Article

On Small Deviation Asymptotics in the L_2 -Norm for Certain Gaussian Processes

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Abstract: The results obtained allow finding sharp small deviations in a Hilbert norm for centered Gaussian processes in the case where their covariances have a special form of the eigenvalues and allow us to describe small deviation asymptotics for certain Gaussian processes.

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MSC: 60G50; 60F99

1. Introduction and Results

Consider a centered Gaussian process $X(t)$, $0 \leq t \leq 1$, with covariance $G(t, s) = \mathbf{E}X(t)X(s)$, $t, s \in [0, 1]$ such that $0 < \int_0^1 G(t, t) dt < \infty$. Set $\|X(t)\|_2^2 = \int_0^1 X^2(t) dt$. Our aim is to obtain sharp asymptotics of the probability $\mathbf{P}(\|X(t)\|_2 < \varepsilon)$ as $\varepsilon \rightarrow 0$. Due to the well-known classical Karhunen–Loève expansion (see, for instance, [1]), the following equality in the distribution:

$$\|X(t)\|_2^2 = \int_0^1 X^2(t) dt = \sum_{n \geq 1} \lambda_n \xi_n^2$$

takes place, where ξ_n , $n \geq 1$, are standard independent Gaussian rv's, while positive summable λ_n are the eigenvalues of the integral equation:

$$\lambda f(s) = \int_0^1 G(t, s) f(t) dt, \quad 0 \leq s \leq 1.$$

Hence, the problem examined is equivalent to studying the asymptotic behavior of the probability $\mathbf{P}(\sum_{n \geq 1} \lambda_n \xi_n^2 < \varepsilon)$ as $\varepsilon \rightarrow 0$ or, in a more general setting, the probability $\mathbf{P}(\sum_{n \geq 1} \lambda_n |\xi_n|^p < \varepsilon)$, $p > 0$.

The problem of small deviations for the norms of Gaussian processes (including one of the simplest ones, the norm in L_2) has been studied quite intensively (see the bibliography in [2–4]). However, due to the fact that explicit formulas for λ_n are known for a few concrete processes, sharp asymptotics often cannot be found. Nevertheless, this was done in [2] for an interesting and rather general case where the coefficients λ_n behave as quotients of powers of two polynomials.

We present this nice result since it is the starting point for our research. To formulate it, we introduce some notation. For $p > 0$ and $V > 1$, set:

$$f(t) = \log \mathbf{E} \exp \{-t|\xi_1|^p\}, \quad t \geq 0; \quad K(V, p) = - \int_0^\infty t^{-1/V} f'(t) dt. \quad (1)$$

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For example, $K(V, 2) = 2^{(1-V)/V} \pi / \sin(\pi/V)$.

Let us define polynomials:

$$\begin{aligned} Q_r(x) &= x^r + A_{r-1} x^{r-1} + \cdots + A_0 = \prod_{i=1}^r (x + \theta_i), \\ P_q(x) &= x^q + B_{q-1} x^{q-1} + \cdots + B_0 = \prod_{i=1}^q (x + \phi_i), \end{aligned} \quad (2)$$

where $r, q \in \mathbb{Z}_+$, $A_i, B_i \in \mathbb{R}$, $\phi_i, \theta_i \in \mathbb{C}$. We assume that for $n = 1, 2, \dots$, the polynomials $Q_r(n)$, $P_q(n)$ are strictly positive and that real constants μ, ν satisfy the condition:

$$V = q\mu - r\nu > 1, \quad (3)$$

as well as that the sequence $a_n = Q_r^\nu(n) / P_q^\mu(n)$ does not increase.

Proposition 1. *If:*

$$\sigma = (\mu B_{q-1} - \nu A_{r-1}) / V > -1, \quad (4)$$

then:

$$\mathbf{P}\left(\sum_{n \geq 1} a_n |\xi_n|^p < \varepsilon\right) \sim \left(\frac{\left(\prod_{j=1}^q \Gamma(1 + \theta_j)\right)^\nu}{\left(\prod_{j=1}^r \Gamma(1 + \phi_j)\right)^\mu} \right)^{1/p} C \varepsilon^A \exp(-B \varepsilon^{-\frac{1}{V-1}}) \quad (5)$$

as $\varepsilon \rightarrow +0$, where (see (1), (3), and (4)):

$$A = \frac{p - (1 + 2\sigma)V}{2p(V-1)}, \quad B = (V-1) \left(\frac{K}{V}\right)^{\frac{V}{V-1}}, \quad K = K(V, p)$$

and:

$$C = \frac{2^{\frac{-3p+2V-2p\sigma}{4p}} V^{\frac{2Vp-V-p-2V\sigma}{2p(V-1)}} K^{\frac{V-Vp+2V\sigma}{2p(V-1)}} \pi^{\frac{-p+2p\sigma+2V}{4p}}}{\Gamma(1 + \frac{1}{p})^{\sigma + \frac{1}{2}} \sqrt{V-1}}.$$

Proposition 1 was proven in [2], Theorem 1. Note that Condition (4) and the assumption on the sequence a_n , which is non-increasing, which is used essentially in the proof, were not explicitly mentioned in the formulation of [2], Theorem 1.

The purpose of the present note is to extend Proposition 1 to a more general class of weights a_n and to omit the constraint (4).

Let us formulate the results. For $n, m \in \mathbb{N}$, $\lambda_j \in \mathbb{R}$, and $z_j \in \mathbb{C}$ ($1 \leq j \leq m$), we set $V = \sum_{j=1}^m \lambda_j > 1$, $a_n = \prod_{j=1}^m |n + z_j|^{-\lambda_j}$.

Note that the weights $Q_r^\nu(n) / P_q^\mu(n)$ in Proposition 1 constitute a subcase of the weights a_n above (one can derive this by taking λ_j to be equal to μ or $-\nu$).

Theorem 1. *Let the integer $k \geq 0$, $z_j \neq -1, -2, \dots$, and:*

$$\sigma := \frac{1}{V} \sum_{j=1}^m \lambda_j \Re z_j > -k - 1. \quad (6)$$

Then, for every $p > 0$:

$$\mathbf{P}\left(\sum_{n \geq 1} a_n |\xi_n|^p < \varepsilon\right) \sim \left(\prod_{j=1}^m |\Gamma(k+1+z_j)|^{-\lambda_j}\right)^{1/p} \left(\prod_{j=1}^k a_j\right)^{-\frac{1}{p}} \cdot C \varepsilon^A \exp(-B \varepsilon^{-\frac{1}{V-1}}) \quad (7)$$

as $\varepsilon \rightarrow +0$, with the same notation as in Proposition 1, except the “new” V and σ .

Note that Equality (7) still holds true if we assume that $z_j \neq -k-1, -k-2, \dots$ and replace the numbers a_j , $1 \leq j \leq k$, in (7) with arbitrary positive constants.

Note also that the minimal k satisfying Condition (6) is equal to $[\max(-\sigma, 0)]$, where $[x]$ stands for the integer part of x .

2. Proofs

Let us prove the statement of Theorem 1 for $k = 0$. We follow the lines of the proof in [2]. Set $b_n = (n + \sigma)^{-V}$. Taking into account the equality $|n + z_j|^2 = n^2 + 2n\Re z_j + |z_j|^2$, it is easy to verify that:

$$|1 - (b_n/a_n)| = O(n^{-2}), \quad n \rightarrow \infty.$$

Therefore,

$$\sum_{n \geq 1} |1 - (b_n/a_n)| < \infty \quad (8)$$

and hence, the product $\prod_{j=1}^{\infty} (b_n/a_n)$ converges. To calculate it, we use the well-known representation for the gamma function:

$$\frac{1}{\Gamma(1+z)} = e^{cz} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}, \quad z \in \mathbb{C},$$

where c is the Euler constant. According to this formula, we have:

$$\prod_{n=1}^{\infty} \frac{n+z_j}{n+\sigma} e^{-(z_j-\sigma)/n} = e^{-c(z_j-\sigma)} \frac{\Gamma(1+\sigma)}{\Gamma(1+z_j)}.$$

Taking into account that $\sum_{j=1}^m \lambda_j (\Re z_j - \sigma) = 0$, and $\sigma > -1$, we get:

$$\left| \prod_{n=1}^{\infty} \prod_{j=1}^m \left(\frac{n+z_j}{n+\sigma}\right)^{\lambda_j} \right| = \left| \frac{\Gamma^V(1+\sigma)}{\prod_{j=1}^m \Gamma^{\lambda_j}(1+z_j)} \right|.$$

Hence,

$$\prod_{j=1}^{\infty} (b_n/a_n) = \Gamma^V(1+\sigma) \prod_{j=1}^m |\Gamma(1+z_j)|^{-\lambda_j}.$$

Applying the comparison theorem ([5], Theorem 2.1) (which is possible due to Condition (8)) and [6], Lemma 1, we conclude that Theorem 1 holds for $k = 0$.

Now, let $k \geq 1$. Represent the probability on the left-hand side of (7) as $\mathbf{P}(X_1 + X_2 < \varepsilon)$, where $X_1 = \sum_{n=1}^k a_n |\xi_n|^p$, $X_2 = \sum_{n \geq 1} a_{n+k} |\xi_{n+k}|^p$, and let us denote $F_1(\cdot)$, $F_2(\cdot)$ the distribution functions of X_1 , X_2 , respectively.

It is easy to see that for any positive a_n :

$$\mathbf{P}(a_n |\xi_n|^p < \varepsilon) \sim \sqrt{\frac{2}{\pi}} \left(\frac{\varepsilon}{a_n} \right)^{\frac{1}{p}}, \quad \varepsilon \rightarrow +0.$$

Hence, using [7], Corollary 1 and the just proven statement of Theorem 1 for $k = 0$, we obtain:

$$F_1(\varepsilon) \sim \frac{\left(\Gamma(1 + \frac{1}{p}) \sqrt{\frac{2}{\pi}} \varepsilon^{\frac{1}{p}} \right)^k}{\Gamma(1 + \frac{k}{p})} \left(\prod_{j=1}^k a_j \right)^{-\frac{1}{p}}, \quad (9)$$

$$F_2(\varepsilon) \sim \left(\prod_{j=1}^m |\Gamma(k+1+z_j)|^{-\lambda_j} \right)^{1/p} \tilde{C} \varepsilon^{\tilde{A}} \exp(-B \varepsilon^{-\frac{1}{p-1}})$$

as $\varepsilon \rightarrow +0$, where $\tilde{C}$ and $\tilde{A}$ are defined similarly to C and A in Proposition 1 with σ being replaced by $k + \sigma$.

From [7], Theorem 5, Equation (1.24), and Remark 3 and (9), it follows that:

$$\mathbf{P}(X_1 + X_2 < \varepsilon) \sim \Gamma(1 + \alpha) F_2(\varepsilon) F_1(1/|g'_2(\varepsilon)|), \quad \varepsilon \rightarrow +0, \quad (10)$$

where $\alpha = k/p$ and $g_2(\varepsilon) = B \varepsilon^{-\frac{1}{p-1}}$. The observation that the right-hand sides in (10) and (7) coincide finally proves Theorem 1.

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