \hat{\psi}_{\mu_2} \quad \text{for any } \mu_1 = 0, 1, \dots, s; \quad \mu_2 = 0, 1, \dots, l. \quad (1)$$

We will prove that the affine system $\{\psi_{\mu_1, \mu_2, j, k}\}_{\mu_1=0,1,\dots,s; \mu_2=0,\dots,l; j \in \mathbb{Z}; k \in \mathbb{Z}^d}$ is also a frame for $L^2(\mathbb{R}^d)$.

Theorem 1. Let $\{\psi_\mu\}_0^l \subset L^2(\mathbb{R}^d)$ and $\{\psi_{\mu,j,k}\}_{\mu=0,\dots,l; j \in \mathbb{Z}; k \in \mathbb{Z}^d}$ be a frame for $L^2(\mathbb{R}^d)$ with bounds A and B , and $\{\psi_{\mu_1, \mu_2}\}_{\mu_1=0,\dots,s; \mu_2=0,\dots,l}$ be defined in (1). Denote the $(s+1) \times 2^d$ matrix

$$\Omega(\omega) = (H_\mu(\omega + \pi v))_{\mu=0,\dots,s; v \in \{0,1\}^d}. \quad (2)$$

The nonzero singular values of the matrix $\Omega(\omega)$ are denoted by

$$\lambda_0(\omega), \quad \lambda_1(\omega), \quad \dots, \quad \lambda_{2^d-1}(\omega)$$

Again denote

$$\lambda(\omega) = \min\{|\lambda_0(\omega)|, \dots, |\lambda_{2^d-1}(\omega)|\},$$

$$\Lambda(\omega) = \max\{|\lambda_0(\omega)|, \dots, |\lambda_{2^d-1}(\omega)|\},$$

and

$$\lambda = \inf_{\omega} \lambda(\omega), \quad \Lambda = \sup_{\omega} \Lambda(\omega). \quad (3)$$

Then $\{\psi_{\mu_1, \mu_2, j, k}\}_{\mu_1=0,1,\dots,s; \mu_2=0,\dots,l; j \in \mathbb{Z}; k \in \mathbb{Z}^d}$ is a frame for $L^2(\mathbb{R}^d)$ with bounds $\lambda^2 A$ and $\Lambda^2 B$.

Proof. For convenience, we use the notation

$$[f, g] = \sum_{k \in \mathbb{Z}^d} f(\cdot + 2\pi k) \overline{g(\cdot + 2\pi k)}.$$

Hence, for $f \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned} (f, \psi_{\mu_1, \mu_2, j, k}) &= (2\pi)^{-d} 2^{-\frac{jd}{2}} \int_{\mathbb{R}^d} \widehat{f}(\omega) \overline{\widehat{\psi}_{\mu_1, \mu_2}}(2^{-j}\omega) e^{\frac{i}{2^j}(k \cdot \omega)} d\omega \\ &= (2\pi)^{-d} 2^{\frac{jd}{2}} \int_{\mathbb{R}^d} \widehat{f}(2^j\omega) \overline{\widehat{\psi}_{\mu_1, \mu_2}}(\omega) e^{ik \cdot \omega} d\omega \\ &= (2\pi)^{-d} 2^{\frac{jd}{2}} \int_{[-\pi, \pi]^d} [\widehat{f}(2^j \cdot), \widehat{\psi}_{\mu_1, \mu_2}](\omega) e^{ik \cdot \omega} d\omega. \end{aligned} \quad (4)$$

By the known Parseval Identity of Fourier series [5,14], we get

$$\sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu_1, \mu_2, j, k})|^2 = 2^{jd} (2\pi)^{-d} \int_{[-\pi, \pi]^d} |[\widehat{f}(2^j \cdot), \widehat{\psi}_{\mu_1, \mu_2}](\omega)|^2 d\omega. \quad (5)$$

By (1), we have

$$\begin{aligned} [\widehat{f}(2^j \cdot), \widehat{\psi}_{\mu_1, \mu_2}](\omega) &= \sum_{\alpha \in \mathbb{Z}^d} \widehat{f}(2^{j+1}(\frac{\omega}{2} + \pi\alpha)) \overline{\widehat{\psi}_{\mu_2}(\frac{\omega}{2} + \pi\alpha)} \overline{H_{\mu_1}(\frac{\omega}{2} + \pi\alpha)} \\ &= \sum_{v \in \{0,1\}^d} \sum_{\beta \in \mathbb{Z}^d} \widehat{f}(2^{j+1}(\frac{\omega}{2} + 2\pi\beta + \pi v)) \overline{\widehat{\psi}_{\mu_2}(\frac{\omega}{2} + 2\pi\beta + \pi v)} H_{\mu_1}(\frac{\omega}{2} + \pi v). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} |[\widehat{f}(2^j \cdot), \widehat{\psi}_{\mu_1, \mu_2}](\omega)|^2 &= \sum_{v, v' \in \{0,1\}^d} \sum_{\beta, \beta' \in \mathbb{Z}^d} \overline{H_{\mu_1}(\frac{\omega}{2} + \pi v)} H_{\mu_1}(\frac{\omega}{2} + \pi v') \widehat{f}(2^{j+1}(\frac{\omega}{2} + \pi v + 2\pi\beta)) \\ &\quad \overline{\widehat{f}(2^{j+1}(\frac{\omega}{2} + \pi v' + 2\pi\beta'))} \overline{\widehat{\psi}_{\mu_2}(\frac{\omega}{2} + \pi v + 2\pi\beta)} \widehat{\psi}_{\mu_2}(\frac{\omega}{2} + \pi v' + 2\pi\beta'). \end{aligned}$$

From this, we deduce that for any $f \in L^2(\mathbb{R}^d)$,

$$\sum_{\mu_1=0}^s |[\widehat{f}(2^j \cdot), \widehat{\psi}_{\mu_1, \mu_2}](\omega)|^2 = \sum_{v, v' \in \{0,1\}^d} p_{v, v'}(\omega) x_v(\omega) \overline{x_{v'}(\omega)}, \quad (6)$$

where

$$\begin{aligned} p_{v, v'}(\omega) &= \sum_{\mu_1=0}^s \overline{H_{\mu_1}(\frac{\omega}{2} + \pi v)} H_{\mu_1}(\frac{\omega}{2} + \pi v'), \\ x_v(\omega) &= \sum_{\beta \in \{0,1\}^d} \widehat{f}(2^{j+1}(\frac{\omega}{2} + \pi v + 2\pi\beta)) \overline{\widehat{\psi}_{\mu_2}(\frac{\omega}{2} + \pi v + 2\pi\beta)} \\ &= [\widehat{f}(2^{j+1} \cdot), \widehat{\psi}_{\mu_2}](\frac{\omega}{2} + \pi v), \quad v \in \{0,1\}^d. \end{aligned}$$

Let $W = \Omega(\frac{\cdot}{2})$, by (2), $W^*(\omega) W(\omega) = (p_{v, v'}(\omega))_{v, v' \in \{0,1\}^d}$, so the quadratic form in (6) can be written into the matrix version

$$\sum_{\mu_1=0}^s |[\widehat{f}(2^j \cdot), \widehat{\psi}_{\mu_1, \mu_2}](\omega)|^2 = X^* W^* W X, \quad (7)$$

where the 2^d -dimensional column vector $X = (x_v)_{v \in \{0,1\}^d}$.

Noticing that W^*W is a positive semi-defined matrix, since $\lambda_1(\frac{\omega}{2}), \dots, \lambda_r(\frac{\omega}{2})$ are the nonzero singular values of the matrix W , the nonzero eigenvalues of W^*W are $|\lambda_1(\frac{\omega}{2})|^2, \dots, |\lambda_{2^d-1}(\frac{\omega}{2})|^2$. Furthermore, there exists a unitary matrix U of order 2^d such that $W^*W = U^*DU$, where D is a diagonal matrix whose diagonal elements are $|\lambda_1(\frac{\omega}{2})|^2, \dots, |\lambda_{2^d-1}(\frac{\omega}{2})|^2$. From this and (7), it follows that

$$\sum_{\mu_1=0}^s |[\hat{f}(2^j \cdot), \hat{\psi}_{\mu_1, \mu_2}](\omega)|^2 = X^*U^*DUX. \quad (8)$$

We denote the column vector $Y = UX$ by $Y = (y_\nu)_{\nu \in \{0,1\}^d}$. So we have $Y^* = X^*U^*$. Since U is a unitary matrix, we have $\|Y\| = \|X\|$. Furthermore,

$$X^*U^*DUX = Y^*DY = \sum_{\nu \in \{0,1\}^d} |\lambda_\nu(\frac{\omega}{2}) y_\nu(\omega)|^2. \quad (9)$$

Again by (3), we get

$$\lambda^2 \sum_{\nu \in \{0,1\}^d} |y_\nu(\omega)|^2 \leq \sum_{\nu \in \{0,1\}^d} |\lambda_\nu(\frac{\omega}{2}) y_\nu(\omega)|^2 \leq \Lambda^2 \sum_{\nu \in \{0,1\}^d} |y_\nu(\omega)|^2.$$

By (9) and $\|Y\| = \|X\|$, we have

$$\lambda^2 \sum_{\nu \in \{0,1\}^d} |x_\nu(\omega)|^2 \leq X^*U^*DUX \leq \Lambda^2 \sum_{\nu \in \{0,1\}^d} |x_\nu(\omega)|^2.$$

From this and (8), it follows that

$$\lambda^2 \sum_{\nu \in \{0,1\}^d} |x_\nu(\omega)|^2 \leq \sum_{\mu_1=0}^s \left| [\hat{f}(2^j \cdot), \hat{\psi}_{\mu_1, \mu_2}](\omega) \right|^2 \leq \Lambda^2 \sum_{\nu \in \{0,1\}^d} |x_\nu(\omega)|^2.$$

Again by (5), we have

$$\begin{aligned} \lambda^2 \int_{[-\pi, \pi]^d} \sum_{\nu \in \{0,1\}^d} |x_\nu(\omega)|^2 d\omega &\leq 2^{-jd} (2\pi)^d \sum_{\mu_1=0}^s \sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu_1, \mu_2, j, k})|^2 \\ &\leq \Lambda^2 \int_{[-\pi, \pi]^d} \sum_{\nu \in \{0,1\}^d} |x_\nu(\omega)|^2 d\omega. \end{aligned} \quad (10)$$

Since $[\hat{f}(2^{j+1} \cdot), \hat{\psi}_{\mu_2}](\omega)$ is a $2\pi\mathbb{Z}^d$ -periodic function and

$$\bigcup_{\nu \in \{0,1\}^d} \left(\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]^d - \pi\nu \right) = \left[-\frac{3\pi}{2}, -\frac{\pi}{2} \right]^d,$$

we have

$$\begin{aligned}
 \int_{[-\pi, \pi]^d} \sum_{\nu \in \{0,1\}^d} |x_\nu(\omega)|^2 d\omega &= \sum_{\nu \in \{0,1\}^d} \int_{[-\pi, \pi]^d} \left| \left[\widehat{f}(2^{j+1} \cdot), \widehat{\psi}_{\mu_2} \right] \left(\frac{\omega}{2} + \pi \nu \right) \right|^2 d\omega \\
 &= 2^d \sum_{\nu \in \{0,1\}^d} \int_{[-\frac{\pi}{2}, \frac{\pi}{2}]^d - \pi \nu} \left| \left[\widehat{f}(2^{j+1} \cdot), \widehat{\psi}_{\mu_2} \right](\omega) \right|^2 d\omega \\
 &= 2^d \int_{[-\pi, \pi]^d} \left| \left[\widehat{f}(2^{j+1} \cdot), \widehat{\psi}_{\mu_2} \right](\omega) \right|^2 d\omega \\
 &= 2^{-jd} (2\pi)^d \sum_{k \in \mathbb{Z}^d} \left| (f, \psi_{\mu_2, j+1, k}) \right|^2 \quad (\text{by (4.5)}).
 \end{aligned}$$

From this and (10), it follows that

$$\lambda^2 \sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu_2, j+1, k})|^2 \leq \sum_{\mu_1=0}^s \sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu_1, \mu_2, j, k})|^2 \leq \Lambda^2 \sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu_2, j+1, k})|^2,$$

$\mu_2 = 0, 1, \dots, l; j \in \mathbb{Z}. \quad (11)$

Summing the above inequalities over $\mu_2 = 0, 1, \dots, l$ and $j \in \mathbb{Z}$, we get

$$\begin{aligned}
 \lambda^2 \sum_{\mu_2=0}^l \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu_2, j, k})|^2 &\leq \sum_{\mu_1=0}^s \sum_{\mu_2=0}^l \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu_1, \mu_2, j, k})|^2 \\
 &\leq \Lambda^2 \sum_{\mu_2=0}^l \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu_2, j, k})|^2.
 \end{aligned}$$

(12)

Since $\{\psi_{\mu_2, j, k}\}$ is a wavelet frame with bounds A and B , we have

$$A \|f\|^2 \leq \sum_{\mu_2=0}^l \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu_2, j, k})|^2 \leq B \|f\|^2 \quad \forall f \in L^2(\mathbb{R}^d).$$

Again by (12), we deduce that for any $f \in L^2(\mathbb{R}^d)$,

$$\lambda^2 A \|f\|^2 \leq \sum_{\mu_1=0}^s \sum_{\mu_2=0}^l \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu_1, \mu_2, j, k})|^2 \leq \Lambda^2 B \|f\|^2.$$

So $\{\psi_{\mu_1, \mu_2, j, k}\}_{\mu_1=0,1,\dots,s; \mu_2=0,1,\dots,l; j \in \mathbb{Z}; k \in \mathbb{Z}^d}$ is a frame with bounds $\lambda^2 A$ and $\Lambda^2 B$. $\square$

In general, we can repeat the above splitting process in Theorem 1. Let

$$\psi_{\mu_1, \dots, \mu_m} = \left(\prod_{\alpha=1}^{m-1} H_{\mu_\alpha}(2^{-\alpha} \cdot) \right) \widehat{\psi}_{\mu_m}(2^{-\alpha} \cdot) \quad (\mu_1, \dots, \mu_m = 0, 1, \dots, l).$$

By Theorem 1, we can deduce that

Theorem 2. *If the conditions of Theorem 1 hold, then $\{\psi_{\mu_1, \dots, \mu_m}\}_{\mu_1, \dots, \mu_m=0, \dots, l}$ is a frame for $L^2(\mathbb{R}^d)$, whose bounds are $\lambda^{2m-2} A$ and $\Lambda^{2m-2} B$.*

Generally, when m is large, the obtained frame in Theorem 2 has finer time-frequency localization, but at the same time, the frame bounds become very large. Therefore, only when $\lambda = \Lambda = 1$, the above splitting trick can be operated for infinite many times, i.e., m can tend to infinity and the bounds of the obtained frames are still A and B .

4. Framelet packets

Generally, since the scaling function of FMRA is discontinuous in the frequency domain, the derived framelet packets cannot possess nice time-domain localization [4–6]. In order to solve the above problems, we remove the restriction of FMRA and square iterative matrix, i.e., we start from a generalized scaling function whose shifts are not a frame of some space and the number of rows in iterative matrix can be any integer number.

Definition 1 ([9,15,16]). If $\varphi \in L^2(\mathbb{R}^d)$ satisfies the following conditions

- (i) $\hat{\varphi}$ is continuous at the origin and $\hat{\varphi}(0) = 1$,
 - (ii) there exists a $M > 0$ such that $\sum_{k \in \mathbb{Z}^d} |\hat{\varphi}(\omega + 2k\pi)|^2 \leq M$, $\omega \in \mathbb{R}^d$,
 - (iii) $\hat{\varphi}(2\omega) = H_0(\omega) \hat{\varphi}(\omega)$, where H_0 is a $2\pi\mathbb{Z}^d$ -periodic bounded function,
- then we call φ a generalized scaling function.

Denote

$$\Omega(\omega) = (H_\mu(\omega + \pi v))_{\mu=0,\dots,l; v \in \{0,1\}^d}. \quad (13)$$

Assume that the matrix $\Omega(\omega)$ has $2^d - 1$ nonzero singular values $\{\lambda_r(\omega)\}_1^{2^d-1}$ and

$$\lambda = \inf_{\omega} \{\min_r |\lambda_r(\omega)|\},$$

$$\Lambda = \sup_{\omega} \{\max_r |\lambda_r(\omega)|\}.$$

In Theorem 1, let $\psi_0 = \varphi$ and $\hat{\varphi}(2\cdot) = H_0 \hat{\varphi}$. Again let $\psi_\mu = \psi_{\mu,0}$, $\mu = 0, 1, \dots, l$, by (11), we obtain the following Lemma:

Lemma 1. Let $\varphi \in L^2(\mathbb{R}^d)$ and $H_0 \in L^\infty(T^d)$ be such that $\hat{\varphi}(2\cdot) = H_0 \hat{\varphi}$. Again let $\{H_\mu\}_1^l \subset L^\infty(T^d)$ and $\{\psi_\mu\}_0^l$ satisfy $\hat{\psi}_\mu(2\cdot) = H_\mu \hat{\varphi}$, $\mu = 0, 1, \dots, l$. Then, for any $f \in L^2(\mathbb{R}^d)$ and $j \in \mathbb{Z}$,

$$\lambda^2 \sum_{k \in \mathbb{Z}^d} |(f, \varphi_{j+1,k})|^2 \leq \sum_{\mu=0}^l \sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu,j,k})|^2 \leq \Lambda^2 \sum_{k \in \mathbb{Z}^d} |(f, \varphi_{j+1,k})|^2.$$

In particular, when $\lambda = \Lambda = 1$, we have

$$\sum_{k \in \mathbb{Z}^d} |(f, \varphi_{j,k})|^2 + \sum_{\mu=1}^l \sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu,j,k})|^2 = \sum_{k \in \mathbb{Z}^d} |(f, \varphi_{j+1,k})|^2 \quad \forall j \in \mathbb{Z}.$$

Before we state our results on framelet packets, we need the following notation:

Notation 1. Let

$$\psi_{\mu_1, \dots, \mu_m} = \left(\prod_{\alpha=1}^{m-1} H_{\mu_\alpha}(2^{-\alpha} \cdot) \right) \hat{\psi}_{\mu_m}(2^{-\alpha} \cdot) \quad (\mu_1, \dots, \mu_m = 0, 1, \dots, l).$$

For each $n \in \mathbb{Z}^+$, define $\omega_n = \psi_{\mu_1, \dots, \mu_m}$, where $n = \mu_1 + (l+1)\mu_2 + \dots + (l+1)^{m-1}\mu_m$, where n , j , and k are said to be the oscillation parameter, the scaling parameter, and the location parameter, respectively.

Theorem 3. Suppose that φ is a generalized scaling function and $\{\psi_\mu\}_0^l$ is such that

$$\hat{\psi}_\mu(2\cdot) = H_\mu \hat{\varphi}, \quad \mu = 0, \dots, l,$$

where $\psi_0 = \varphi$ and $\{H_\mu\}_0^l \subset L^\infty(T^d)$. Again let the matrix $\Omega(\omega) = (H_\mu(\omega + \pi v))_{\mu=0,\dots,l; v \in \{0,1\}^d}$ satisfy $\Omega^*(\omega) \Omega(\omega) = I$. Then the sequence $\{\omega_n(\cdot - k)\}_{n=0,1,\dots; k \in \mathbb{Z}^d}$ is a tight frame for $L^2(\mathbb{R}^d)$.

Proof. From the condition $\Omega^*(\omega) \Omega(\omega) = I$, we know that the nonzero singular values of Ω are $\lambda_r(\omega) = 1$ ($r = 1, \dots, 2^d - 1$), so $\lambda = \Lambda = 1$. By Lemma 1, we have

$$\sum_{\mu=0}^l \sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu,j,k})|^2 = \sum_{k \in \mathbb{Z}^d} |(f, \varphi_{j+1,k})|^2.$$

A similar argument shows that

$$\sum_{\mu_1=0}^l \sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu_1, \mu_2, j, k})|^2 = \sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu_2, j+1, k})|^2. \quad (14)$$

So we have

$$\sum_{\mu_1, \mu_2=0}^l \sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu_1, \mu_2}(\cdot - k))|^2 = \sum_{\mu_2=0}^l \sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu_2, 1, k})|^2 = \sum_{k \in \mathbb{Z}^d} |(f, \varphi_{2,k})|^2.$$

In general, we have

$$\sum_{\mu_1, \dots, \mu_s=0}^l \sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu_1, \dots, \mu_s}(\cdot - k))|^2 = \sum_{k \in \mathbb{Z}^d} |(f, \varphi_{s,k})|^2. \quad (15)$$

Similar to (5), we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d} |(f, \varphi_{s,k})|^2 &= (2\pi)^{-d} 2^{sd} \int_{[-\pi, \pi]^d} |\widehat{f}(2^s \cdot), \widehat{\varphi}|(\omega)|^2 d\omega \\ &= (2\pi)^{-d} 2^{sd} \int_{[-\pi, \pi]^d} \left(\sum_{v \in \mathbb{Z}^d} |\widehat{f}(2^s(\omega + 2\pi v)) \widehat{\varphi}(\omega + 2\pi v)|^2 \right) d\omega. \end{aligned} \quad (16)$$

Since $\lim_{\omega \rightarrow 0} \widehat{\varphi}(\omega) = 1$, for any $\epsilon > 0$, there is a δ ($0 < \delta < \pi$) such that $1 - \epsilon \leq |\widehat{\varphi}(\omega)|^2 \leq 1 + \epsilon$. If $\widehat{f}$ is compactly supported, there exists $s_0 > 0$ such that

$$\text{supp } \widehat{f}(2^s \cdot) \subset [-\delta, \delta] \quad \text{if } s > s_0.$$

When $s > s_0$, $|v| \geq 1$, and $\omega \in [-\pi, \pi]$, we have $\widehat{f}(2^s(\omega + 2\pi v)) = 0$. From this and (16), we deduce that

$$\sum_{k \in \mathbb{Z}^d} |(f, \varphi_{s,k})|^2 = (2\pi)^{-d} 2^{sd} \int_{[-\pi, \pi]^d} |\widehat{f}(2^s \omega) \widehat{\varphi}(\omega)|^2 d\omega = (2\pi)^{-d} 2^{sd} \int_{[-\delta, \delta]^d} |\widehat{f}(2^s \omega) \widehat{\varphi}(\omega)|^2 d\omega.$$

Furthermore, we get

$$(1 - \epsilon)(2\pi)^{-d} 2^{sd} \int_{[-\delta, \delta]^d} |\widehat{f}(2^s \omega)|^2 d\omega \leq \sum_{k \in \mathbb{Z}^d} |(f, \varphi_{s,k})|^2 \leq (1 + \epsilon)(2\pi)^{-d} 2^{sd} \int_{[-\delta, \delta]^d} |\widehat{f}(2^s \omega)|^2 d\omega. \quad (17)$$

However,

$$(2\pi)^{-d} 2^{sd} \int_{[-\delta, \delta]^d} |\widehat{f}(2^s \omega)|^2 d\omega = (2\pi)^{-d} 2^{sd} \int_{\mathbb{R}^d} |\widehat{f}(2^s \omega)|^2 d\omega = (2\pi)^{-d} \int_{\mathbb{R}^d} |\widehat{f}(\omega)|^2 d\omega = \|f\|^2.$$

Again, by (17), we deduce that for $s \geq s_0$,

$$(1 - \epsilon) \|f\|^2 \leq \sum_{k \in \mathbb{Z}^d} |(f, \varphi_{s,k})|^2 \leq (1 + \epsilon) \|f\|^2.$$

So we have

$$\lim_{s \rightarrow \infty} \sum_{k \in \mathbb{Z}^d} |(f, \varphi_{s,k})|^2 = \|f\|^2. \quad (18)$$

By Definition 1 and (15), we have

$$\sum_{n=0}^{(l+1)^s-1} \sum_{k \in \mathbb{Z}^d} |(f, \omega_n(\cdot - k))|^2 = \sum_{k \in \mathbb{Z}^d} |(f, \varphi_{s,k})|^2.$$

From this and (18), it follows that if $\hat{f}$ is compactly supported

$$\sum_{n=0}^{\infty} \sum_{k \in \mathbb{Z}^d} |(f, \omega_n(\cdot - k))|^2 = \|f\|^2. \quad (19)$$

Noticing that any $f \in L^2(\mathbb{R}^d)$ can be written as the limit of the sequence of functions whose Fourier transforms are compactly supported, (19) holds for all $f \in L^2(\mathbb{R}^d)$. This implies that the system $\{\omega_n(\cdot - k)\}_{n=0,1,\dots; k \in \mathbb{Z}^d}$ is a tight frame. $\square$

More generally, we can deduce the following:

Theorem 4. Under the conditions of Theorem 3, for $n, j \in \mathbb{Z}^+$, define

$$I_{n,j} = \{\tau \in \mathbb{Z}^+ : (l+1)^j n \leq \tau < (l+1)^j(n+1)\}.$$

Then, the sequence

$$\{\omega_{n,j,k}\}_{(n,j) \in S, k \in \mathbb{Z}^d} = \{2^{\frac{jd}{2}} \omega_n(2^j \cdot - k)\}_{(n,j) \in S, k \in \mathbb{Z}^d}$$

is a tight frame for $L^2(\mathbb{R}^d)$ if the set S of index pair (n, j) is such that $\{I_{n,j}\}_{(n,j) \in S}$ is a partition of positive integers $\mathbb{Z}^+$, i.e., $\mathbb{Z}^+ = \bigcup_{(n,j) \in S} I_{n,j}$ is a disjoint union.

Proof. Similar to the argument of Lemma 1, for $j \in \mathbb{Z}^+$, we have

$$\sum_{\mu_1=0}^l \sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu_1, \mu_2, \dots, \mu_s}(\cdot - k))|^2 = \sum_{k \in \mathbb{Z}^d} |(f, \psi_{\mu_2, \dots, \mu_s, 1, k})|^2.$$

Let

$$n = \sum_{r=0}^{s-2} (l+1)^r \mu_{r+2}, \quad n' = \sum_{r=0}^{s-1} (l+1)^r \mu_{r+1}.$$

By Definition 1, we have

$$\sum_{\mu'_1=0}^l \sum_{k \in \mathbb{Z}^d} |(f, \omega_{n'}(\cdot - k))|^2 = \sum_{k \in \mathbb{Z}^d} |(f, \omega_{n,1,k})|^2 \quad \text{and} \quad n' = (l+1)n + \mu_1.$$

Furthermore, let $n'' = (l+1)n' + \mu_2 = (l+1)^2 n + (l+1)\mu_1 + \mu_2$, then

$$\sum_{\mu_1=0}^l \sum_{k \in \mathbb{Z}^d} |(f, \omega_{n''}(\cdot - k))|^2 = \sum_{k \in \mathbb{Z}^d} |(f, \omega_{n',1,k})|^2.$$

So we have

$$\sum_{\mu_1, \mu_2=0}^l \sum_{k \in \mathbb{Z}^d} |(f, \omega_{n''}(\cdot - k))|^2 = \sum_{k \in \mathbb{Z}^d} |(f, \omega_{n,2,k})|^2.$$

In general,

$$\sum_{\mu_1, \dots, \mu_\tau=0}^l \sum_{k \in \mathbb{Z}^d} |(f, \omega_{n^{(\tau)}}(\cdot - k))|^2 = \sum_{k \in \mathbb{Z}^d} |(f, \omega_{n, \tau, k})|^2, \quad (20)$$

where $n^{(\tau)} = (l+1)^\tau n + (l+1)^{\tau-1} \mu_1 + (l+1)^{\tau-2} \mu_2 + \dots + \mu_\tau$. Since

$$\{(l+1)^{\tau-1} \mu_1 + (l+1)^{\tau-2} \mu_2 + \dots + \mu_\tau, \mu_1, \dots, \mu_\tau = 0, 1, \dots, l\} = \{0, 1, \dots, (l+1)^\tau - 1\},$$

we have

$$\{n^{(\tau)} : \mu_1, \mu_2, \dots, \mu_\tau = 0, 1, \dots, l\} = [(l+1)^\tau n, (l+1)^\tau(n+1)) \cup \mathbb{Z}^+ = I_{n, \tau}.$$

By (20), we get

$$\sum_{p \in I_{n, \tau}} \sum_{k \in \mathbb{Z}^d} |(f, \omega_p(\cdot - k))|^2 = \sum_{k \in \mathbb{Z}^d} |(f, \omega_{n, \tau, k})|^2. \quad (21)$$

Since $\mathbb{Z}^+ = \bigcup_{(n, \tau) \in S} I_{n, \tau}$ is a disjoint union, we have

$$\sum_{n=0}^{\infty} \sum_{k \in \mathbb{Z}^d} |(f, \omega_n(\cdot - k))|^2 = \sum_{(n, \tau) \in S} \sum_{p \in I_{n, \tau}} \sum_{k \in \mathbb{Z}^d} |(f, \omega_p(\cdot - k))|^2.$$

Again, by (21),

$$\sum_{n=0}^{\infty} \sum_{k \in \mathbb{Z}^d} |(f, \omega_n(\cdot - k))|^2 = \sum_{(n, \tau) \in S} \sum_{k \in \mathbb{Z}^d} |(f, \omega_{n, \tau, k})|^2.$$

Finally, by (19), we have

$$\sum_{(n, \tau) \in S} \sum_{k \in \mathbb{Z}^d} |(f, \omega_{n, \tau, k})|^2 = \|f\|^2,$$

i.e., $\{2^{\frac{\tau d}{2}} \omega_n(2^\tau \cdot - k)\}_{(n, \tau) \in S, k \in \mathbb{Z}^d}$ is a tight frame for $L^2(\mathbb{R}^d)$. $\square$

Since a lot of frame generated by affine system are constructed in Theorems 3 and 4 the set of these frames is called a framelet packet. Below we give an example.

Example 1. Let $\varphi = N_4$ be the cube B-spline and

$$P(z) = \left(\frac{1+z}{2}\right)^4, \quad Q_1(z) = zP(-z),$$

$$Q_2(z) = \frac{1}{16}(1-z^2)(1-2\sqrt{7}z+z^2), \quad Q_3(z) = zQ_2(-z).$$

Then $\varphi \in L^2(\mathbb{R})$ is a compactly supported generalized scaling function and the corresponding filter $H_0(\omega)$ satisfies $H_0(\omega) = P(e^{-i\frac{\omega}{2}})$ [9]. Again, let

$$H_\mu(\omega) = Q_\mu(e^{-i\frac{\omega}{2}}), \quad \mu = 1, 2, 3$$

Then, the matrix $\Omega(\omega) = (H_\mu(\omega + \pi v))_{\mu=0, \dots, J; v \in \{0, 1\}}$ satisfy $\Omega^*(\omega) \Omega(\omega) = I$ [9]. By using Notation 1, we can define ω_n . It is clear that each ω_n is compactly supported and smooth. By Theorem 4, the sequence

$$\{\omega_{n, j, k}\}_{(n, j) \in S, k \in \mathbb{Z}^d} = \{2^{\frac{j d}{2}} \omega_n(2^j \cdot - k)\}_{(n, j) \in S, k \in \mathbb{Z}^d}$$

is a tight frame for $L^2(\mathbb{R}^d)$ if the set S of index pair (n, j) is such that $\{I_{n,j}\}_{(n,j) \in S}$ is a partition of positive integers $\mathbb{Z}^+$, i.e., $\mathbb{Z}^+ = \bigcup_{(n,j) \in S} I_{n,j}$ is a disjoint union.

5. Conclusions

In this study, the role of frame multiresolution analysis and square iterative matrices in the construction of frame packets is removed. Two novel tricks are proposed to construct framelets with better time-frequency localization features than those of known framelets. One approach is to split known framelets by using various non-square iterative matrices. The other approach is to start from a generalized scaling function. Moreover, the iterative process in these two approaches can be operated an infinite amount of times.

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