



Article Buckling of Tapered Heavy Columns with Constant Volume

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Abstract: This paper studies the buckling of standing columns under self-weight and tip load. An emphasis is placed on linearly tapered columns with regular polygons cross-section whose volume is constant. Five end conditions for columns are considered. The differential equation governing the buckling shapes of the column is derived based on the equilibrium equations of the buckled column elements. The governing equation is numerically integrated using the direct integration method, and the eigenvalue is obtained using the determinant search method. The accuracy of the method is verified against the existing solutions for particular cases. The effects of side number, taper ratio, self-weight, and end condition on the buckling load and mode shape are investigated. The contribution of self-weight acting alone to the buckling response is also explored. For a given column volume, especially, the buckling length and its stress distribution of the columns with different geometries and end conditions are estimated.

Keywords: heavy column; buckling load; bucking length; self-weight; tapered column; constant volume



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1. Introduction

Columns are elements of the structures in various engineering fields that are subjected to external compressive loads. Long and slender columns have been erected for highways, bridges, offshore facilities, plant structures, etc. In the design of slender mega-columns, selfweight effects are important and must be included in buckling analysis. Such columns are also referred to as heavy columns [1,2]. Tapered members behave differently than uniform members because their variable cross-sections create effective coupling between internal forces and efficient stress distributions [3]. Based on their space utilization, esthetics, safety, optimization, and economic benefits, tapered members are commonly used in engineering practice. Because a tapered member is controlled by its cross-sectional shape and column volume, which affect structural behaviors, various shapes of cross-section are frequently used in practical engineering. Over the past few decade, many efforts have been made to improve structural analyses, including column analysis based on the topics described above.

A short literature review of these topics is provided below. Wang and Drachman [1] investigated the self-weight buckling of a cantilever heavy column with an end load based on a second-order differential equation in terms of the arc length of the buckled column. Interestingly, they applied an inverted cantilever column, which is a column hanging from its fixed end that is subjected to an upward end load. Greenhill [4] studied the maximum stable column lengths (i.e., buckling lengths) of heavy columns such as mast poles. As indicated by the title of the paper, column buckling length was compared to the maximum height at which trees considered as cantilever columns could grow. Since then, small amounts of impactful research have been performed on the buckling loads of cantilever columns: Grishcoff [5] used the infinite series to study the buckling loads of cantilever columns by combining the effects of self-weight and axial loading; Wang and Ang [6]

derived buckling load equations for a heavy column subjected to an axial compressive load and restrained by internal supports. Chai and Wang [7] determined the minimum critical buckling load of self-weighted heavy columns under various end conditions using the differential transformation technique. Duan and Wang [2] derived the exact buckling loads of heavy columns under various end conditions in terms of generalized hypergeometric functions. Lee and Lee [8] studied the buckling of a prismatic heavy column under various end conditions, where the buckling length of the column was calculated by considering only its self-weight (without any axial compressive load). Regarding the optimization of heavy columns, tall columns with variable cross-sections and constant volumes were investigated by Keller and Niordson [9], Atanackovic and Glavardanov [10], and Sadiku [11].

For tapered beam/column analysis, various taper functions [3,12,13] along the column axis, including linear, parabolic, sinusoidal, and exponential functions, have been considered. The effects of various cross-sectional shapes [14,15], including rectangular, circular, elliptical, and regular polygons, on the optimization of column buckling have been examined. Additionally, the initial imperfection affecting column behavior was discussed in the open literature [16,17]. The stability of standing heavy column with the intermediate supports, i.e., laterally braced column, was discussed by Wang [18].

Despite the considerable works discussed above, no buckling solutions have been presented in the open literature with a focus on tapered heavy columns and self-weight with regular polygon cross-sections and constant volumes. This study focused on the buckling loads and buckling self-weights of columns under various end conditions. Based on the small deflection theory, a differential equation is derived from the equilibrium equations of the buckled column elements. A direct integral method is developed for integrating the governing equation and the determinant search method is adopted for determining eigenvalues. The predicted results for the buckling load and buckling selfweight are compared to reference values. Numerical results for the buckling load, buckling length, and buckling stress with corresponding mode shapes are presented.

2. Mathematical Formulation

Figure 1a presents an ideal and linear elastic column of span length l placed in a Cartesian coordinate (x, y) system originating at the toe end t. The toe end t (x = 0) is either hinged or clamped and the head end h (x = l) is either free, hinged, or clamped. Therefore, five end condition combinations are possible: "hinged-hinged (H-H)", "hinged-clamped (H-C)", "clamped-free (C-F)", "clamped-hinged (C-H)", and "clamped-clamped (C-C)", where the former end represents the toe end and the latter end represents the head end. Columns with H-F end condition were not considered in this study because they are unstable in the structural mechanism from an engineering point of view.

The target columns are linearly tapered with cross-sectional shapes of $k (\geq 3)$ -sided regular polygons with circumradii r measured from the centroid to a vertex at any coordinate x. At the toe end t, r is represented as r_t . At the head end h, r is represented as r_h . The column volume V is always constant. The cross-sectional area and second moment of the plane area at x are denoted as A and I, respectively. In the buckling analysis in this study, self-weight effects were included. Such effects are a major concern in the analysis of heavy columns. The internal self-weight intensity, which is the downward self-weight per unit of axial length induced by column mass and gravity, is represented as F_w (= γA), where γ is the weight density of the column material. The column is subjected to an external compressive load P at the head end and its own self-weight W (= γV). When Pincreases and reaches the buckling load B, the column with a buckling length l buckles and forms the buckled-mode shape represented by the solid curve. After column buckling, the internal forces of the axial force N, shear force Q, and bending moment M are applied to the buckled column at x.



Figure 1. (a) Schematic diagram of a buckled heavy column with a *k*-sided regular polygon cross-section and (b) forces imposed on a buckled element.

To express the taper function of r at x mathematically, the taper ratio n, which is defined as the ratio of the head radius r_h to the toe radius r_t is introduced.

$$n = \frac{r_h}{r_t}.$$
 (1)

The linear taper function, which is one of the most practical functions in field engineering, of r is expressed in terms of x as follows:

$$r = r_t F_1, \ F_1 = n_1 \frac{x}{l} + 1,$$
 (2)

where $n_1 = n - 1$.

By using r in Equation (2), the variable functions of A and I for the k-sided regular polygon at x can be obtained as follows [19]:

$$A = c_1 r^2 = c_1 r_t^2 F_1^2, (3)$$

$$I = c_2 r^4 = c_2 r_t^4 F_1^4, (4)$$

where c_1 and c_2 are

$$c_1 = k \sin\left(\frac{\pi}{k}\right) \cos\left(\frac{\pi}{k}\right),\tag{5}$$

$$c_2 = \frac{k}{12} \sin\left(\frac{\pi}{k}\right) \cos^3\left(\frac{\pi}{k}\right) \left[3 + \tan^2\left(\frac{\pi}{k}\right)\right],\tag{6}$$

where $k (\geq 3)$ is the integer side number and $k = \infty$ for the circular cross-section.

The column volume V is determined as

$$V = \int_0^l A dx = c_1 r_t^2 \int_0^l F_1^2 dx = c_1 c_3 r_t^2 l,$$
(7)

where c_3 is

by

$$c_3 = \frac{1}{3} \left(n^2 + n + 1 \right). \tag{8}$$

Note that the length *l* in Equation (7) is the buckling length of the column subjected to an external buckling load *B* and self-weight $W (= \gamma V)$.

Based on Equation (7), the circumradius r_t can be obtained in terms of V as

$$r_t = \sqrt{\frac{V}{c_1 c_3 l}}.$$
(9)

By using Equations (3) and (4) with Equation (9), A and I can be obtained in terms of V as

$$A = \frac{V}{c_3 l} F_1^2,$$
 (10)

$$I = \frac{c_2 V^2}{c_1^2 c_3^2 l^2} F_1^4.$$
(11)

Figure 1b presents a free-body diagram with an infinitesimal length dx for a buckled column element, which is in an equilibrium state based on the internal forces (N,Q,M) and self-weight $F_w(=\gamma A)$. By setting $\sum F_x = 0$, $\sum F_y = 0$, and $\sum M = 0$, the three equilibrium equations can be established as

$$\frac{dN}{dx} + F_w = 0, (12)$$

$$\frac{dQ}{dx} = 0, (13)$$

$$\frac{dM}{dx} - N\frac{dy}{dx} - Q = 0. \tag{14}$$

Differentiating Equation (14) yields the second derivative d^2M/dx^2 as

$$\frac{d^2M}{dx^2} - \frac{dN}{dx}\frac{dy}{dx} - N\frac{d^2y}{dx^2} - \frac{dQ}{dx} = 0.$$
 (15)

Substituting Equations (12) and (13) into Equation (15) yields

$$\frac{d^2M}{dx^2} + F_w \frac{dy}{dx} - N \frac{d^2y}{dx^2} = 0.$$
 (16)

The self-weight intensity F_w at x caused by the γ value of the column material is given

$$F_w = \gamma A = \frac{\gamma V}{c_3 l} F_1^2. \tag{17}$$

Considering *B* and F_w in Equation (17), the axial force *N* at *x* is obtained as

$$N = B + \gamma V - \int_0^x F_w dx = B + \gamma V \left(1 - \frac{F_2}{c_3} \right), \ F_2 = \frac{n_1^2}{3} \frac{x^3}{l^3} + n_1 \frac{x^2}{l^2} + \frac{x}{l}, \tag{18}$$

where the term γV is equal to the total column weight *W*.

The bending moment *M* is given by the relationship between load and deformation based on the small deflection theory [19] as

$$M = -EI\frac{d^2y}{dx^2}.$$
(19)

Differentiating Equation (19) twice yields

$$\frac{d^2M}{dx^2} = -E\frac{d^2I}{dx^2}\frac{d^2y}{dx^2} - 2E\frac{dI}{dx}\frac{d^3y}{dx^3} - EI\frac{d^4y}{dx^4}.$$
(20)

Substituting Equations (17), (18), and (20) into Equation (16) yields

$$\frac{d^4y}{dx^4} = -\frac{2}{I}\frac{dI}{dx}\frac{d^3y}{dx^3} - \frac{1}{I}\frac{d^2I}{dx^2}\frac{d^2y}{dx^2} - \frac{1}{EI}\left[B + \gamma V\left(1 - \frac{F_2}{c_3}\right)\right]\frac{d^2y}{dx^2} + \frac{\gamma V}{c_3l}\frac{F_1^2}{EI}\frac{dy}{dx}.$$
 (21)

From Equation (11), the first and second derivatives of *I* are determined, respectively:

$$\frac{dI}{dx} = \frac{4n_1c_2V^2}{c_1^2c_3^2l^3}F_1^3,$$
(22a)

$$\frac{d^2I}{dx^2} = \frac{12n_1^2c_2V^2}{c_1^2c_3^2l^4}F_1^2.$$
(22b)

Substituting Equation (22a,b) into Equation (21) yields

$$\frac{d^4y}{dx^4} = -\frac{8n_1}{lF_1}\frac{d^3y}{dx^3} - \frac{12n_1^2}{l^2F_1^2}\frac{d^2y}{dx^2} - \frac{c_1^2c_3^2}{c_2}\frac{l^2}{EV^2} \left[B + \gamma V\left(1 - \frac{F_2}{c_3}\right)\right]\frac{1}{F_1^4}\frac{d^2y}{dx^2} + \frac{c_1^2c_3}{c_2}\frac{\gamma l}{EV}\frac{1}{F_1^2}\frac{dy}{dx}.$$
 (23)

To facilitate numerical analysis and obtain the most general results for this class of problems, the following system parameters are cast into non-dimensional forms:

$$\xi = \frac{x}{l},\tag{24}$$

$$\eta = \frac{y}{l},\tag{25}$$

$$\beta = \frac{Bl^4}{EV^2},\tag{26}$$

$$\lambda = \frac{\gamma l^4}{EV},\tag{27}$$

where (ξ, η) are non-dimensional Cartesian coordinates, β is the buckling load parameter, and λ is the self-weight parameter.

By using Equations (24)–(27), Equation (23) in dimensional units can be transformed into the non-dimensional differential Equation (28), which governs the buckled shape of the heavy column as

$$\frac{d^4\eta}{d\xi^4} = -\frac{8n_1}{f_1}\frac{d^3\eta}{d\xi^3} - \frac{12n_1^2}{f_1^2}\frac{d^2\eta}{d\xi^2} - \frac{c_1^2c_3^2}{c_2}\left[\beta + \lambda\left(1 - \frac{f_2}{c_3}\right)\right]\frac{1}{f_1^4}\frac{d^2\eta}{d\xi^2} + \frac{c_1^2c_3}{c_2}\frac{\lambda}{f_1^2}\frac{d\eta}{d\xi},$$
(28)

where $f_1 = n_1\xi + 1$ and $f_2 = (n_1^2/3)\xi^3 + n_1\xi^2 + \xi$. The eigenvalues of (β, λ) in Equation (28) are conjugated with each other. This means that for a given λ value, the eigenvalue β is unique, and vice versa.

Now, consider the boundary conditions in Equation (28). At the top free end (x = l), M in Equation (19) and Q in Equation (14) are both equal to zero. Therefore, the nondimensional boundary conditions of the head free end ($\xi = 1$) are obtained as follows:

$$\frac{d^2\eta}{d\xi^2} = 0, \ \frac{d^3\eta}{d\xi^3} + \frac{c_1^2 c_3^2}{n^4 c_2} \frac{d\eta}{d\xi} \beta = 0.$$
(29)

For the toe and head hinged ends (x = 0 and x = l), y and M are both zero and the non-dimensional boundary conditions at $\xi = 0$ and $\xi = 1$ are obtained:

$$\eta = 0, \ \frac{d^2\eta}{d\xi^2} = 0. \tag{30}$$

For the toe and head clamped ends (x = 0 and x = l), y and dy/dx are both zero and the non-dimensional boundary conditions at $\xi = 0$ and $\xi = 1$ are obtained:

$$\eta = 0, \ \frac{d\eta}{d\xi} = 0. \tag{31}$$

By using the differential equation in Equation (28) subjected to the selected boundary conditions in Equations (29)–(31), the conjugate eigenvalues of (β , λ) can be computed using appropriate numerical solution methods for a given set of column parameters for the end conditions (*k* and *n*).

It is possible for a column to buckle under its self-weight W, even if no external load P = 0 is applied. The buckling self-weight parameter Γ for P = 0 was introduced using Equation (27) and can be formulated as

$$\Gamma = \frac{\gamma L^4}{EV},\tag{32}$$

where *L* is the self-weight buckling length for which the column buckles under self-weight alone. Setting $\beta = 0$ and using Γ instead of λ in Equation (28) yields the following equation:

$$\frac{d^4\eta}{d\xi^4} = -\frac{8n_1}{f_1}\frac{d^3\eta}{d\xi^3} - \frac{12n_1^2}{f_1^2}\frac{d^2\eta}{d\xi^2} - \frac{c_1^2c_3^2}{c_2}\Gamma\left(1 - \frac{f_2}{c_3}\right)\frac{1}{f_1^4}\frac{d^2\eta}{d\xi^2} + \frac{c_1^2c_3}{c_2}\frac{\Gamma}{f_1^2}\frac{d\eta}{d\xi},\tag{33}$$

where the buckling self-weight parameter Γ is the eigenvalue in the differential equation of Equation (33).

After calculating the conjugate eigenvalues (β , λ) from Equation (28) for a given set of *E*, *V*, and γ , the buckling length *l* is calculated using Equation (26) or Equation (27), and the buckling stress σ at *x* is obtained as

$$l = \sqrt[4]{\frac{EV^2}{B}\beta},\tag{34}$$

$$\sigma = \frac{N}{A} = \frac{c_3 EV}{l^3} \left[\frac{\beta}{f_1^2} + \left(1 - \frac{f_2}{c_3} \right) \frac{\lambda}{f_1^2} \right],$$
(35)

where *A* and *N* in Equation (35) are given by Equations (10) and (18), respectively. In particular, the self-weight buckling length *L* and self-weight buckling stress σ caused only by the self-weight *W* with *P* = 0 are obtained using Equations (36) and (37), respectively.

$$L = \sqrt[4]{\frac{EV}{\gamma}}\Gamma,\tag{36}$$

$$\sigma = \frac{c_3 EV}{l^3} \left(1 - \frac{f_2}{c_3} \right) \frac{\Gamma}{f_1^2}.$$
(37)

3. Solution Methods

Based on the mathematical formulations above, two FORTRAN computer programs were written to solve the conjugate eigenvalues of (β , λ) in Equation (28) and the eigenvalue Γ in Equation (33). The input column parameters are the end conditions, as well as the side number *k* and taper ratio *n* for Equations (28) and (33), respectively. To calculate the

mode shape (ξ, η) , Equations (28) and (33), which are boundary problems subjected to the end conditions selected from Equations (29)–(31), are integrated numerically using a direct integration method such as the Runge–Kutta method [20]. The eigenvalues (β, λ) and Γ are calculated using the determinant search method [15,19] enhanced by the Regula– Falsi method [20]. By using these solution methods, the eigenvalues (β, λ) and Γ with their corresponding mode shapes (ξ, η) for various end conditions can be calculated. Five different end conditions are considered, as discussed in Section 2. These types of solution methods for boundary and eigenvalue problems such as Equations (28) and (33) have been described in detail in [15,21], and interested readers should refer to these previous studies.

Before executing the numerical methods described above, it is important to choose a suitable step size $\Delta \xi$ when applying the Runge–Kutta scheme, which is computed using the following equation for a given number of dividing elements n_e for the unit buckling length:

$$\Delta \xi = \frac{1}{n_e}.\tag{38}$$

Convergence analysis considering the buckling load parameter β was performed to obtain a suitable n_e (= 1/ $\Delta\xi$) and the results of the H-H column with a circular cross-section ($k = \infty$) with $\lambda = 1$ and n = 0.5 are presented in Figure 2. The solution β with $n_e = 10$ converges at a ratio of 0.9996 (= 0.26865/0.26876) to the solution β with $n_e = 200$, meaning that β with $n_e = 10$ is sufficiently converged. Additionally, the solution β with $n_e = 20$ agrees well with that with $n_e = 200$ (within four significant figures). All computations in this study were conducted on a PC with a GPU. The solutions for β with $n_e = 20$ were computed within one-third of a second.



Figure 2. Convergence analysis for a suitable n_e (= $1/\Delta\xi$) in the Runge–Kutta scheme.

4. Results and Discussion

Numerical experiments on the effects of column parameters on the conjugate eigenvalues of (β, λ) in Equation (28) and the eigenvalue Γ in Equation (33) with their corresponding mode shapes (ξ, η) were performed. For validation purposes, the buckling loads *B* and buckling self-weight parameters Γ in this study and various references [2,6,8,22] are compared in Tables 1 and 2, respectively. First, the *B* values for a concrete column with V = 15 m³, E = 20 GPa, and $\lambda = 0$ (i.e., without self-weight, with varying end conditions, a side number *k*, and taper ratio *n*) are compared. The results of this study and those presented by Riley [22] are in good agreement (0.3% error). Second, the buckling self-weight parameters $c_1^2 c_3^2 \Gamma/c_2$ (= $\gamma AL^3/EI$) for n = 1 (i.e., uniform column) in this study and previous studies [2,6,8] with various end conditions are compared. Note that the parameters of $c_1^2 c_3^2 \Gamma/c_2$ for the buckling self-weight parameters

 Γ have also been adopted in the literature [2,6]. If n = 1, then the parameters are the same, regardless of k. The results of this study and the references [6] agree well, and the results of this study and the references [2,8] are the same to within five significant figures. Thus, the analytical theories and numerical methods developed in this study are validated when considering all of the column parameters, including the end conditions, k, and n.

	Bu	ckling Load B in MN	J
End Condition, k and n	This Study	Riley [22]	% Error
H-H, $k = 3, n = 0.4$	49.95	49.95	0.
H-C, $k = 4$, $n = 0.5$	109.88	109.72	0.15
C-F, $k = 5$, $n = 0.6$	22.07	22.01	0.27
C-H, $k = 6$, $n = 0.7$	132.39	132.33	0.05
C-C, $k = \infty$, $n = 0.8$	270.17	270.17	0.

Table 1. Comparisons of buckling load B * for $\lambda = 0$.

* l = 15 m, $V = 15 \text{ m}^3$, and E = 20 GPa for the concrete column.

Table 2. Comparisons of buckling self-weight parameter Γ in terms of $c_1^2 c_3^2 \Gamma/c_2$ for $n = 1^*$.

End Condition			$c_1^2 c_3^2 arGamma/c_2$		
Life Contaition	H-H	H-C	C-F	C-H	C-C
This study	18.5687	30.0094	7.8373	52.5007	74.6286
Duan and Wang [2]	18.5687	-	7.8373	52.5007	74.6286
Wang and Ang [6]	18.58	-	7.84	53.91	78.96
Lee and Lee [8]	18.5687	30.0094	7.8373	52.5007	74.6286

* If n = 1, $c_1^2 c_3^2 \Gamma / c_2 (= \gamma A L^3 / EI)$ are identical regardless of k.

Table 3 shows the effects of the side number *k* on the buckling load parameter β with a conjugate eigenvalue of $\lambda = 1$ and n = 0.5 for each end condition. As *k* increases, β decreases. One can see that an equilateral triangle (k = 3) column is the strongest column with the largest β value for a given column volume. This is because the area is the same regardless of *k* in the same volume, but the circumradius *r* and the second moment of the plane area *I* depend on *k* and are greater from k = 3 to $k = \infty$ (see the ratio of $I_k/I_{k=3}$ in the last column of the table). The β value of the equilateral triangle column is 1.464 (= 0.3934/0.2688) times larger than the circular column ($k = \infty$) for the H-H condition. The value of β depends heavily on the end conditions, as indicated by C-C column maximum and C-F column minimum. For the circular cross-section, the β value of the C-C column is 19.79 (= 2.0759/0.1049) times larger than that of the C-F column.

Table 3. Effects of side number *k* on β with $\lambda = 1$ and n = 0.5.

k		Ratio of				
R	H-H	H-C	C-F	С-Н	C-C	$I_k/I_{k=3}$
3 (triangle)	0.3934	1.0123	0.1578	1.2814	2.5595	1.0
4 (square)	0.2970	1.8193	0.1170	1.0896	2.1850	0.8660
5 (pentagon)	0.2789	0.7833	0.1092	1.0538	2.1152	0.8410
6 (hexagon)	0.2734	0.7722	0.1069	1.0428	2.0936	0.8333
∞(circular)	0.2688	0.7630	0.1049	1.0337	2.0759	0.8270

Figure 3 presents β versus *n* curves for a conjugate eigenvalue of $\lambda = 1$ and circular cross-section. Columns subjected to an external load *p* are in the stability domain under the β versus *n* curves (i.e., p < b), meaning they are not buckled. As *n* increases, β increases, reaches a peak coordinate (β , *n*) marked with \blacktriangle , and then decreases. At the peak point of (β , *n*) of each curve, the taper ratio *n* is optimized, implying that the column with the optimized *n* has the maximum β . For example, for the C-H column, the column achieves

the maximum $\beta = 1.2814$ with an optimized n = 0.8501. One can see that the β values with n = 1, excluding the C-F column, are nearly identical to the β values with the optimized n. Figure 3 also highlights the stability region of n. For the C-F column, the columns with 0 < n < 0.7383 are stable (i.e., not buckled), unless p < b. In contrast, the columns with n > 0.7383 are unconditionally unstable (i.e., buckled), even if p = 0, implying that the columns are buckled by the self-weight parameter $\lambda = 1$. For the C-C and C-H columns, the lower limit of stability for n is n = 0 (see marks of \blacksquare) and the upper limit of n does not appear until n = 1. For the H-C and H-H columns, the lower limits of stability for n are n = 0.0949 and n = 0.1426 (see marks of \blacksquare), respectively, and upper limits of n do not appear until n = 1.



Figure 3. Curves of β versus *n*.

Figure 4 presents a graphical chart of the conjugated eigenvalues of the buckling load parameter and self-weight parameter (β , λ) for a circular cross-section with n = 0.5. In the governing differential equation, namely, Equation (28), there are two conjugated eigenvalues of (β , λ) that are unique. As λ increases, β decreases. β is the largest at $\lambda = 0$ when excluding the self-weight effect and the effect of λ on β is significant. For example, β with $\lambda = 2$ is 25.5% smaller than β with $\lambda = 0$ (2.3035/1.8353 = 1.255; see marks of •). Eventually, β becomes zero at $\lambda = 8.6443$ (i.e., the buckling self-weight parameter $\Gamma = 8.6443$). Therefore, the column with $\lambda = 8.6443$ buckles under the column self-weight alone, without any external load. In this figure, values of Γ marked by \blacksquare are presented for a given set of column parameters.



Figure 4. Chart of conjugate eigenvalues of (β, λ) .

Figure 5 presents the buckled mode shapes (ξ, η) for each end condition with a circular cross-section, $\lambda = 1$, and n = 0.5. In this figure, the buckling load parameters β shown in Table 3 and the positions ξ of the maximum deflection for each end condition are also presented. Note that the coordinate η of the deflection represents relative deflection, rather than absolute deflection. The buckling length parameter $\lambda = 1$ for each end condition. The location of the maximum deflection of the end condition. The location of the maximum deflection of a column like a utility pole may be controlled by guywires to prevent unexpected buckling stemming from undesirable column imperfections.



Figure 5. Example buckling mode shapes for $\lambda = 1$, $k = \infty$, and n = 0.5.

Table 4 shows the effects of the side number k on the buckling self-weight parameter Γ with n = 0.5 for each end condition. As k increases, Γ decreases. An equilateral triangle (k = 3) column is the strongest column with the largest Γ value for a given column volume. The Γ value of the triangle column is 1.209 (= 2.1405/1.7701) times larger than that of the circular column ($k = \infty$) for the H-H condition. The value of Γ depends heavily on the end conditions, as indicated by the C-C column maximum and C-F column minimum. For the circular cross-section, the Γ value of the C-C column is 5.257 (= 8.6443/1.6443) times greater than that of the C-F column. Therefore, selecting proper end conditions is one of the most important design criteria for heavy column design, as discussed previously regarding Table 3.

k	Buckling Self-Weight Parameter Γ					
<i>n</i> –	Н-Н	H-C	C-F	С-Н	C-C	
3 (triangle)	2.1405	3.2497	1.9883	8.0144	10.453	
4 (square)	1.8537	2.8143	1.7219	6.9407	9.0523	
5 (pentagon)	1.8002	2.7331	1.6772	6.7403	8.7911	
6 (hexagon)	1.7837	2.7080	1.6569	6.6787	8.7106	
∞(circular)	1.7701	2.6874	1.6443	6.6278	8.6443	

Table 4. Effect of side number *k* on Γ for n = 0.5.

Figure 6 presents Γ versus n curves for the circular cross-section, where the values of Γ with n = 0.5 listed in Figure 4 are also represented as \blacksquare marks. Columns with the self-weight parameter λ are in the stability domain under the Γ versus n curves (i.e., $\lambda < \Gamma$) and are not buckled by self-weight. As n increases, Γ increases, reaches a peak at the coordinates (Γ , n) marked with \blacktriangle , and then decreases. At the peak point of (Γ , n) on each curve, the taper ratio n is optimized to avoid buckling under self-weight, implying that





Figure 6. Curves of *Γ* versus *n*.

Figure 7 presents the buckling stresses σ in dimensional units for columns subjected to (a) self-weight without an external load (P = 0) and (b) an external buckling load of B = 5MN, where the buckling column length l, stress σ_t at the column toe ($\xi = 0$), and stress σ_t at the pile head ($\xi = 1$) are presented. The column parameters considered are a circular cross-section, n = 0.5, V = 10 m³, E = 20 GPa, and $\gamma = 23$ kN/m³ for a concrete material. In the case of (a) self-weight, σ decreases along the column axis and σ is maximized as $\sigma_{max} = \sigma_t$ at $\xi = 0$, which is the expected behavior. For the buckling column length L, the C-C column is the longest and the C-F column is the shortest, which is the expected behavior. Considering the ultimate stress of $\sigma_u = 40$ MPa for the concrete material, σ_t values between 0.825 and 1.249 MPa are relatively small compared to σ_u , meaning heavy column ruptures are caused by buckling, rather than fracturing. In the case of (b), the external load of B = 5 MN, σ increases along the column axis, where σ is minimized as $\sigma_{min} = \sigma_t$ and maximized as $\sigma_{max} = \sigma_h$ because the column is subjected to an external load and the column area decreases (i.e., n = 0.5). Additionally, the buckling length l of the C-C column is the largest and that of the C-F column is the smallest. Even when an external load is applied to the column, the column ruptures as a result of buckling, rather than fracturing, just as in the case of self-weight buckling.

Tables 5 and 6 summarizes the tallest non-buckling column lengths of (L, l) provided in Figure 7. These tables also include numerical results for a steel heavy column with a square cross-section, E = 210 GPa, and $\gamma = 77$ kN/m³, with the other parameters kept constant. The buckling behavior of steel columns is similar to that of concrete columns. It is noteworthy that the self-eight buckling length L (see Equation (36)) and buckling length l (see Equation (34)) of the steel column do not increase significantly beyond those of the concrete column, despite the Young's moduli of E = 20 GPa for the concrete column and E = 210 GPa for the steel column. Note that under the same column parameters given above, if the length of a particular column is shorter than the tallest length L or l shown in Tables 5 and 6, the column is safe from column buckling. For example, the H-H column with a specific column length of 10 m (< L = 62.64 m) will not be buckling. The corresponding circumradii of the column are $r_t = 0.752$ m and $r_h = 0.376$ m (n = 0.5, V = 10 m³ and column length = 10 m), which are practical in real engineering systems. The column stress $\sigma_t = \gamma V/A_t$ at the toe end is computed as $\sigma_t = 0.129$ MPa ($< \sigma_{all} = 2$ MPa, approximately), and therefore this column is safe from self-weight buckling.



Figure 7. Examples of buckling stress σ for a concrete column with a circular cross-section, n = 0.5, $V = 10 \text{ m}^3$, E = 20 GPa, and $\gamma = 23 \text{ kN/m}^3$: (a) self-weight buckling (P = 0) and (b) external load buckling (B = 5 MN).

Table 5. Tallest buckling length L and buckling stresses σ_t of heavy columns * without compressive load.

L and σ_t	H-H	H-C	C-F	С-Н	C-C
	(a) Circular ($k = \infty$) concrete colum	າກ	
<i>L</i> (m)	62.64	69.52	61.49	87.13	93.11
σ_t (MPa)	0.840	0.933	0.825	1.169	1.249
		(b) Square ($k =$	4) steel column		
<i>L</i> (m)	84.32	93.60	82.78	117.3	125.3
σ_t (MPa)	3.787	4.204	3.718	5.269	5.630

* Buckling due to self-weight W.

Table 6. Tallest buckling length * *l* and buckling stresses σ_h of heavy columns * with compressive load.

<i>l</i> and σ_h	Н-Н	H-C	C-F	С-Н	C-C
		(a) Circular co	ncrete column		
<i>l</i> (m)	21.53	25.87	17.26	25.94	30.77
σ_h (MPa)	25.11	30.18	20.14	30.26	35.90
		(b) Square s	teel column		
<i>l</i> (m)	39.41	46.97	31.66	47.34	55.94
σ_h (MPa)	45.98	54.80	36.93	55.24	65.26

* Buckling due to external compressive load P = 5 MN including self-weight W.

5. Concluding Remarks

This paper presents the buckling of heavy column included its own self-weight. The column is linearly tapered, the cross-section is a regular polygon, and the column volume is held in constant. Five end conditions of the column are considered. Using equilibrium equations of the buckled column element based on the small deflection beam theory, the fourth-order ordinary differential equation governing the buckled mode shape of such column is derived. For integrating the differential equation, the direct integration method such as Runge–Kutta method is used and for calculating eigenvalues, i.e., buckling load and self-weigh buckling length, is applied as the solution methods. Predicted buckling loads and self-weight buckling lengths agree well with those of references. Numerical results of the buckling load, self-weight buckling length, buckled mode shape, and buckling stress are presented herein and are extensively discussed. The results of this study are expected to be utilized in the design of heavy columns including the self-weight effect.

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